

A LEISURELY ELEMENTARY TREATMENT OF STIRLING'S FORMULA

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ABSTRACT. A self-contained account of Stirling's formula for $n!$ is presented that is based on definitions of the constants e and π that appear in it, and uses only the rudiments of the analysis of the convergence of numerical sequences, infinite series, and infinite products.

1. INTRODUCTION

Stirling's formula tells us that $n!$ is asymptotically equal to $n^n e^{-n} \sqrt{2\pi n}$, implying that

$$\lim_{n \rightarrow \infty} \frac{n!}{n^{n+\frac{1}{2}} e^{-n}} = \sqrt{2\pi}. \quad (1)$$

This formula is remarkable because it provides an approximation of $n!$ that consists of a non-integral expression involving the irrational numbers e and π . In this respect, to somebody seeing it for the first time, it must be just as amazing as Euler's identity linking the numbers $-1, i, e$ and π via the equation $e^{i\pi} = -1$.

Over the years, many proofs of equation 1 have been published, and the literature is replete with ones of many different kinds. A sample of these can be found in [2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. With one exception—and the present offering is no different in this regard—these articles contain *ad-hoc* proofs of equation 1. Reference [6] is the exception: it is Walter Hayman's most cited article, and his proof illustrates a powerful method for asymptotically estimating the coefficients of a class of power series of analytic functions, termed *admissible* by him.

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In this note we outline a derivation of equation 1 that could be taught at the end of a first course of analysis that includes a thorough treatment of the theory of convergence of sequences, series and infinite products, and the basic properties of the exponential and trigonometrical functions. It eschews integration completely.

We begin by establishing that there are positive constants a, b such that

$$an^{n+\frac{1}{2}}e^{-n} \leq n! \leq bn^{n+\frac{1}{2}}e^{-n}, \quad n = 1, 2, \dots \quad (2)$$

This is very often all that is needed for many applications. Next, we show that the positive sequence

$$S_n = \frac{n!}{n^{n+\frac{1}{2}}e^{-n}}, \quad n = 1, 2, \dots$$

is decreasing, and so convergent; thereafter, we concentrate on showing that $\sqrt{2\pi}$ is its limit. This is the heart of the matter, and, to deal with it, we establish Wallis's product formula for π from first principles. Along the way we encounter the central binomial coefficients and derive an asymptotic expression for them; whence, as an application, we evaluate $\int_0^\infty e^{-s^2} ds$.

2. PROOF OF STATEMENT 2

A rigorous course on sequences and series would surely include a definition of the number e , and proofs that the sequences

$$u_n = (1 + 1/n)^n, \quad v_n = (1 - 1/n)^n, \quad n = 1, 2, \dots$$

are both strictly increasing, and converge to e and e^{-1} , respectively, and hence that

$$\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}, \quad n = 1, 2, \dots$$

By considering the Geometric Means of the two lists of positive numbers u_1, u_2, \dots, u_n , and v_2, v_3, \dots, v_n , and utilising the last pair of displayed inequalities, it can be confirmed that the following crude estimates hold for $n!$, viz.,

$$(n+1)^n e^{-n} < n! < en^{n+1}e^{-n}, \quad n = 1, 2, \dots,$$

which go someway towards explaining the presence of e in equation 1. In particular, $\frac{1}{\sqrt{n}} < S_n < e\sqrt{n}$, $n = 1, 2, \dots$

To obtain a sharper lower bound for S_n , we use the power series expansion of the exponential function.

Lemma 2.1.

$$S_n > e^{-1}, \quad n = 1, 2, \dots$$

Proof. Fix $n \geq 1$. Using the power series expansion for $\exp x$ with $x = n$, we have that, for any natural number m ,

$$e^n > \sum_{k=n}^{n+m} \frac{n^k}{k!} = \frac{n^n}{n!} \left(1 + \sum_{k=1}^m \frac{n^k}{\prod_{j=1}^k (n+j)} \right).$$

Now, if $1 \leq k \leq m$,

$$\begin{aligned} \frac{n^k}{\prod_{j=1}^k (n+j)} &= \frac{1}{\prod_{j=1}^k (1 + \frac{j}{n})} \\ &\geq \prod_{j=1}^k \exp(-\frac{j}{n}) \\ &= \exp(-\frac{k(k+1)}{2n}) \\ &\geq \exp(-\frac{m(m+1)}{2n}). \end{aligned}$$

Hence, for any integer $m \geq 1$,

$$\frac{e^n n!}{n^n} > (m+1) \exp(-\frac{m(m+1)}{2n}).$$

The stated result now follows from this upon setting $m = \lfloor \sqrt{n} \rfloor$. \square

Theorem 2.2. *The sequence (S_n) is strictly monotonic decreasing and converges to a positive number.*

Proof. Evidently, (S_n) is strictly monotonic decreasing if and only if

$$e < (1 + \frac{1}{n})^{n+\frac{1}{2}}, \quad n = 1, 2, \dots \quad (3)$$

But, $2^{n-1} \leq n!$, $n = 1, 2, \dots$, and so, if $0 \leq x < 1$, then

$$e^{2x} = 1 + 2 \sum_{n=1}^{\infty} \frac{2^{n-1}}{n!} x^n \leq 1 + 2 \sum_{n=1}^{\infty} x^n = \frac{1+x}{1-x},$$

with equality if and only if $x = 0$. Hence, inequality 3 follows on setting $x = 1/(2n+1)$. Thus, (S_n) is monotonic decreasing. Since it is bounded below by e^{-1} , it converges to a positive number as claimed. \square

Corollary 2.3.

$$e^{-1}n^{n+\frac{1}{2}}e^{-n} \leq n! \leq en^{n+\frac{1}{2}}e^{-n}, \quad n = 1, 2, \dots \quad (4)$$

Proof. This is an immediate consequence of Lemma 2.1, and the fact that $S_n \leq S_1 = e$, $n = 1, 2, \dots$ \square

This result implies statement 2, and suffices in many situations where one is interested in crude but tight estimates for sequences involving the factorial function. For instance, one can use it to obtain good bounds for the sequence of central binomial coefficients $\binom{2n}{n}$, $n = 1, 2, \dots$. Indeed, it easily follows from statement 4 that

$$e^{-3}\sqrt{2}4^n n^{-\frac{1}{2}} \leq \binom{2n}{n} \leq e^3\sqrt{2}4^n n^{-\frac{1}{2}}, \quad n = 1, 2, \dots$$

In passing, we note that the sequence (C_n) defined by

$$C_n = \frac{\sqrt{n} \binom{2n}{n}}{4^n}, \quad n = 1, 2, \dots$$

is strictly increasing, and so converges to a positive number, which we denote by C ; we'll determine its value in Section 4.

3. THE LIMIT OF THE SEQUENCE $(C_n S_n)$

It follows from Theorem 2.2 that the limit $S \equiv \lim_{n \rightarrow \infty} S_n$ exists, and $S \geq e^{-1}$. We proceed to identify the product CS .

Theorem 3.1. $CS = \sqrt{2}$.

Proof. To see this, note that

$$\begin{aligned} S_{2n} &= \frac{(2n)!}{(2n)^{2n} e^{-2n} \sqrt{2n}} \\ &= \frac{(2n)! \sqrt{n}}{(n!)^2 2^{2n+1/2}} \left(\frac{n!}{n^n e^{-n} \sqrt{n}} \right)^2 \\ &= \frac{C_n S_n^2}{\sqrt{2}}. \end{aligned}$$

Hence, by the product and subsequence rules for limits of sequences, $S = \frac{CS^2}{\sqrt{2}}$, and so $CS = \sqrt{2}$, as claimed. \square

Thus, once we know C , we know S , and *vice versa*.

4. THE VALUE OF C

Starting with the definition of π as the smallest positive root of the sine function, in this section we determine C by relating it to the value of the Wallis product.

4.1. **The Wallis product.** Since, for every positive integer n ,

$$\begin{aligned} C_n^2 &= \left(\frac{(2n)! \sqrt{n}}{(n!)^2 2^{2n}} \right)^2 \\ &= \frac{n}{2n+1} \frac{((2n)!)^2 (2n+1)}{2^{4n} (n!)^4} \\ &= \frac{n}{2n+1} \prod_{k=1}^n \left(1 - \frac{1}{4k^2} \right), \end{aligned}$$

we have that

$$2C^2 = \prod_{k=1}^{\infty} \left(1 - \frac{1}{4k^2} \right).$$

We proceed to relate the value of this infinite product ¹ to the limit of another sequence, which comes from a factorisation of $\sin(2n+1)x$. To obtain this, fix the positive integer n , and let $m = 2n+1$, for convenience. Let $\omega = e^{2\pi i/m}$. Then, for any complex number z ,

$$\begin{aligned} z^{2m} - 1 &= \prod_{r=1}^m (z^2 - \omega^r) \\ &= (z^2 - 1) \prod_{r=1}^n (z^2 - \omega^r) \prod_{r=n+1}^{2n} (z^2 - \omega^r) \\ &= (z^2 - 1) \prod_{r=1}^n (z^2 - \omega^r)(z^2 - \bar{\omega}^r) \\ &= (z^2 - 1) \prod_{r=1}^n (z^4 - 2z^2 \Re \omega^r + 1). \end{aligned}$$

¹Of course, if, at this point, we accept as known that this is $2/\pi$, we can delete the rest of this section. However, rather than do so, we prefer to prove this result *ab initio*.

Hence, letting $z = e^{ix}$, where x is any real number, it follows that

$$\begin{aligned}\sin(mx) &= 2^n \sin x \prod_{r=1}^n (\cos(2x) - \cos(2r\pi/m)) \\ &= 2^{2n} \sin x \prod_{r=1}^n (\sin^2(r\pi/m) - \sin^2 x).\end{aligned}$$

In particular, we conclude from this that, first of all,

$$m = 2^{2n} \prod_{r=1}^n \sin^2(r\pi/m),$$

and thence that

$$\sin(mx) = m \sin x \prod_{r=1}^n \left(1 - \frac{\sin^2 x}{\sin^2(r\pi/m)}\right).$$

Choose $x = \pi/2m$, and consider the limiting behaviour of

$$\frac{1}{m \sin(\pi/2m)} = \prod_{r=1}^n \left(1 - \frac{\sin^2(\pi/2m)}{\sin^2(r\pi/m)}\right)$$

as $n \rightarrow \infty$. Clearly, the sequence on the left side converges to $2/\pi$. To handle the sequence on the right side, we need a variant of Tannery's theorem [1, 3], tailored to the situation in hand. Since this may not be familiar to many people, we include a proof for completeness.

Lemma 4.1. *Suppose $a_m(n)$, $m, n = 1, 2, \dots$, is a double sequence of complex numbers, with the following properties:*

- (1) *there is a sequence of positive numbers m_r , $r = 1, 2, \dots$, such that $\prod_{r=1}^{\infty} (1 + m_r)$ is convergent, and*

$$\sup_{n \geq 1} |a_r(n)| \leq m_r, \quad r = 1, 2, \dots;$$

- (2) *for each m , the sequence $a_m(n)$, $n = 1, 2, \dots$, converges to a_m ;*

- (3) *the sequence P_n , defined by*

$$P_n = \prod_{m=1}^n (1 + a_m(n)), \quad n = 1, 2, \dots,$$

converges to P , say.

Then

$$P = \prod_{m=1}^{\infty} (1 + a_m).$$

Proof. Fix m and let $n > m$. Then

$$\begin{aligned} \left| P_n - \prod_{r=1}^m (1 + a_r(n)) \right| &= \left| \prod_{r=1}^m (1 + a_r(n)) \right| \left| \prod_{r=m+1}^n (1 + a_r(n)) - 1 \right| \\ &\leq \prod_{r=1}^m (1 + m_r) \left(\prod_{r=m+1}^n (1 + m_r) - 1 \right) \\ &= Q_n - Q_m, \end{aligned}$$

where

$$Q_n = \prod_{r=1}^n (1 + m_r), \quad n = 1, 2, \dots$$

Under our assumptions, the sequence Q_n converges to the infinite product

$$Q = \prod_{m=1}^{\infty} (1 + m_r),$$

while $\lim_{n \rightarrow \infty} a_m(n) = a_m$, and $\lim_{n \rightarrow \infty} P_n = P$. Hence, letting $n \rightarrow \infty$, we deduce from the inequality just established that

$$\left| P - \prod_{r=1}^m (1 + a_r) \right| \leq Q - Q_m, \quad m = 1, 2, \dots,$$

whence the result follows. \square

We apply Lemma 4.1 with

$$a_r(n) = -\frac{\sin^2(\pi/2m)}{\sin^2(r\pi/m)}, \quad r = 1, 2, \dots, n, \quad n = 1, 2, \dots$$

Since $2x/\pi \leq \sin x \leq x$, if $0 \leq x \leq \pi/2$, it follows that

$$\sup\{|a_r(n)| : n = 1, 2, \dots\} \leq \frac{\pi^2}{16r^2}, \quad r = 1, 2, \dots$$

Also

$$\lim_{n \rightarrow \infty} a_r(n) = -\frac{1}{4r^2}.$$

Hence

$$\lim_{n \rightarrow \infty} \prod_{r=1}^n \left(1 - \frac{\sin^2(\pi/2m)}{\sin^2(r\pi/m)} \right) = \prod_{r=1}^{\infty} \left(1 - \frac{1}{4r^2} \right).$$

Combining our results we may conclude that

$$2C^2 = \prod_{r=1}^{\infty} \left(1 - \frac{1}{4r^2}\right) = \frac{2}{\pi},$$

and so $C = 1/\sqrt{\pi}$. It follows that $S = \sqrt{2\pi}$.

5. THE VALUE OF THE INTEGRAL $\int_0^{\infty} e^{-s^2} ds$

As a quick look at the references reveals, many proofs of equation 1 rely on knowing the value of this integral. Here, we reverse matters and essentially derive its value from equation 1, by identifying it directly with $C\pi/2$. Conversely, of course, making this identification, we can dispense entirely with the previous section and obtain equation 1 very quickly.

Since $\binom{2n}{n}$ is the constant term in the trigonometrical polynomial $(e^{ix} + e^{-ix})^{2n}$, it's clear that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \cos^{2n} x dx = 2^{-2n} \binom{2n}{n}, \quad n = 0, 1, \dots$$

Hence, for $n = 1, 2, \dots$,

$$\begin{aligned} C_n &= \frac{\sqrt{n} \binom{2n}{n}}{2^{2n}} \\ &= \frac{\sqrt{n}}{2\pi} \int_{-\pi}^{\pi} \cos^{2n} x dx \\ &= \frac{2\sqrt{n}}{\pi} \int_0^{\pi/2} (1 - \sin^2 x)^n dx \\ &= \frac{2\sqrt{n}}{\pi} \int_0^1 (1 - t^2)^{n-\frac{1}{2}} dt \\ &= \frac{2}{\pi} \int_0^{\sqrt{n}} \left(1 - \frac{s^2}{n}\right)^{n-\frac{1}{2}} ds. \end{aligned}$$

But, if $0 \leq s \leq \sqrt{n}$, then

$$\left(1 - \frac{s^2}{n}\right)^{n-\frac{1}{2}} \leq e^{-(n-\frac{1}{2})s^2/n} = e^{-s^2} e^{s^2/2n} \leq e^{-s^2} \sqrt{e}.$$

Furthermore,

$$\lim_{n \rightarrow \infty} \left(1 - \frac{s^2}{n}\right)^{n-\frac{1}{2}} = e^{-s^2}.$$

Hence, by the variant of Tannery's theorem for integrals, or by Lebesgue's dominated convergence theorem,

$$C = \lim_{n \rightarrow \infty} C_n = \frac{2}{\pi} \int_0^{\infty} e^{-s^2} ds.$$

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