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Counting Commutativities in Finite Algebraic Systems

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Abstract. We examine the total possible number of commutativities in a finite algebraic system, concentrating on groups, but also examining rings and semigroups. Numerical restrictions are found and bounds for the total number of commutativities in subgroups and factor groups are derived. Finally, a curious connection with group representations is explored.

1. Introduction

Consider the Cayley table of a finite group G. For $a, b, \in G$, if $ab = ba$, we place a 1 in each of the boxes corresponding to ab and ba. This is called a commutativity in G. Otherwise we put a 0 in each of these boxes, indicating a non-commutativity in G. If G is an abelian group, there will be a 1 in each box, so we disregard this uninteresting case.

We call this matrix of 1's and 0's the commutativity chart for G. Here for example is the commutativity chart for S_3, the group of all permutations on the set $\{1, 2, 3\}$ under composition. S_3 is in fact the smallest non-abelian group.

	e	(123)	(132)	(12)	(13)	(23)
e	1	1	1	1	1	1
(123)	1	1	1	0	0	0
(132)	1	1	1	0	0	0
(12)	1	0	0	1	0	0
(13)	1	0	0	0	1	0
(23)	1	0	0	0	0	1

We denote by $I(G)$ the number of times that 1 appears in the commutativity chart and by $O(G)$ the number of times that 0 appears. Thus $I(S_3) = 18$ and $O(S_3) = 18$ also.

In general we see that $I(G) + O(G) = |G|^2$ and $O(G) > 0$ since we are assuming G is non-abelian. Also we have $I(G) > 0$ since for

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example $xx = xx$ for all $x \in G$. One of our objectives of this note will be to discuss the possible values of $I(G)$ and $O(G)$, where G is a finite non-abelian group and to investigate the values of $I(S)$ and $O(S)$ for other non-commutative algebraic systems S.

Since if $ab \neq ba$ then $ba \neq ab$ and $xx = xx$ for all x, we see that $O(G)$ is always an even number, but there are examples to show that $I(G)$ can be either even or odd. For example, $I(A_4) = 48$, where A_4 is the alternating group of order 12, while $I(G(21)) = 105$, where $G(21)$ is the non-abelian group of order 21. We emphasise that throughout, G denotes a finite non-abelian group.

2. Some Elementary Results

Let us recall some facts from elementary group theory. Two elements x and y in G are said to be conjugate if there exists $w \in G$ with $y = w^{-1}xw$. The relation of conjugacy is easily seen to be an equivalence relation on G, under which G is partitioned into disjoint conjugacy classes. For example, in the group S_3, the conjugacy classes are $\{e\}, \{(123), (132)\}$ and $\{(12), (13), (23)\}$.

In general, let G have exactly $k(G)$ conjugacy classes and let $Cl(x)$ be the class containing x. Let $C_G(x)$, the centralizer of x in G, be the subgroup of G given by $C_G(x) = \{a \in G \mid ax = xa\}$. There is a nice connection between conjugacy classes and centralizers viz. $|Cl(x)| = (G : C_G(x))$, i.e. the number of cosets of $C_G(x)$ in G, and both these numbers are divisors of $|G|$.

From the definition, we have that

$$I(G) = \sum_{x \in G} |C_G(x)| = \sum_{x \in G} \frac{|G|}{|Cl(x)|}$$

$$= |G| \sum_{x \in G} \frac{1}{|Cl(x)|} = |G|k(G). \text{ See [5].}$$

It follows that $O(G) = |G|^2 - I(G) = |G|(|G| - k(G))$. Thus in the case of S_3, since $k(S_3) = 3$, we have $I(G) = 6 \cdot 3 = 18$ and $O(G) = 6 \cdot (6-3) = 18$, in agreement with our previous calculations.

Theorem 2.1. *If $|G|$ is odd, then $k(G)$ is odd.*

Proof. If $|G|$ is odd, since $O(G)$ is even and $O(G) = |G|(|G| - k(G))$, we see that $|G| - k(G)$ must be even, so $k(G)$ is odd. $\square$

We note that the converse of this result is not true; $k(S_3) = 3$, but $|S_3| = 6$.

Theorem 2.2. *$I(G)$ is odd if and only if $|G|$ is odd.*

Proof. If $|G|$ is odd then by Theorem 2.1 $k(G)$ is odd, so $I(G) = |G|k(G)$ is odd. Conversely, if $I(G)$ is odd then $|G|$ clearly must be odd. □

In fact the smallest possible odd value of $I(G) = 105 = 21 \cdot 5$, arising from $G(21)$, which is the smallest odd-order non-abelian group. We remark that Theorem 2.1, which says that if $|G|$ is odd, then $|G| - k(G) \equiv 0 \pmod 2$, can be improved upon considerably using the theory of matrix group representations. A lovely theorem of Burnside [3] states that if $|G|$ is odd, then $|G| - k(G) \equiv 0 \pmod{16}$.

Again $G(21)$ shows that this result is the best possible. Since $O(G) = |G|(|G| - k(G))$ we have

Theorem 2.3. *If $|G|$ is odd, then $O(G)$ is a multiple of $16|G|$.*

Again, $O(G(21)) = 336 = 16 \cdot 21$, shows that this result is the best possible.

We now investigate the possible values of $I(G)$ and $O(G)$ as G ranges over all finite non-abelian groups. For a given group G it is easy, if tedious, to calculate the value of $k(G)$, and for certain classes of groups, and for groups of small order, this information is readily available from a variety of sources.

In particular let D_n be the dihedral group of order $2n$ $(n > 2)$ given by

$$< a, b \mid a^n = 1 = b^2; b^{-1}ab = a^{-1} >$$

Then if $n(= 2m)$ is even, we have $k(D_{2m}) = m+3$, making $I(D_{2m}) = 4m(m+3) = 4m^2 + 12m$.

If $n(= 2m+1)$ is odd, then $k(D_{2m+1}) = m+2$, so $I(D_{2m+1}) = (4m+2)(m+2) = 4m^2 + 10m + 4$.

The values of $O(D_n)$ can be found from $O(G) = |G|^2 - I(G)$.

The symmetric group S_n of order $n!$ has exactly $p(n)$ conjugacy classes, where $p(n)$ is the (integer) partition function, so $I(S_n) = n!p(n)$ and $O(S_n) = n!(n! - p(n))$.

For distinct odd primes p and q, with $p < q$ where $p|(q-1)$, there is a unique non-abelian group $G(pq)$ of order pq. Easy calculations show that $G(pq)$ has exactly $p + \frac{q-1}{p}$ conjugacy classes, so that $I(G(pq)) = q(p^2 + q - 1)$ and $O(G(pq)) = p^2q^2 - I(G) = q(q-1)(p^2-1)$.

We now present a chart with three columns. In the first column are the possible orders of a finite non-abelian group G. In the second and third columns we give the values of $I(G)$ and $O(G)$ for each non-abelian group of order $|G|$. Since it is known that there are only finitely many groups with a given order and also only finitely many groups with a given number of conjugacy classes ([6], [9]) , we see that there are just finitely many (maybe zero) groups with a given $I(G)$ or a given $O(G)$. Note that there may be several different groups of order $|G|$ with the same $k(G)$ and hence the same $I(G)$ and $O(G)$.

$\boldsymbol{\lvert G\rvert}$	$\boldsymbol{I(G)}$	$\boldsymbol{O(G)}$
6	18	18
8	40	24
10	40	60
12	48	96
12	72	72
14	70	126
16	112	144
16	160	96
18	108	216
18	162	162
20	100	300
20	160	240
21	105	336
22	154	330
24	120	456
24	168	408
24	192	384
24	216	360
24	288	288
24	360	216
26	208	468
27	297	432
28	280	504
30	270	630
30	360	540
30	450	450
32	352	672
32	448	576

$\boldsymbol{\lvert G\rvert}$	$\boldsymbol{I(G)}$	$\boldsymbol{O(G)}$
32	544	480
34	340	816
36	216	1080
36	324	972
36	360	936
36	432	864
36	648	648
38	418	1026
39	273	1248
40	400	1200
40	520	1080
40	640	960
40	1000	600
42	294	1470
42	420	1344
42	504	1260
42	630	1134
42	882	882
44	616	1320
46	598	1518
48	384	1920
48	480	1824
48	576	1728
48	672	1632
48	720	1584
48	768	1536
48	864	1440
48	1008	1296

$\boldsymbol{\lvert G\rvert}$	$\boldsymbol{I(G)}$	$\boldsymbol{O(G)}$
48	1152	1152
48	1440	864
50	700	1800
50	1000	1500
52	364	2340
52	832	1872
54	540	2376
54	810	2106
54	972	1944
54	1188	1728
54	1458	1458
55	385	2640
56	448	2688
56	952	2184
56	1120	2016
56	1960	1176
57	513	2736
58	928	2436
60	300	3300
60	540	3060
60	720	2880
60	900	2700
60	1080	2520
60	1200	2400
60	1440	2160
60	1800	1800

We note that for direct products of groups G_1 and G_2, $I(G_1 \times G_2) = I(G_1)I(G_2)$ and $k(G_1 \times G_2) = k(G_1)k(G_2)$. However, $O(S_3)O(S_3) = 18 \cdot 18 = 324 \neq 972 = O(S_3 \times S_3)$.

By [7] we have $\frac{k(G)}{|G|} \leq \frac{5}{8}$ so $I(G) \leq \frac{5}{8}|G|^2$, and $O(G) \geq \frac{3}{8}|G|^2$.

Also, by examining Cayley tables, it is clear that $I(G) \geq 3|G| - 2$, so that $O(G) \leq |G|^2 - 3|G| + 2$.

Thus, consulting the above charts, we see that the allowable values for $I(G)$ are: 18, 40, 48, 70, 72, 100, 105, 108, 112, 120, 154, 160, 162, 168, 192, 208, 216, 270, 273, 280, 288, 294, 297, 300, 324, 340, 352, 360, 364, 384, 385, 400, 418, 432,...

Similarly the allowable values for $O(G)$ are: 18, 24, 60, 72, 96, 126, 144, 162, 216, 240, 288, 300, 330, 336, 360, 384, 408, 432, 450, 456, 468, 480, 504, 540, 576, 600, 630, 648, 672,...

We mention that the function $|G| - k(G)$ is examined in considerable detail in [1] . Also, one can show that $I(G) = O(G)$ if and only if $G/Z(G) = S_3$, where $Z(G)$ is the centre of G.

3. Subgroups and Factor Groups

Gallagher [4] gives elementary proofs of the following results for all finite groups G, where H is a subgroup of G and N is a normal subgroup of G.

(i) $k(H) < (G : H)k(G)$, for $H \neq G$;
(ii) $k(G) \leq (G : H)k(H)$;
(iii) $k(G) \leq k(G/N)k(N)$.

In our notation, these results immediately become

Theorem 3.1. *(i) $I(H) < I(G)$ if $H \neq G$;*
(ii) $I(G) \leq (G : H)^2 I(H)$;
(iii) $I(G/N) \geq I(G)/I(N)$.

4. Other Algebraic Systems

Let $S = \{a, b\}$ be a set of cardinality 2. Define a binary operation $*$ on S as follows

$*$	a	b
a	a	b
b	a	b

Easy calculations show that S is a non-commutative semigroup with $I(S) = 2 = O(S)$, so the sequences of allowable value of $I(S)$ and $O(S)$ for semigroups are different from those of $I(G)$ and $O(G)$ for groups.

The reader is invited to determine the sequences of allowable values of $I(S)$ and $O(S)$ for non-commutative semigroups.

Moving on to rings, consider the following set of 2×2 matrices over $\mathbb{Z}_2$ under matrix addition and multiplication mod 2:

$$R = \{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)\}$$

It is easy to see that $\{R, +, \cdot\}$ is a non-commutative ring of order 4. The commutativity chart for $\{R, \cdot\}$ looks as follows:

	$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$
$\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$	1	1	1	1
$\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right)$	1	1	0	0
$\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)$	1	0	1	0
$\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$	1	0	0	1

Thus $I(R) = 10$ and $O(R) = 6$. This single example shows that the sequences of allowable values of $I(R)$ and $O(R)$ for finite rings are different from those for finite groups.

Again the reader is invited to investigate this problem for other algebraic systems such as near–rings, loops, quasigroups etc.

We remark that if S is a set with $|S| = n$ we can always choose closed binary operations $*$ and $\circ$ on S such that $I(S, *) = n$ $(n > 1)$, and $O(S, \circ) = 2n$ (n arbitrary).

For example, if $S = \{a, b, c\}$ define $*$ by

$*$	a	b	c
a	a	a	c
b	b	b	b
c	a	c	c

to achieve $I(S, *) = 3$ and similarly for the general case.

$\circ$	a	b	c
a	a	a	a
b	b	a	a
c	b	b	c

Also in the second example $O(S, \circ) = 6$ and similarly for the general case.

5. A Connection with Matrix Representations of Groups

There is a surprising connection between $I(G)$ and matrix representations of G. For definitions we refer the interested reader to [5].

Let $d_i, 1 \leq i \leq k$, be the degrees of the irreducible complex matrix representations of a finite group G i.e. the sizes of the square matrices involved. There are $k(G)$ of these where G has $k(G)$ conjugacy classes.

$$\text{Let } T(G) = \sum_{i=1}^{k(G)} d_i.$$

[For example, for $S_3, (d_1, d_2, d_3) = (1, 1, 2)$ so $T(S_3) = 4$.]
Using the Cauchy-Schwarz inequality on $(1, 1, 1, \ldots, 1)$ and $(d_1, d_2, d_3, \ldots, d_k)$ as in [8] , and remembering that $\sum_{i=1}^{k} d_i^2 = |G|$, we find that

$$(T(G))^2 < k(G)|G| = I(G). \text{ (G non–abelian)}$$

Let us see how this inequality looks for some specific groups of small order.
[We use the notation Q_n for the dicyclic group of order $4n$ for $n > 1$ where $Q_n = \langle a, b | a^{2n} = 1; b^2 = a^n, b^{-1}ab = a^{-1} \rangle$].

Group	$\boldsymbol{(T(G))^2}$	$\boldsymbol{I(G)}$	
S_3	16	18	
D_4	36	40	
Q_2	36	40	(quaternion group)
D_5	36	40	
D_6	64	72	
Q_3	64	72	
A_4	36	48	
D_7	64	70	
S_4	100	120	

When we write $T(G) < \sqrt{I(G)}$ in a particular case such as D_4, we get $T_4 < \sqrt{I(D_4)} = \sqrt{40} = 6.3245$. Now $T(D_4)$ is an integer so $T(D_4) \leq 6$ and 6 is actually the correct answer!

Similarly in the case of S_4, we get $T(S_4) < \sqrt{120} = 10.95445$. Again $T(S_4)$ is an integer, so $T(S_4) \leq 10$ which gives the correct value of $T(S_4) = 10$.

It is remarkable that such a basic function as $I(G)$, whose values can be read from the Cayley table, can be used to find information about $T(G)$, which would appear to be a much more advanced group theoretic concept.

6. Analogues of $I(G)$ and $O(G)$

There are so many analogies between $k(G)$ and $T(G)$ (as defined in section 5) that we make the following definitions:

For a finite non–abelian group G, let $N(G) = |G|T(G)$ and $M(G) = |G|(|G| - T(G))$.

It is not immediately clear what the interpretations of $N(G)$ and $M(G)$ are, but these functions have many properties analogous to $I(G)$ and $O(G)$. To save space we state results only, but methods of proof are very similar to those for results concerning $I(G)$ and $O(G)$. We remark that the properties of $|G| - T(G)$ are examined in some detail in [2] .

Theorem 6.1. *$I(G) < N(G)$ and $O(G) > M(G)$.*

Theorem 6.2. *There are only finitely many groups G (maybe zero) with a given $N(G)$ or a given $M(G)$.*

Theorem 6.3. *$N(G)$ is odd if and only if $|G|$ is odd.*

Theorem 6.4. *If $|G|$ is odd, $M(G)$ is a multiple of $4|G|$.*

Theorem 6.5. *If H is a proper subgroup of G, then $N(H) < N(G)$.*

Theorem 6.6. *$M(G)$ is always even.*

Theorem 6.7. *$N(G) < |G|^{\frac{3}{2}}(k(G))^{\frac{1}{2}}$.*

Theorem 6.8. *$N(G_1 \times G_2) = N(G_1) \cdot N(G_2)$.*

Theorem 6.9. *For the non–abelian group $G(pq)$, we have $N(G) = pq(p+q-1)$ and $M(G) = pq(p-1)(q-1)$, where p and q are distinct odd primes.*

Theorem 6.10. *$N(G) \leq \frac{3}{4}|G|^2$ and $M(G) \geq \frac{1}{4}|G|^2$.*

Finally, we give a chart of values of $N(G)$ and $M(G)$ for non–abelian groups G of small order which leads to information about the sequences of allowable values of $N(G)$ and $M(G)$.

$\|G\|$	$N(G)$	$M(G)$
6	24	12
8	48	16
10	60	40
12	72	72
12	96	48
14	112	84
16	120	136
16	192	64
18	120	204
20	160	240
20	240	160
21	189	252

$\|G\|$	$N(G)$	$M(G)$
22	264	220
24	240	336
24	288	288
24	336	240
24	384	292
24	432	144
26	364	312
27	405	324
28	448	336
30	480	420
30	540	360
30	600	300

The sequence of allowable values of $N(G)$ thus begins $24, 48, 60, 72, 96, 112, 120, 160, 189, 192, 240, 264, 288, \ldots$

The sequence of allowable values of $M(G)$ thus begins $12, 16, 40, 48, 64, 72, 84, 136, 144, \ldots$

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