EDITORIAL

The present issue of the Bulletin includes an article by Patrick Browne about printing 3-D mathematical objects. Due to the expense of colour printing, the images in the print edition are black-and-white. They may be viewed in colour in the online edition. The online edition is now hosted at the new IMS website

irishmathsoc.org

(courtesy of TCD and Richard Timoney), but also continues to be available at the old address, at least for the time being.

This Boole Centenary year was much celebrated worldwide, but most splendidly in Cork. As it ends, we print a couple of articles about Boolean rings, including an entertaining gem by the Cork *maestro*, and Boole biographer, Des MacHale.

Ari Laptev, Director of the Institut Mittag-Leffler has asked us to inform IMS members that the 27th Nordic Congress of Mathematics in Stockholm on March 16-20 will mark the centenary of the foundation of the institute, the oldest Mathematics research institute in the world. The Congress is free to attend, and further details are available at mittag-leffler.se/congress-2016.

Colm Mulcahy's tireless efforts to document the history and scope of Irish mathematics have produced remarkable results in just over a year. His Annals site cardcolm.org/AIMM.html has grown beyond all expectation. Members will probably want to have a copy of the Irish mathematical calendar, compiled by Colm and Eoin Gill (of Maths Week fame).

Material from the National Forum for Teaching and Learning is available at the link weusemaths.ie.

The next main scientific meeting of the Society will take place in TCD in April 2016, and will mark the fortieth anniversary of our foundation.

Helge Holden wrote that the IMU asks us to inform members about the The 4th Heidelberg Laureate Forum (HLF), see heidelberg-laureate-forum.org, which will take place on September 18-23, 2016. At HLF all winners of the Fields Medal, the Abel Prize, the Alan Turing Award and the Nevanlinna Medal are invited to attend. In addition, young and talented computer scientists and mathematicians are invited to apply for participation. The three previous HLFs have been an exceptional success. The HLF serves as a great platform for interaction between the masters in the fields of mathematics and computer science and young talents.

Sidney Kolpas wrote to draw attention to his article with Sue Hawes about the Irish Mathematician Oliver Byrne, available at the MAA website. The link is maa.org/press/periodicals/convergence/-oliver-byrne-the-matisse-of-mathematics-biography-1810-1829

The December issue of the EMS Newsletter is on line:
ems-ph.org/journals/journal.php?jrn=news

The series of papers on the 25th anniversary of the EMS continues, with articles on more recent ECMs and the editorial by R. Elwes about the celebration of the EMS Jubilee in Paris.

In this issue you can read the interview of the Abel laureate L. Nirenberg, the interview of M. Bhargava, a written version of the lecture given by S. Burris in occasion of the bicentennial of George Boole.

The December issue also contains the "Problems for Children" by the late V. Arnold, courtesy of the AMS and the MSRI.

AOF. Department of Mathematics and Statistics, NUI, Maynooth, Co. Kildare

E-mail address: ims.bulletin@gmail.com

LINKS FOR POSTGRADUATE STUDY

The following are the links provided by Irish Schools for prospective research students in Mathematics:

DCU: (Olaf Menkens)
http://www.dcu.ie/info/staff_member.php?id_no=2659
DIT: (Chris Hills)
mailto://chris.hills@dit.ie
NUIG:
mailto://james.cruickshank@nuigalway.ie
NUIM:
http://www.maths.nuim.ie/pghowtoapply
QUB:
http://www.qub.ac.uk/puremaths/Funded_PG_2016.html
TCD:
http://www.maths.tcd.ie/postgraduate/
UCD:
mailto://nuria.garcia@ucd.ie
UU:
http://www.compeng.ulster.ac.uk/rgs/

The remaining schools with Ph.D. programmes in Mathematics are invited to send their preferred link to the editor, a url that works. All links are live, and hence may be accessed by a click, in the electronic edition of this Bulletin[1].

E-mail address: ims.bulletin@gmail.com

[1] http://www.irishmathsoc.org/bulletin/

NOTICES FROM THE SOCIETY

Officers and Committee Members 2015

President	Dr M. Mackey	University College Dublin
Vice-President	Prof S. Buckley	Maynooth University
Secretary	Dr R. Quinlan	NUI Galway
Treasurer	Prof G. Pfeiffer	NUI Galway

Dr P. Barry, Prof J. Gleeson, Dr B. Kreussler, Dr R. Levene, Dr M. Mac an Airchinnigh, Dr M. Mathieu, Dr A. Mustata, Dr J. O'Shea, .

Local Representatives

Belfast	QUB	Dr M. Mathieu
Carlow	IT	Dr D. Ó Sé
Cork	IT	Dr D. Flannery
	UCC	Dr S. Wills
Dublin	DIAS	Prof T. Dorlas
	DIT	Dr C. Hills
	DCU	Dr M. Clancy
	SPD	Dr S. Breen
	TCD	Prof R. Timoney
	UCD	Dr R. Higgs
Dundalk	IT	Mr Seamus Bellew
Galway	UCG	Dr J. Cruickshank
Limerick	MIC	Dr B. Kreussler
	UL	Mr G. Lessells
Maynooth	NUI	Prof S. Buckley
Tallaght	IT	Dr C. Stack
Tralee	IT	Dr B. Guilfoyle
Waterford	IT	Dr P. Kirwan

Applying for I.M.S. Membership

(1) The Irish Mathematical Society has reciprocity agreements with the American Mathematical Society, the Deutsche Mathematiker Vereinigung, the Irish Mathematics Teachers Association, the Moscow Mathematical Society, the New Zealand Mathematical Society and the Real Sociedad Matemática Española.

(2) The current subscription fees are given below:

Institutional member	€160
Ordinary member	€25
Student member	€12.50
DMV, I.M.T.A., NZMS or RSME reciprocity member	€12.50
AMS reciprocity member	$15

The subscription fees listed above should be paid in euro by means of a cheque drawn on a bank in the Irish Republic, a Eurocheque, or an international money-order.

(3) The subscription fee for ordinary membership can also be paid in a currency other than euro using a cheque drawn on a foreign bank according to the following schedule:

If paid in United States currency then the subscription fee is US$ 30.00.
If paid in sterling then the subscription is £20.00.
If paid in any other currency then the subscription fee is the amount in that currency equivalent to US$ 30.00.

The amounts given in the table above have been set for the current year to allow for bank charges and possible changes in exchange rates.

(4) Any member with a bank account in the Irish Republic may pay his or her subscription by a bank standing order using the form supplied by the Society.

(5) Any ordinary member who has reached the age of 65 years and has been a fully paid up member for the previous five years may pay at the student membership rate of subscription.

(6) Subscriptions normally fall due on 1 February each year.

(7) Cheques should be made payable to the Irish Mathematical Society. If a Eurocheque is used then the card number should be written on the back of the cheque.

(8) Any application for membership must be presented to the Committee of the I.M.S. before it can be accepted. This Committee meets twice each year.

(9) Please send the completed application form with one year's subscription to:

The Treasurer, IMS
School of Mathematics, Statistics and Applied Mathematics
National University of Ireland
Galway
Ireland

Deceased Member

It is with regret that we report that Professor Wilhelm Kaup of Tübingen died on 16 June 2015. He was an Honorary Member of the Society.

E-mail address: subscriptions.ims@gmail.com

IMS President's Report 2015

Let me begin by thanking my predecessor Martin Mathieu for his diligent steering of the society as president through 2013-2014. Changes to the committee for 2015 included the election of Stephen Buckley (Maynooth) as vice-president and the departure of Cora Stack (ITT) after six years service.

At the 2014 AGM, a proposal to establish an Education sub-Committee was approved and I am grateful to the members of committee who brought this proposal to fruition in 2015. The sub-committee is chaired by A. O'Farrell (Maynooth). The other current members are C. Mac an Bhaird, C. Stack, A. Ní Shé, E. Oldham, D. Malone, R. Flatley, A. Cronin, C. Lundon, J. Grannell, M. O'Reilly, A. McCluskey, Aoibhinn ní Shúilleabháin and R. Quinlan. The sub-committee will play its part in gathering and representing the views of mathematicians in the Irish education system and in working with other bodies. (For example, the society became a partner of the National Forum for the Enhancement of Teaching and Learning this year.) We wish the members of our education sub-committee well in their work.

Also proposed at the 2014 AGM was a working group to examine the possibility of a new National Committee for Mathematics. This follows dissolution of the RIA Mathematics Committee and its replacement by a combined Physical, Chemical and Mathematical Sciences Committee. Through early 2015, members of the working group teased out some of the issues that might arise in forming a national committee. It became clear that if research and education interests at all levels were to be fairly represented then a national committee could swell to a large number. There would almost inevitably be overlaps both in mission and personnel amongst the new RIA PCMS committee, the IMS committee itself (and the IMS education sub-committee) and the proposed body. Moreover, the necessity of the proposed body is not fully apparent. The idea may be revisited when arguments in favour become stronger.

The current situation for Irish funding of mathematics research, particularly of doctoral students, is a topic of concern to many members. I can report that in 2015 the society was active in presenting these concerns to policy makers and arguing for a fairer deal for the mathematical sciences when funding policy is reviewed. I want to thank members S. Gardiner, G. McGuire and vice-president S. Buckley for their active involvement in this area.

For its part, the IMS was able to support the following meetings in 2015:

- Vertex Operator Algebras (NUIG)
- Irish Geometry Conference (MIC)
- SumTop30/GalTop18 (NUIG)
- Annual Irish Workshop on Mathematics Learning (UCD)
- Groups in Galway (NUIG)

The society's own annual meeting was held in UCC as part of the extensive and wonderful celebrations around George Boole's bicentenary. Thanks are due particularly to S. Wills who organised the IMS strand of the conference. Next year is the 40th anniversary of the IMS as our constitution was adopted at a meeting in TCD in April 1976. To celebrate, we return to Trinity college for the annual meeting on April 15-16th. Ironically perhaps, our constitution stipulates the AGM cannot be held before July and so the 2016 AGM will be held later in the year.

A new internet address for the society has been registered: irishmathsoc.org. The website continues to be kindly maintained by R. Timoney and hosted by TCD but we felt it appropriate that the society should have its own domain, independent of any one institution. New on the web page this year, is a roll of past officers of the society. I am also most grateful to Richard for his help with completing the online archival of the Bulletin which was achieved this year. I encourage members, new and old, to rediscover and enjoy those early issues of the IMS Bulletin/Newsletter. Happily, the current issues maintain the standard and so it is appropriate to finish with a thank you to editor Tony O'Farrell, his editorial team, problem page maintainer Ian Short, and Gordon Lessells who continues to quietly oversee printing and distribution of the bulletin.

Michael Mackey, President
21 December 2015

Minutes of the Meeting of the Irish Mathematical Society
Annual General Meeting
UCC August 28 2015

The meeting commenced at 11.30. There were 15 members present.

(1) **Minutes of 2014 AGM**
The minutes of the 2014 AGM were approved and signed.

(2) **Matters Arising**
- A list of past presidents of the Society has now been assembled.
- The plan to establish a "national committee" for mathematics is in abeyance for the time being, pending clarity on the necessity for such a body. The Physical, Chemical and Mathematical Sciences Committee of the Royal Irish Academy has been in existence for 18 months now and is operating on a cooperative and open basis, with matters that are of interest to a particular discipline being left mostly to the representatives of that discipline. A serious concern was that mathematics would be left without an adequate voice as a minority discipline in a much broader committee; this has not happened so far. At present there is some intersection between the RIA Committee and the IMS Committee but this is not by design. It will be important to maintain channels of communication between the two groups, and M. Mackey will write to the RIA Committee to express this on behalf of the IMS. Should it become desirable to reopen discussions towards the establishment of a "national committee", it will be possible to do so without delay.

(3) **Correspondence**
- A request for a statement of support from the IMS for a bid to host the 2022 International Congress of Mathematicians in Strasbourg has been received. M. Mackey will respond positively.
- The National Forum for Teaching and Learning has invited the IMS to become a partner. This invitation has been accepted.
- C. Hogg of DIAS has requested that the IMS endorse the project `weusemaths.ie` which aims to raise awareness of

the versatility and applicability of mathematics. This was agreed.

- Colm Mulcahy has done extensive work on collecting and organising data for an archive on Irish mathematicians, which can be viewed at `http://cardcolm.org/AIMM.html`. He has invited the IMS to assist with the maintenance of this resource after it is established. The Society is positively disposed towards this idea but the practical details are unclear as yet. It was proposed by S. Dineen that the IMS communicate to Dr Mulcahy its appreciation of the value of this project and of the work that has been done.

(4) **Membership Applications**
Since the 2014 Annual General Meeting, applications from the following persons have been received and approved:

Patrick Browne	NUI Galway
Bernard Hanzon	University College Cork
Fionnán Howard	St Patrick's College
Claus Koestler	University College Cork
Natalia Iyudu	University of Edinburgh
James McTigue	NUI Galway
Elizabeth Oldham	Trinity College Dublin
Henry Ricardo	City University of New York
Stanislav Shkarin	Queen's University Belfast

(5) **President's Business**
The content of the President's report was already covered under Items 2. and 3.

(6) **Treasurer's Report**
The report was accepted, with thanks to the Treasurer.

(7) **Bulletin**
IMS members are encouraged to consider submitting articles of general interest to the Bulletin. Organizers of conferences that received financial support from the IMS are reminded to submit short reports to the Bulletin. Abstracts of completed PhD theses should also be sent to the editor for inclusion in the Bulletin. Thanks were expressed to Gordon Lessells for his sustained efforts on the printing and circulation of the Bulletin.

(8) **Education sub-Committee**

The IMS has established an Education sub-committee as initially proposed by C. Stack, who was thanked for her efforts in this regard. The present membership of the subcommittee consists of:

Anthony Cronin	UCD
Ronan Flatley	Mary Immaculate College
Jim Grannell	UCC
Clare Lundon	Galway-Mayo IT
Ciarán Mac an Bhaird	Maynooth University
David Malone	Maynooth University
Aisling McCluskey	NUI Galway
Anthony O'Farrell	Maynooth University
Elizabeth Oldham	TCD
Maurice O'Reilly	St. Patrick's College
Rachel Quinlan	NUI Galway
Áine ní Shé	Cork IT
Aoibhinn ní Shúilleabháin	UCD
Cora Stack	IT Tallaght

The subcommittee is chaired by A. O'Farrell and the secretary is R. Quinlan. It has terms of reference (agreed at the 2014 AGM) and operating rules. It is currently in the process of conducting its first meeting, by email.

The following matters of relevance to education were raised.

- Ireland is a member of ICMI (the International Commission on Mathematical Instruction, a commission of the International Mathematical Union), with the Royal Irish Academy as the articulating organisation. Many countries have ICMI subcommissions, and it was suggested by M. O'Reilly that Ireland should consider establishing one.
- The 10th Congress on European Research in Mathematics Education (CERME 10) will take place in Dublin in February 2017. This is a major international event which is likely to have up to 700 delegates. The co-chairs are Maurice O'Reilly and Therese Dooley of St Patrick's College.

- M. O'Reilly will present a report on developments and activities in Mathematics Education in Ireland at the International Congress on Mathematics Education (ICME) in Hamburg in July 2016. ICME is the quadrennial conference of the ICMI. Work is proceeding on the preparation of this report, with the support of various individuals and organisations.

(9) **Elections to Committee**
Nominations were invited for the position of Secretary, commencing on January 1, 2016. R. Timoney nominated David Malone for this position, and was seconded by C. Stack. There were no other nominations and D. Malone was elected.

The numbers of years that each continuing member will have served on the IMS Committee as of January 2016 are tabulated below.

P. Barry	2
J. Gleeson	4
B. Kreussler	4
R. Levene	2
M. Mac an Airchinnigh	3
M. Mackey	6
M. Mathieu	5
A. Mustata	2
G. Pfeiffer	2
J. O'Shea	2

(10) **Future Meetings**
The IMS will celebrate its 40th anniversary in 2016. R. Timoney and V. Dotsenko have undertaken to host a scientific meeting of the Society in Trinity College Dublin on April 15 and 16, 2016. A date for the annual general meeting is to be decided and must be after July 31.
Expressions of interest in hosting the 2017 September meeting are welcome.

(11) **Any Other Business**
C. Stack raised concerns about the low levels of funding for mathematical activity in Ireland, and mentioned that she has raised this issue with Mary Mitchell-O'Connor TD, who would be willing to meet for discussions. C. Stack also proposed that the IMS consider holding an annual public lecture.

The meeting concluded at 12.55.

Rachel Quinlan
NUI Galway

The Annual September Meeting of the IMS

University College Cork

27–28 August 2015

Thursday 27 August	
10.30–11.00	IMS opening
11.00–12.00	Stanley Burris
12.00–13.00	Barbara Fantechi
13.00–14.00	*Lunch*
14.00–15.00	Rajarshi Roy
15.00–15.30	Adele Marshall
15.30–16.00	David Henry
16.00–16.30	*Coffee & Tea*
16.30–17.00	Rupert Levene
17.00–17.30	Ivan Todorov
17.30–18.30	Chris Rogers

Friday 28 August	
09.00–10.00	Alice Niemeyer
10.00–10.30	*Coffee & Tea*
10.30–11.00	Aoife Hennessy
11.00–11.30	Boole2School
11.30–13.00	IMS AGM
13.00–14.00	*Lunch*
14.00–15.00	Franz Pedit
15.00–16.00	Muriel Medard
16.00–16.30	*Coffee & Tea*
16.30–17.30	Marius Junge

The meeting was funded in part by the European Mathematical Society, Science Foundation Ireland and University College Cork, in conjunction with the IMS. There were 52 registered participants. The timetable of the talks is above, and the abstracts are given below.

Titles and Abstracts

Stanley Burris — A Primer on Boole's Algebra of Logic

I will describe how Boole managed to squeeze an algebra of logic into the algebra of numbers. The only background needed is high school algebra and the distributive law from basic set theory: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. With this setup Boole proved his elegant results on Elimination and Solution. Boole's approach led to his models being partial algebras since addition and subtraction were only partially defined, and this led to his use of uninterpretables. Boole's main results were admired, but not his algebra-of-numbers approach. Within a decade of his publication of Laws of Thought his system was being replaced by the modern Boolean algebra which had total algebras and was able to derive the same results.

Barbara Fantechi — Counting Curves on Algebraic Varieties

Enumerative geometry is one of the oldest parts of mathematics, with entry-level problems that can be explained to the layman (given four lines in space, how many lines meet/intersect all of them?). Yet in recent decades stunning progress has come thanks to a synergy of algebraic and complex geometry, string theory and function theory. In this talk we give a very partial overview, highlighting the interplay between concrete problems and theoretical advances.

Aoife Hennessy — Riordan Arrays and Bijections of Weighted Lattice Paths

This talk introduces Riordan arrays and concerns paths counted by Riordan arrays arising from the decomposition of certain Hankel matrices and bijective relationships between them. We consider certain Hankel matrices, H under two decompositions, $H = L_M D L_M^T$ with L_M a Riordan array of generating functions that count weighted Motzkin paths and a $H = L_L SDSL_L^T$ decomposition, with L_L a Riordan array with generating functions that count weighted Łukasiewicz paths. A bijection is introduced between these paths.

References

[1] P. Flajolet, Combinatorial aspects of continued fractions, *Discrete Mathematics* **32** (1980), pp. 125–161.

[2] A. Hennessy, A study of Riordan arrays with applications to continued fractions, orthogonal polynomials and lattice paths, PhD Thesis, Waterford Institute of Technology, 2011.

[3] F. Lehner, Cumulants, lattice paths and orthogonal polynomials, *Discrete Mathematics* **270** (2003), pp. 177–191.

David Henry — Exact Solutions of the Water Wave Problem

The equations of motion which govern fluid dynamics are highly intractable to mathematical analysis for a variety of reasons, among them being the inherent nonlinearity of the governing system of partial differential equations coupled with the fact that one is often dealing with a free-boundary problem whereby the fluid domain is a priori unknown.

Recently, modern functional analytic techniques have enabled major progress to be made in proving the existence of exact water wave solutions to the fully nonlinear free-boundary value problem. In this talk, which consists of recent work with various collaborators, I will present some exact solutions to the governing equations of water wave problems in two- and three-dimensions which have an explicit form in the Lagrangian formulation, describing briefly how these solutions are particularly amenable to a hydrodynamical stability analysis.

Obligatory Boole reference: one of the major figures in the area of hydrodynamical stability, Sir Geoffrey Ingram Taylor (1886-1975), was none other than George Boole's grandson!

Marius Junge — Analysis on Noncommutative Spaces

Noncommutative tori are simplest examples for noncommutative mainfolds in noncommutative geometry. For these concrete spaces and their noncompact analogues, we will discuss basic concepts from noncommutative geometry and finite dimensional approximation in the Gromov–Haussdorff sense. Using a semigroup approach one can show that also many tools in classical harmonic analysis concerning convergence of Fourier series and singular integral operators remain valid in the context of noncommutative deformations of classical spaces.

Rupert Levene — The Boolean Lattice of Schur Idempotents

A Schur idempotent is an infinite matrix A of ones and zeros with the property that if the matrix of any bounded Hilbert space operator B is changed by replacing b_{ij} with 0 whenever $a_{ij} = 0$, then we obtain another bounded operator. We will introduce these objects and consider an open problem concerning the generation of the Boolean lattice of all Schur idempotents, touching on joint work with Ivan Todorov (QUB) and Georgios Eleftherakis (Patras).

Adele Marshall — Predictive Analytics and Healthcare Modelling Using Coxian Phase-type Distributions

Muriel Medard — Stormy Clouds: Security in Distributed Cloud Systems

As massively distributed storage becomes the norm in cloud networks, they contend with new vulnerabilities imputed by the presence of data in different, possibly untrusted nodes. In this talk, we consider two such types of vulnerabilities. The first one is the risk posed to data stored at nodes that are untrusted. We show that coding alone can be substituted to encryption, with coded portions of data in trusted nodes acting as keys for coded data in untrusted ones. In general, we may interpret keys as representing the size of the list over which an adversary would need to generate guesses in order to recover the plaintext, leading to a natural connection between list decoding and secrecy. Under such a model, we show that algebraic block maximum distance separable (MDS) codes can be constructed so that lists satisfy certain secrecy criteria, which we define to generalise common perfect secrecy and weak secrecy notions. The second type of vulnerability concerns the risk of passwords being guessed over some nodes storing data, as illustrated by recent cloud attacks. In this domain, the use of guesswork as a metric shows that the dominant effect on vulnerability is not necessarily from a single node, but that it varies in time according to the number of guesses issued. We also introduce the notion of inscrutability, as the growth rate of the average number of probes that an attacker has to make, one at a time, using his best strategy, until he can correctly guess one or more secret strings from multiple randomly chosen strings.

Joint work with Ahmad Beirami, Joao Barros, Robert Calderbank, Mark Christiansen, Ken Duffy, Flavio du Pin Calmon, Luisa Lima, Paulo Oliveira, Stefano Tessaro, Mayank Varia, Tiago Vinhoza, Linda Zeger.

Alice Niemeyer — The Divisibility Graph of a Finite Group

For a set of positive integers X, A. Camina and R. Camina introduced the Divisibility Graph of X as the directed graph with vertex set $X\setminus\{1\}$ and an edge from vertex a to vertex b whenever a divides b. For a group G let $cs(G)$ denote the set of conjugacy class lengths of non-central elements in G. Camina and Camina asked how many components the Divisibility Graph of $cs(G)$ has. In joint work with Abdolghafourian and Iranmanesh we determine the connected components of the Divisibility Graph of the finite groups of Lie type in odd characteristic.

The answer to this question is closely related to another kind of graph defined for groups, namely the Prime Graph. The vertex set of the Prime Graph of a finite group G is the set of primes dividing the order of the group and two vertices r and s are adjacent if and only if G contains an element of order rs. Williams investigated the Prime Graph of finite simple groups and determined its connected components.

Franz Pedit — Integrable Surface Geometry for Surfaces of Non-abelian Topology

We give a short historical introduction to the theory of constant mean curvature surfaces and discuss recent advances for surfaces whose fundamental groups are non-abelian. We explain how to use the Riemann-Hilbert correspondence between local systems, representation varieties and holomorphic bundles to give a description of those surfaces in terms of algebro-geometric data. Computer experiments and visualisations support and guide the theoretical investigations making the talk quite accessible to non-experts.

Chris Rogers — Fundamental Fallacies of Finance

The Concise Oxford Dictionary defines a fallacy as "A mistaken belief esp. based on unsound argument", and the history and current practice of the finance industry provides egregious examples. This talk will fearlessly expose some of these, beat up the unsound arguments, and, unexpectedly, offer some practical suggestions on how to avoid such errors. These suggestions are unlikely ever to be adopted, for reasons that will also be explained.

Rajarshi Roy — Seeing the Light: Visual Illusions and Reference Frames

The eye and brain work together to provide us with perspectives of reality. Signals from the eye to the brain determine what we see and how we interpret images and the dynamical, changing world around us. We will explore simple and complex aspects of "seeing the light" - from the formation of images to their interpretation based on frames of reference that lead us to impressions of the world around us. Visual illusions and demonstrations with simple apparatus will be used to illustrate how eyes and brain work together to help us navigate our way through life with balance and poise.

Ivan Todorov — Positive Extensions of Schur Multipliers

In this talk I will describe the positive completion problem for matrices, give a summary of known results and then discuss generalisations of these results in infinite dimensions. I will highlight the operator space approach to the topic, in which completely positive maps play a decisive role. Time permitting, I will give some applications of the results to the problem of extending positive definite functions defined on subsets of locally compact groups.

Joint work with R. Levene and Y.-F. Lin,

Report by Stephen Wills, University College Cork

Reports of Sponsored Meetings

Infinite Dimensional Function Theory: Present Progress and Future Problems 8-9 January 2015, UCD

An informal meeting on *Infinite Dimensional Function Theory: Present Progress and Future Problems* was held at UCD during 8th-9th January, 2015. The meeting received financial support from the Fulbright Commission and the IMS as well as from NUI Galway, TCD, and UCD. The meeting was organised by Chris Boyd and Michael Mackey (UCD), Richard Timoney (TCD), and Richard Aron (Kent State University).

Given that the study of Infinite Dimensional Holomorphy has been based in Ireland for many decades, it was appropriate that this meeting take place in Dublin. There were a number of foreign attendees, including five from Spain and two from the US, all of whom have strong mathematical connections with the infinite dimensional analysis group in Ireland. In addition, there were speakers from all major Irish universities.

The following list gives the 19 speakers and titles of the half-hour talks.

José M. Ansemil: Property LF of $(H(U), \tau_\delta)$
Richard M. Aron: Fiber structure in the maximal ideal space of H^∞
James Boland: Operators approximable by hypercyclic operators
Anthony Brown: Norms of mappings between spaces of polynomials
Gerard Buskes: Complex vector lattices
Sean Dineen: Distance between Hilbertian operator spaces
Pablo Galindo: Group-symmetric holomorphic functions on a Banach space
Domingo García: The Bohr radius for Dirichlet series
Ilja Gogic: On derivations and elementary operators on C^* algebras
Robin Harte: Residual Quotients

Finbarr Holland: Convolution of almost periodic functions
Rupert Levene: Hyperreflexivity and idempotent Schur multipliers
Manuel Maestre: Banach spaces of Dirichlet series
Martin Mathieu: Interplay Between Spectrally Bounded Operators and Complex Analysis
A. G. O'Farrell: Boundary smoothness of holomorphic functions
M. Pilar Rueda: Automatic surjectivity of isometries of weighted spaces of analytic functions
Raymond Ryan: The Kantorovich Theorem for Polynomials
Richard Smith: Lipschitz-free spaces and the metric approximation property
Milena Venkova: Polynomials on James Tree-type spaces

Ample time was devoted to informal and fruitful discussions.

Participants availed themselves of excellent dining at the appropriately named UCD restaurant Pi as well as at the Conference Banquet at the Beaufield Mews in Stillorgan.

Report by Richard M. Aron, Kent State University
aron@math.kent.edu

Irish Maths Learning Support Network 9th Annual Workshop
29 May 2015, University College Dublin

The theme of the 9th Annual Workshop of the Irish Maths Learning Support Network was *Maximizing the impact of digital supports in Mathematics Learning Support in Higher Education.*

This workshop was motivated by recent calls of the National Forum for the Enhancement of Teaching and Learning in Higher Education for building digital capacity for both staff and learners across third level education in Ireland. The workshop was hosted in the Science Centre at University College Dublin (UCD) on May 29th 2015.

The day began with a Welcome Address by the Registrar and Deputy-President of UCD, Professor Mark Rogers. This was then followed by the first of the keynote speakers, Professor Chris Sangwin (then at Loughborough University now at the University of Edinburgh). Professor Sangwin gave an interactive presentation on supporting mathematics students' problem solving skills via the free software GeoGebra. He outlined his use of the "Moore Method" when teaching the many diverse classes he has used GeoGebra with, including Maths Learning Centre (MLC) students. Following this was a short presentation by Dr Jonathan Cole of Queens University Belfast who gave a very engaging talk around the development of an app for use by second-level students studying vectors which was developed by a student at Queens University Belfast. The second keynote speaker Shazia Ahmed of the University of Glasgow then discussed an interesting study she undertook involving the use of Facebook Groups to provide academic support for first year mathematics and computer science students at Glasgow.

The next address was delivered by Nuala Curley of UCD, who spoke about the challenges of collecting useful qualitative feedback data based on the difficulties students present with at a maths support centre. The purpose of this research is to identify university students' mathematical "trouble-spots" in a maths support centre setting and to develop effective supports. The final keynote speaker was Dr Madonna Herron of Ulster University, who spoke about using the Livescribe 3 smarten to produce pencasts which, she reported, can extend and enrich the student learning experience in mathematics and engineering programmes at her university. Dr Cormac Breen of DIT was the next speaker (describing work done in collaboration with Ciarán O Sullivan of IT Tallaght and Damien Cox of IT Blanchardstown) and he informed us about a trial of virtual mathematics drop-in support for students across the three IoTs in question. The technology used here was Wacom Intuos Tablets and the Adobe Connect software package. Dr Gerry Golding of the Open University (OU) was the final speaker of the day and he gave a very interesting presentation on the virtual mathematics support centre at OU. This is a pilot project delivered over the Moodle platform for Irish students studying (service course) mathematics with the OU.

The day concluded with a discussion session regarding various small-scale initiatives undertaken by maths support practitioners and the possibility of sharing support materials between MLCs .

A total of 39 delegates from 23 different institutions, including international visitors from the United States (Stanford University), Canada (University of British Columbia), Norway (University of

Agder), as well as keynote and invited speakers from the UK (Loughborough University, University of Glasgow, Ulster University and the Open University). There were 12 Irish Higher Education Institutions represented at the workshop encompassing five universities (DCU, MU, NUIG, UCD and UL) and seven Institutes of Technology, (Dublin IT, Dundalk IT, IT Tallaght, Limerick IT, IT Carlow, IADT and Cork IT). Companies such as Google, Folens, KPMG and Infonalis were also represented.

This workshop was funded by the National Forum's Seminar Series, the School of Mathematical Sciences in UCD and the Irish Mathematical Society.

All abstracts, slides and videos from each presentation can be found at: http://supportcentre.maths.nuim.ie/mathsnetwork/ucd2015

Report by Anthony Cronin, University College Dublin
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Thirteenth Irish Geometry Conference
15-16 May 2015, MIC Limerick

The Irish Geometry Conference is a research meeting that has taken place annually since its inauguration at NUI Galway in 2003. Its 2015 edition took place at MIC Limerick and was organized by Norbert Hoffmann (MIC), Bernd Kreussler (MIC) and Clifford Nolan (UL). This conference was funded by the Research and Graduate School, MIC Limerick; the Department of Mathematics, University of Limerick; and the Irish Mathematical Society.

There were 22 participants from 12 institutions in Ireland and abroad. The speakers, titles and abstracts were as follows.

Hans-Christian Graf v. Bothmer (*Universität Hamburg*)
Rationality of hypersurfaces

I will review classical and modern results about the rationality of hypersurfaces and present our results (with Christian Böhning and Pawel Sosna) regarding Kuznetsov's derived-category approach to the rationality question of cubic 4-folds.

Ulrich Derenthal (*Leibniz Universität Hannover*)

Cox rings over nonclosed fields

For a wide class of varieties over algebraically closed fields, Cox rings were defined and studied by Cox, Hu, Keel, Hausen, Hassett and others. We give a new definition of Cox rings for suitable varieties over arbitrary fields that is compatible with universal torsors, which were introduced by Colliot-Thélène and Sansuc. We study their existence and classification, and we make their relation to universal torsors precise. This is joint work with Marta Pieropan.

Brendan Guilfoyle (*IT Tralee*)

Flowing a classical surface by its mean radius of curvature

In this talk I will present joint work with Wilhelm Klingenberg on the flow of a convex surface in Euclidean 3-space by its mean radius of curvature. Under this expanding flow, it is well known that the surface runs out to infinity, becoming round as it does so. In the talk I will outline our proof that the centre of this "sphere at infinity" can be computed from the spectral data of the surface. This result can be viewed in a number of ways: convergence of the normal lines of the flowing surface or a definition of a "centre" for an arbitrary convex surface which is conserved under mean radius of curvature flow.

Nobuhiro Honda (*Tokyo Institute of Technology*)

Some examples of twistor spaces of algebraic dimension one

It has been known that twistor spaces provide nice examples of compact complex 3-fold whose algebraic dimension takes all values from zero to three.

Most compact twistor spaces are of algebraic dimension zero, and also a lot of examples are already known of twistor spaces of algebraic dimension three. Also, twistor spaces of K3 surfaces, complex tori (and also some Hopf surfaces) form a good class of twistor spaces whose algebraic dimension is one.

In this talk, I will present twistor spaces of algebraic dimension one with a different flavour; namely I will present a series of simply connected twistor spaces of algebraic dimension one whose general fibre of the algebraic reduction is birational to an elliptic ruled surface. In these examples, a pair of Hopf surfaces are contained as a reducible fibre of the algebraic reduction.

Daniel Huybrechts (*Universität Bonn*)

The K3 category of a cubic fourfold

The derived category of a smooth cubic hypersurface of dimension four determines the cubic. However, due to a result of Kuznetsov the category contains a full subcategory that behaves in many respects like the derived category of a K3 surface. In this talk, I will explain what is known about it from a purely categorical point of view but also from a more Hodge theoretic perspective.

Benjamin McKay (*University College Cork*)

Bending metal sheets, Riemann surfaces and integrable systems

When you bend a metal sheet, without stretching, it deforms through isometric immersions of a Riemannian metric. Problem: for which surfaces is the differential equation of isometric immersion an integrable system? We find the first examples. We use ideas of Darboux relating complex geometry and integrable systems. Joint work with Jeanne Clelland, Tom Ivey and Peter Vassiliou.

Sergey Mozgovoy (*Trinity College Dublin*)

Counting Higgs bundles

In this talk I will discuss a problem of counting semistable twisted Higgs bundles over a smooth projective curve defined over a finite field. I will also introduce the Donaldson-Thomas invariants for this problem and explain their relation to counting of indecomposable vector bundles over a curve. I will discuss an explicit formula for the above problem and its relation to the conjectural formula of Hausel-Rodriguez-Villegas. This is a joint project with Olivier Schiffmann.

Fabian Reede (*MIC Limerick*)

Vector bundles and Arakelov geometry

We study vector bundles on the projective line over the integers and apply concepts of Arakelov geometry to these bundles. For example we compute their arithmetic Chern classes and derive the arithmetic Hirzebruch-Riemann-Roch theorem from the arithmetic

Riemann-Roch theorem due to Gillet and Soulé. As an application we will compute the Ray Singer analytic torsion for all line bundles on the Riemann sphere.

DAVID WRAITH (*Maynooth University*)

Positive Ricci curvature on highly connected manifolds

This talk concerns the existence of positive Ricci curvature metrics on compact $(2n-2)$-connected $(4n-1)$-manifolds. The focus will be largely topological: we will describe new constructions of these objects to which existing curvature results can be applied. The constructions are based on the technique of plumbing disc bundles. This is joint work with Diarmuid Crowley.

IGC Participants

Report by Bernd Kreussler, Mary Immaculate College, Limerick
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THEORETICAL AND NUMERICAL ANALYSIS OF RIGID-BODY IMPACTS WITH FRICTION

SHANE J. BURNS

This is an abstract of the PhD thesis *Theoretical and Numerical Analysis of Rigid-body Impacts with Friction* written by S. Burns under the supervision of Petri Piiroinen at the School of Mathematics, Statistics, and Applied Mathematics, National University of Ireland, Galway and submitted in September 2015.

This thesis gives a flavour of the area of rigid body impacts with friction, an area which has far reaching applications in engineering, sports science and every day life. The focus of this work will be on the two main streams of this field, theoretical and numerical. This thesis will present an overview of the general subject of rigid-body impact, including discussion and analysis of the validity of ones choice of impact law and the numerical techniques required for the simulation of rigid-body impacting systems.

Two impact laws will be introduced in Chapter 3 and a direct comparison will be made in order to examine the varying dynamics that can be achieved using both a basic and a complex impact law and to explore some of the problems that can occur with a more basic formulation. It will be demonstrated that for certain regions in parameter space the two formulations are equivalent, however, for many other regions the two formulations can vary greatly.

A hybrid event-driven numerical scheme is one in which smooth dynamics are described by differential equations, which can be solved numerically using standard techniques, and non smooth events which are described by maps. In Chapter 5, a hybrid event-driven numerical scheme for the implementation of the Energetic Impact Law

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Key words and phrases. Impact with friction, Nonsmooth, Bifurcation.
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described in Chapter 3 is presented. Moreover, the framework necessary for the long term simulation of mechanical systems with impacts and chatter is derived.

This thesis also gives an overview of the phenomena known as the Painlevé Paradox in Chapter 6 and presents a numerical experiment to show the occurrence of the paradox for a mechanical system.

References

[1] Burns, S. J. and Piiroinen, P. T.: *Simulation and long-term behaviour of unconstrained planar rigid bodies with impact and friction*. International Journal of Nonlinear Mechanics. 2015.

[2] Burns, S. J. and Piiroinen, P. T.: *The complexity of a basic impact mapping for rigid bodies with impact and friction*. Journal of Regular and Chaotic Dynamics. 2014.

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MATHEMATICAL MODELS OF SEASONALLY MIGRATING POPULATIONS

JOHN G. DONOHUE

This is an abstract of the PhD thesis *Mathematical models of seasonally migrating populations* written by J. Donohue under the supervision of Dr. P. T. Piiroinen at the School of Mathematics, Statistics, and Applied Mathematics, National University of Ireland, Galway and submitted in September 2015.

The phenomenon of seasonal migration has attracted a wealth of attention from biologists. However, the dynamics of migratory populations have been little considered. In this thesis, we use differential equations to model the variation in abundance of seasonally migrating populations.

Our contribution to the field begins with a representation of seasonal breeding. We use piecewise-smooth differential equations to model the variation in the size of a population that has a short interval each year during which successful reproduction is possible. We first consider a one-species model which illustrates the dynamics of a population of specialist feeders over the course of a single breeding season and use it to examine how reproductive success depends on the population's distribution of breeding dates. We then introduce time-dependent switches to extend the model to a broader class of species. This allows us to consider the effect of climate change on populations that annually travel long distances.

We then shift focus to consider interactions between migrants and species at higher levels in the food web. Predatory pressure influences almost all populations to some extent. Here, however, interactions may occur for just a brief period each year before the populations involved become spatially separated. The range of a migrating population may overlap with that of a population of predators for a

2010 *Mathematics Subject Classification.* 92D25, 34C25, 37G15.

Key words and phrases. dynamics, populations, seasonality, periodic.

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single season. We outline a framework for examining how this kind of "transient" predation influences the dynamics of the prey population. We are then able to examine how a migratory population may be overwhelmed by the fleeting influence of members of other species.

Finally, as an alternative to the aforementioned models, we outline a different approach to modelling migration, namely using partial differential equations instead of ordinary differential equations. In this way, we provide two distinct templates for the future exploration of the dynamical features of such populations.

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TOPICS IN COCYCLIC DEVELOPMENT OF PAIRWISE COMBINATORIAL DESIGNS

RONAN EGAN

This is an abstract of the PhD thesis *Topics in Cocyclic Development of Pairwise Combinatorial Designs* written by Ronan Egan under the supervision of Dane Flannery at the School of Mathematics, Statistics, and Applied Mathematics, National University of Ireland, Galway and submitted in July 2015. This thesis is a compilation of results dealing with cocyclic development of pairwise combinatorial designs.

Motivated by a classification of the indexing and extension groups of the Paley Hadamard matrices due to de Launey and Stafford, we investigate cocyclic development of the so-called generalized Sylvester (or Drake) Hadamard matrices. We describe the automorphism groups and derive strict conditions on possible indexing groups, addressing research problems of de Launey and Flannery in doing so.

The shift action, discovered by Horadam, is a certain action of any finite group on the set of its 2-cocycles with trivial coefficients, which preserves both cohomological equivalence and orthogonality. We answer questions posed by Horadam about the shift action, in particular regarding its fixed points. One of our main innovations is the concept of linear shift representation. We give an algorithm for calculating the matrix group representation of a shift action, which enables us to compute with the action in a natural setting. We prove detailed results on reducibility, and discuss the outcomes of some computational experiments, including searches for orthogonal cocycles.

Using the algorithms developed for shift representations, and other methods, we classify up to equivalence all cocyclic BH(n, p)s where p is an odd prime (necessarily dividing n) and $np \leq 100$. This was

2010 *Mathematics Subject Classification.* 05B20, 20B25, 20J06.
Key words and phrases. Cocyclic development, generalized Hadamard matrix.
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achievable with the further aid of our new non-existence results for a wide range of orders.

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INCREMENTAL ELASTIC SURFACE WAVES AND STATIC WRINKLES

ARTUR L. GOWER

This is an abstract of the PhD thesis *Incremental elastic surface waves and static wrinkles* written by Artur L. Gower under the supervision of Prof. Michel Destrades at the School of Mathematics, Statistics, and Applied Mathematics, National University of Ireland, Galway and submitted in September 2015.

This *article-based thesis* comprises a collection of four articles, each of which constitutes a chapter written and formatted in manuscript form. The general aim underlying these articles is to understand and predict how incremental elastic surface waves propagate or static wrinkles form on a deformed elastic substrate. The formation of these small-amplitude disturbances can be the end goal, such as in sending signals or creating functional coatings, or they can be used to measure and characterise the underlying elastic substrate. This thesis focuses on using surface waves or static wrinkles to characterise soft solids, such as biological tissues.

For the complete thesis see [1].

Here we summarize the *main conclusion of the thesis.* Chapter 1 predicts a new phenomenon: oblique wrinkles, which should appear in a large range of materials. Yet oblique wrinkles have not been seen experimentally so far on soft solids. Another issue raised was why are the predicted critical strains greater than the experimentally observed critical strains? We showed that this is likely due to a skin effect caused by dehydration.

In Chapter 2 the effects of a stiffer skin on an elastic substrate on surface wrinkles was initially studied, and therein we also studied the

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Key words and phrases. Elastic waves, incremental elasticity, elastic wrinkles, surface wrinkles.

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possibility of using surface wrinkles to characterise fibre reinforced materials.

The results from Chapter 3 show that measuring the propagation speed of surface waves only along the principal directions of deformation leads to many challenges in non-destructive evaluation of strain and stress, because these directions are not necessarily aligned with the directions of fastest and slowest propagation. However, the methods for calculating surface waves along any direction presented in that chapter are now sufficiently mature and robust to be able to use the full Rayleigh wave-field in order to characterise solids. There are now experiments in place that measure surface waves on tissue, and a wide range of techniques to infer the surface elastic properties from these measurements for a range of materials. Yet, to date, surface wave measurements have not been adequately linked to the elastic properties of soft tissue, such as the residual stress or the reinforcing fibre properties.

Chapter 4 shows a surprisingly simple relationship between the angle of the surface wrinkle wave-front and the fibre orientation, a trend which becomes stronger the stiffer the fibres. Yet predicting how these wrinkles appear on soft fibre reinforced solids required a highly technical and involved numerical method. A promising alternative model is that of a soft tissue reinforced by fibres idealized to be infinitely stronger than the surrounding soft matrix.

References

[1] A. L. Gower (2015) Incremental Elastic Surface Waves and Static Wrinkles. Available at:http://hdl.handle.net/10379/5303

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COMPLETIONS OF PARTIAL MATRICES

JAMES MCTIGUE

This is an abstract of the PhD thesis *Completions of Partial Matrices* written by J. McTigue under the supervision of Rachel Quinlan at the School of Mathematics, Statistics, and Applied Mathematics, National University of Ireland, Galway and submitted in March 2015.

A *partial matrix* over a field $\mathbb{F}$ is a matrix whose entries are either elements of the field or independent indeterminates. A *completion* of a partial matrix is any matrix that results from assigning a field element to each indeterminate. The set of completions of an $m \times n$ partial matrix forms an affine subspace of $M_{m\times n}(\mathbb{F})$.

This thesis investigates partial matrices whose sets of completions satisfy particular rank properties - specifically partial matrices whose completions all have ranks that are bounded below and partial matrices whose completions all have the same rank. The maximum possible number of indeterminates in such partial matrices is determined, and the partial matrices that attain these bounds are fully characterized for all fields. These characterizations utilize a duality between properties of affine spaces of matrices that are related by the trace bilinear form.

Precise conditions (based on field order, rank and size) are provided to determine if a partial matrix whose completions all have rank r must possess an $r \times r$ partial submatrix whose completions are all nonsingular.

Finally a characterization of maximal nonsingular partial matrices is provided - a maximal nonsingular partial matrix is a square partial matrix each of whose completions has full rank, with the property

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Key words and phrases. Partial matrix, Completion, Rank.

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that replacement of any constant entry with an indeterminate results in a partial matrix having a singular completion.

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PRECONDITIONING TECHNIQUES FOR SINGULARLY PERTURBED DIFFERENTIAL EQUATIONS

THÁI ANH NHAN

This is an abstract of the PhD thesis *Preconditioning techniques for singularly perturbed differential equations* written by Thái Anh Nhan, under the supervision of Dr Niall Madden, at the School of Mathematics, Statistics, and Applied Mathematics, National University of Ireland, Galway and submitted in July 2015.

This dissertation is concerned with the numerical solution of linear systems arising from finite difference and finite element discretizations of singularly perturbed reaction-diffusion problems. Such linear systems present several difficulties that make computing accurate solutions efficiently a nontrivial challenge for both direct and iterative solvers.

The poor performance of direct solvers, such as Cholesky factorization, is due to the presence of *subnormal floating point numbers* in the factors. This thesis provides a careful analysis of this phenomenon by giving a concrete formula for the magnitude of the fill-in entries in the Cholesky factors in terms of the perturbation parameter, ε, and the discretization parameter, N. It shows that, away from the main diagonal, the magnitude of fill-in entries decreases exponentially. Furthermore, with our analysis, the location of corresponding fill-in entries associated with some given magnitude can also be determined. This can be used to predict the number and location of subnormals in the factors.

Since direct solvers scale badly with ε, one must use iterative solvers. However, the application of finite difference and finite element discretizations on layer-adapted meshes results in ill-conditioned

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Key words and phrases. Shishkin mesh, singularly perturbed, preconditioning, conjugate gradients.

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linear systems. The use of suitable preconditioners is essential. In this thesis we analyze several preconditioning techniques. They include the diagonal and incomplete Cholesky preconditioners for finite difference discretized systems, and a specially designed *boundary layer preconditioner* for a finite element discretized system. The study of the diagonal and incomplete Cholesky preconditioners focuses on the simplicity and robustness of these techniques; while that of the boundary layer preconditioner is concerned with optimality.

Finally, a novel contribution of this thesis is a pointwise uniform convergence proof for one-dimensional singularly perturbed problems. The central idea of the proof is based on the the preconditioning of the discrete system.

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$3D$ PRINTING A ROOT SYSTEM.

PATRICK J. BROWNE

Abstract. In this short note we describe how a 3d printer was used to make a model of a root system.

1. Introduction

A 3D printer is a device used to make three dimensional objects. In 3D printing, additive processes are used, in which successive layers of material are laid down under computer control, as opposed to established techniques such as injection moulding. These objects can take on a wide variety of shapes not possible by traditional techniques, and are produced from a computer model. This allows for great accuracy and is very suited to producing objects with a mathematical origin. In recent years there has been interest and excitement about these machines, since they are now more affordable and easier than ever to use. In this short note, we describe how use was made of one of these printers to print a 3 dimensional mathematical object, in this case a root system. The choice of object was chosen since it is well known to the author. Pictures of root systems are clear, easy to understand, and can be used to great effect in the class room environment to engage a student in a rich new topic that may seem rather abstract at first glance. The author feels the contents of this note could make an excellent addition to an undergraduate student project. Other interesting shapes with mathematical origins that a potenial supervisor or student may wish to explore can be found in [4]. We will describe what a root system is, how to construct a virtual model of it on a computer and then how to export this model and print it. To assuage any curiosity the printed object can be seen in Figure 1.

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Key words and phrases. 3d printer, root systems.
Received on 18-6-2015; revised 15-12-2015.
The author wishes to acknowledge, Dr G. Brychkova in the Discipline of Botany and Plant Science at NUIG, for use of their 3d printer.

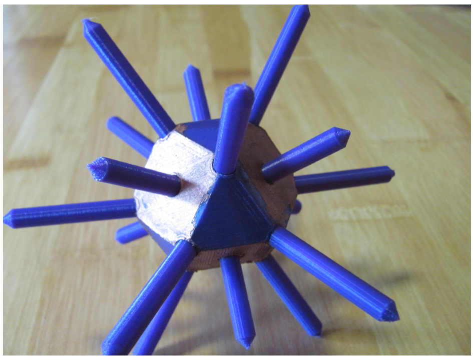

FIGURE 1. The finished 3d print.

2. Root systems

For the purposes of this note we will adopt the axiomatic approach to root systems. All basic facts and definitions can be found in Chapter 3 of [1] and Chapter 8 of [3]. Let $\mathbf{E}$ be a finite dimensional Euclidean vector space over $\mathbb{R}$, endowed with a positive definite symmetric bilinear form $(\cdot,\cdot)$. We define a reflection in $\mathbf{E}$ as an invertible linear transformation leaving pointwise fixed some hyperplane and sending any vector orthogonal to that hyperplane into its negative. Any non zero vector α in $\mathbf{E}$ defines reflection σ_α with reflecting hyperplane $P_\alpha = \{\beta \in \mathbf{E} | (\beta,\alpha) = 0\}$. An explicit formula for reflecting is given by,

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha.$$

The term $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ is often written as $\langle \beta,\alpha\rangle$. A subset Φ of $\mathbf{E}$ is called a *crystallographic root system* in $\mathbf{E}$ if the following axioms are satisfied:

(1) Φ is finite and spans E and does not contain 0.
(2) If $\alpha \in \Phi$ the only multiples of α in Φ are $\pm\alpha$.
(3) If $\alpha \in \Phi$ the reflection σ_α leaves Φ invariant.
(4) If $\alpha,\beta \in \Phi$ then $\langle\alpha,\beta\rangle \in \mathbb{Z}$.

Henceforth we will just refer to this as a *root system.* Axiom (4) limits the possible angles occurring between pairs of roots. To see this, recall that the cosine of the angle, θ, between two roots α and β is given by $\|\alpha\|\|\beta\|cos\theta = (\alpha,\beta)$. Therefore $\langle\beta,\alpha\rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} = 2\frac{\|\beta\|}{\|\alpha\|}cos\theta$. So that $\langle\beta,\alpha\rangle\langle\alpha,\beta\rangle = 4cos^2\theta$. This is a non negative integer, $0 \leq cos^2\theta \leq 1$, and $\langle\alpha,\beta\rangle, \langle\beta,\alpha\rangle$ have the same sign. If we also assume that $\alpha \neq \pm\beta$ and $\|\beta\| \geq \|\alpha\|$, enumeration of the possible values of $\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle$, yields $\theta \in \{\frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{6}, \frac{5\pi}{6}\}$.

FIGURE 2. The A_1 root system.

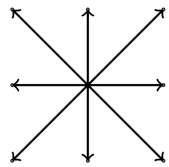

FIGURE 3. The B_2 root system.

We refer to the dimension of **E** as the *rank* of the root system. For example in rank 1, Axiom (2) means there is only one choice of root system, we represent this with Figure 2. The informed reader will of course recognise this as the root system in Lie theory that belongs to $\mathfrak{sl}(2, F)$. In rank 2 there are more choices, one of which is the root system B_2, shown in Figure 3. This root system consists of 2 spanning vectors and two distinct root lengths, where the ratio of these lengths is $\sqrt{2}$. The roots correspond to the vertices and midpoints of the edges of a square, the ones of greater length are denoted *long roots* and the others *short roots.*
The above drawings of root systems are clear and easy to understand. Unfortunately representations of rank 3 root systems by pictures can be crowded and difficult to see on the page. With some skill they can be drawn on the page in a pleasing manner, see Figures 8 and 9 in [5]. As an aside we also mention the excellent computer aided drawings seen after page 162 in [3]. Unfortunately these drawings are subject to a fixed point of view. For this reason we choose to construct a computer model of a rank 3 root system and then print it, allowing one to view the roots from any perspective. There are only three choices of irreducible root systems in rank 3, namely A_3, B_3 and C_3. The root systems of B and C are dual to each other and the root system of A has only one root length. For this reason we choose to model the B_3 root system since it captures most of the interesting details.
To construct the B_3 root system we follow the construction outlined in section 2.10 of [2] (alternatively section 12 of [1]). Take $\mathbb{R}^n$ with standard bases vectors $\{\epsilon_i\}$, define Φ to be the set of all vectors of squared length 1 or 2 in the standard lattice of $\mathbb{R}^n$. So Φ consists of the $2n$ short roots $\pm\epsilon_i$ and the $2n(n-1)$ long roots $\pm\epsilon_i \pm \epsilon_j$ where $i < j$.
For example in rank 2 there are 4 short roots $\{(1,0), (-1,0), (0,1), (0,-1)\}$, and 4 long roots $\{(1,1), (1,-1), (-1,1), (-1,-1)\}$, as

shown in Figure 3. In rank 3 we have the a total of 18 roots summarised in Table 1. Examination of the vectors in Table 1 will show

Short roots	Long roots
$(\pm 1, 0, 0)$	$(\pm 1, \pm 1, 0)$
$(0, \pm 1, 0)$	$(0, \pm 1, \pm 1)$
$(0, 0, \pm 1)$	$(\pm 1, 0, \pm 1)$

TABLE 1. The 12 long and 6 short root vectors for B_3.

that the long roots correspond to the vertices of a regular cube-octahedron, that is the intersection of a cube and an octahedron, with the short roots being the centres of the square sides. One can best picture the cube-octahedron as a cube with its corners cut off. For a visual depiction of this please see Figure 9 in [5] and Figure 4.

3. Computer modeling

We now have our root system that we wish to print. To do this we need to construct it as a computer model and ready it for export. There is a wide choice of software available. This ranges from the very capable *Blender*[1] to the point and click web interfaces such as *Tinkercad*[2]. The former has the advantage of allowing precise manipulation, but the disadvantage of a very steep learning curve. While the latter is easy to pick up, it lacks the precision needed for our task.

The software we will use is OpenSCAD [6], a versatile package that allows precise control and is easy to pick up for anyone that is familiar with functions or scripting languages, ideal for an undergraduate student. To illustrate this, we present a sample of some code:

```
union()\\ form the union of two shapes
{cylinder(r = a,h=b,$fs=c);
\\ cylinder with radius
\\and fs is a measure of its smoothness
translate([x,y,z]){
\\[x,y,z] being the translation vector
```

[1]Blender is the free and open source 3D creation suite, https://www.blender.org/

[2]Tinkercad is a free, easy-to-learn online app anyone can use to create and print 3D models, https://www.tinkercad.com/

```
rotate(a=180,v=[d,e,f]){
\\[d,e,f] being the axis of rotation along with
\\angle a
cylinder(h=g,r1=a,r2=0,center=false);}
```

The root system of B_3 is easiest viewed as vectors extending from a cube-octahedron as seen in Figure 4. While we could construct the cube and octahedron ourselves, we opt to use the community resource of the *Thingiverse solids package*[7]. This is simply done as follows:

```
use <maths_geodesic.scad> // package to create
// platonic solids
include <test_platonic.scad>//
module cubeocto(){// a cubeoctahedron where the
// roots sit
intersection(){ // the intersection of two shapes
display_polyhedron(octahedron(90));
color("red")cube(90,center=true);}}
```

Once this package is loaded into our file we place the roots at the various points. Each root will simply be a cylinder joined to a cone. Since each root is also accompanied by its negative, we only need draw 9 roots and then use rotations to place their negatives. After some small effort the model in Figure 1 was constructed. At this point all that is left to do is to export the file to *.stl* format to ready it for printing.

4. Printing of the model

Once in possession of the .stl file, we check that it is physically possible to print. Any 3d printer will have software to do this, but there are also resources online such as *Willit 3D Print.*[3] When making a computer model it is best to try to avoid parts that are extended in free space, as the printer will need to print a removable support. Removing this support can sometimes damage delicate parts of your model. If possible try to design your models to avoid this. For this reason the author printed the cube octahedron with

[3]Willit 3D Print is the website using javascript and webgl, where you can analyse your 3D design (STL or AMF files) before you 3D print it, http://www.willit3dprint.com/

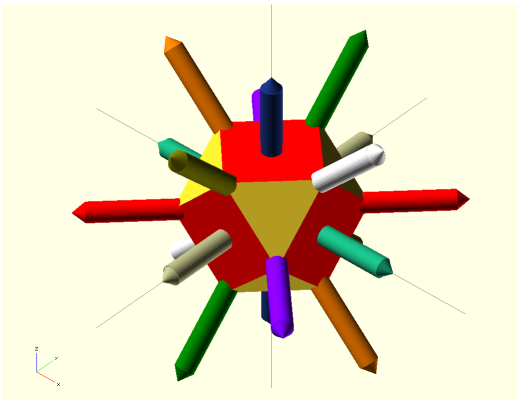

FIGURE 4. Computer model of the B_3 root system, with cube-octahedron.

holes for roots and the roots separately, see Figure 5. This was accomplished by taking the difference of the roots and the cube-octahedron:

```
 difference () {
cubeocto();\\ The order of the arguments
long_a3roots();\\determines how the difference
\\ is taken
short_b3roots();}
```

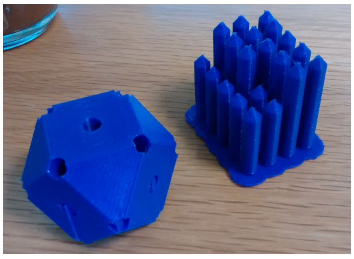

FIGURE 5. The printed model parts, with two spare roots.

This was purely to avoid having to remove the supports which may have led to damage. The printed model was approximately 12cm from one end to the other. The 3d printer that was used was the *maker bot replicator 2*. We were generously allowed use of the printer by the Discipline of Botany and Plant Science at NUIG and the author wishes to thank Dr G. Brychkova for her help.

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THE MAXIMUM FOR $\mathbf{\Delta u + f(u) = 0}$ ON AN ISOSCELES TRIANGLE

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ABSTRACT. We use the moving planes method to prove that if u is a positive solution to the equation $\Delta u + f(u) = 0$, on an isosceles triangle T in R^2, with $u = 0$ on ∂T and f Lipschitz continuous and restoring, then u has a unique maximum value on the axis of symmetry of T. We conjecture that the location of the maximum is independent of f, and extend the result to a set in R^3.

1. Introduction

In recent years the use of moving planes and maximum principles for thin domains, have produced a number of interesting results concerning the solutions of elliptic partial differential equations on certain special classes of bounded domains in R^n . Most of these ideas can be found in the works of Berestycki and Nirenberg [1], Du [2], Fraenkel [3], and Gidas, Ni, and Nirenberg [4].

A recent paper by Cima and Derrick [5] explored the use of these methods on an isosceles triangle in the plane R^2 . In [5], we proved that if a bounded convex domain Ω in R^2 has two (or more) axes of symmetry, then the solution of the equation $\Delta u + f(u) = 0$, with $u|_{\partial\Omega} = 0$, where f is *restoring* ($f(u) > 0$ when $u > 0$), must have its maximum value where the axes of symmetry meet. Thus, for example, if Ω is an equilateral triangle, the maximum value of u will occur at the intersection of the angle bisectors, while for a circle it will be at the center. For an isosceles triangle, we were only able to prove that the maximum is achieved on a subset of its line of symmetry, since the use of moving planes shows that $u(x, y) = u(x, -y)$, implying that $u_y(x, 0) = 0$, $u_x(x, y) = u_x(x, -y)$, and $u_y(x, -y) = u_y(x, y)$. We were unable to show that the maximum was unique, nor were we

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able to find precise locations for maxima. However, we did provide numerical evidence that suggested two things: (1) the maximum was unique and (2) the maximum occurred at the same place regardless of the function $f(u)$. This suggests that it is possible that for convex domains Ω there may be a unique maximum whose location depends on Ω and not on $f(u)$.

In this paper we will prove that the maximum is unique for an isosceles triangle. We are still unable to show that the location of the maximum is independent of $f(u)$. In Section 2 we will give a short description of the moving plane method, and present some properties of analytic solutions of the problem described in the paragraph above. In particular we will show that the number of maximum points is finite. In Section 3 we will use eigenfunction theory to show that certain functions $f(u)$ have a single maximum on an isosceles triangle. This proof is based on work done by Payne [7] and Sperb [8]. In [6] Chanillo and Cabre have proved for smoothly bounded, strictly convex domains, that a unique critical point (maximum or minimum) exists for such problems. We produce an analogous result for isosceles triangles using continity and compactness tools. In Section 5 we extend these ideas to a triangular shaped region in R^3 .

2. Preliminaries

The partial differential equation that we shall consider in this paper is the elliptic equation of the form

$$\Delta u(x,y) + f(u(x,y)) = 0 \quad \text{on}, \quad \text{with} \quad u(x,y) = 0 \quad \text{on} \quad \partial\Omega, \quad (1)$$

where $u > 0$ in Ω, and f is Lipschitz continuous and restoring ($f(u) > 0$ when $u > 0$).

Definition. A bounded simply connected domain Ω in R^2 is *Symmetric-Convex* (S-C) when a pair of orthogonal straight lines m and n exist such that

(1) Ω is symmetric with respect to line m, and

(2) every straight line parallel to n (including n) that intersects Ω, must intersect Ω on a single open line segment. This property defines *n-convexity.*

For a given domain there may be many such pairs of orthogonal lines.

For the special case of the isosceles triangle, we will denote the domain Ω by T, symmetric in the x-axis and of height c, with base $|y| \leq b$ on y-axis; that is $T = \{(x, y) \in R^2 \mid \ |y| < b(1 - (x/c)), 0 < x < c\}$. Observe that T is S-C. We shall denote by W_a the symmetric trapezoid with vertices at $(0, \pm b)$ and $(c, \pm a)$. Note that $W_0 = T$, and that most of the results we obtain for T are equally valid for W_a .

Note that a domain Ω is *convex* if for any two points a and b in Ω the line segment Λ joining a and b lies entirely in Ω.

We sketch the way the "moving planes" method is used in showing (as we did in [5]) that there is an interval $\Lambda = (x', x^*)$, along the x-axis, with $0 < x' < x^* \leq (1/2)\cdot c$ in which the maximum of u on T is attained. For $0 < a < b$, the horizontal line $L_{-a} = \{(x, y) : y = -a\}$ meets T and cuts a small triangular "cap" (an open subset of T), say $\Sigma(-a) = \{(x, y) \in T : y < -a\}$ from T. Reflecting the domain $\Sigma(-a)$ about L_{-a} we obtain the reflection $\Sigma(-a)^{\perp} \subset T$, each point $P \in \Sigma(-a)$ having a reflected point $P^{\perp} \in \Sigma(-a)^{\perp}$. Define

$$w(P; -a) \equiv u(P) - u(P^{\perp}), \tag{2}$$

for $P \in \bar{\Sigma}(-a)$. Since f is Lipschitz, w satisfies $\Delta w + (P; -a)w = 0$ on $\Sigma(-a)$ with

$$\gamma(P; -a) = \begin{cases} \dfrac{f(u(P)) - f(u(P^{\perp}))}{u(P) - u(P^{\perp})}, & P \in \Sigma(-a), \\ \\ 0, & P \in L_{-a}. \end{cases} \tag{3}$$

By the Maximum principle, it follows that $w \leq 0$ in $\bar{\Sigma}(-a)$, which can then be extended to $w < 0$ on $\Sigma(-a)$. Thus, $u(P) < u(P^{\perp})$ for all P in $\Sigma(-a)$, and hence that $u_y(P) > 0$ in $\Sigma(-a)$. Letting $-a \to 0$, it follows that $u_y > 0$ for $y < 0$ in T. A similar analysis shows that $u_y < 0$ for $y > 0$ in T. Thus $u_y(x, 0) = 0$.

The vertical line $L_k = \{(x, y) : \ x = k > (1/2)\cdot c\}$ cuts a triangular cap from T. Reflecting this cap about L_k we obtain in a similar manner as above, that $u(P) < u(P^{\perp})$ for P in the cap, and that $u_x < 0$ in this cap. Letting k decrease to $c/2$, it follows that $u_x < 0$ for $x > c/2$. Finally, bisecting the angle at $(0, b)$ in T, we show that points P below the line of bisection satisfy $u(P) < u(P^{\perp})$, where $P^{\perp}$ is the reflection of P in, and similarly for the angle of bisection at $(0, -b)$. Let x' be the point where these two lines of angle bisection meet, then $u_x(x,0) > 0$ for all $0 < x \le x'$. Hence, the maxima lie in the interval $\Lambda = (x', c/2)$.

Lemma 2.1. *If the solution u of problem (1) is real analytic, there are at most a finite number of maximum points of u on Λ.*

Proof. Assume there are infinitely many points in the open interval Λ , where u attains it's maximum M. Since $\bar{T}$ is compact, there must be an accumulation point in Λ. Let ξ be the value of the largest such accumulation point. Then there is an increasing sequence $x_j \to \xi \le c/2$ for which $u(x_j, 0) = M$ for $j = 1, 2, 3, \ldots$. It follows by our assumption, and by considering the series expansion at ξ, that $u(x, 0) \equiv M$ in a neighborhood of ξ which is a contradiction. □

Remark. Notice that $u_y(x, 0) = 0$ on T for $0 < x < c$, so that for all partials of u, where one of the partials is in the y direction, such as $u_{x\ldots xy}(x,0) = 0$ in $0 < x < c$. Thus, $u_{yx}(x, 0) = 0$ and $u_{xy}(x, 0) = 0$, so that the Hessian at any maximum point P, has the form

$$H(P) = \begin{bmatrix} u_{xx}(P) & 0 \\ 0 & u_{yy}(P) \end{bmatrix},$$

and has rank at least 1, because $\Delta u(P) = -f(u(P)) < 0$, with $u(P) > 0$ and f is restoring.

Lemma 2.2. *If the solution u to problem (1) is analytic, then the diagonal entries of the Hessian $H(P)$ at any maximum point P are nonpositive.*

Proof. Let x_* be the smallest positive value such that the point $P = (x_*, 0)$ satisfies $u(P) = M$, where M is the maximum of the solution u of problem (1) on T. From the remark above, at least one of terms $u_{xx}(P)$ and $u_{yy}(P)$ is negative. To reach a contradiction assume the lemma fails. That is, suppose to the contrary that $u_{xx}(P) > 0$, making $u_{yy}(P) < 0$. Let $(x, 0)$ be a point close to P. Expanding $u(x, 0)$ as a Taylor series centered at P, we get

$$u(x,0) = M + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(P)(x - x_*)^2 + \ldots,$$

and for sufficiently small $|x - x_*|$, the second term dominates the rest of the Taylor series terms, so that $u(x, 0) > M$, a contradiction. Thus $u_{xx}(P)$ is nonpositive. If $u_{yy}(P) > 0$, then expand $u(x_*, y)$ as a Taylor series centered at P:

$$u(x_*, y) = M + \frac{1}{2}\frac{\partial^2 u}{\partial y^2}(P)y^2 + ..., \tag{4}$$

and the second term will dominate the rest of the series again leading to a contradiction. Hence the diagonal terms of $H(P)$ are nonpositive, and at least one of them is negative. □

Remark. Observe that if $u_{yy}(P) = 0$ then no odd powered y-derivative of u can be the first nonzero such term in the Taylor series (4). The first such term will dominate further terms in a sufficiently small neighborhood of P. Hence, if $u_{yyy}(P) > 0$, the expansion of u,

$$u(x_*, y) = M + \frac{1}{3!}\frac{\partial^3 u}{\partial y^3}(P)y^3 + ...,$$

with $y > 0$, will contradict the maximality of M. Similarly, if $u_{yyy}(P) < 0$, $y < 0$ will yield a contradiction. Let $2k$ be the first nonzero even powered y-derivative. Then, by the proof in Lemma 2.2,

$$\frac{\partial^{2k} u}{\partial y^{2k}}(P) < 0.$$

Lemma 2.3. *Let P be a point on the x-axis in T, where the positive solution u of problem (1) has a (relative) maximum value. The level curves $u > K$, for $M - K > 0$ and small, are convex.*

Proof. Let $P = (x_*, 0)$ and consider the Taylor expansion of $u(x, y)$ in a neighborhood of $P = u(x_*, 0) = M$, $u_x(x_*, 0) = u_y(x_*, 0) = 0$. If the Hessian H has rank 2, then by Lemma 2.2, $u_{xx}(x_*, 0) = -\alpha$, $u_{yy}(x_*, 0) = -\beta$, with α, $\beta > 0$, so that the Taylor series has the form

$$u(x, y) = M - \alpha(x - x_*)^2 - \beta y^2 + O(|\epsilon|^3),$$

where $\epsilon^2 = (x - x_*)^2 + y^2$. If the Hessian has rank 1, the first nonzero terms in x and y will dominate the remaining terms in a small neighborhood, so the Taylor series is either

$$u(x, y) = M - \alpha(x - x_*)^2 - \beta y^{2j} + O(|\epsilon|^3),$$

or

$$u(x, y) = M - \alpha(x - x_*)^{2j} - \beta y^2 + O(|\epsilon|^3), \quad j \geq 2.$$

Letting $u(x,y)=k$, with $M-k>0$ and small, then the level curve of height k through (x,y) has one of the following three forms:

$$\alpha(x-x_*)^2+\beta y^2+O(|\epsilon|^3)=M-K,$$
$$\alpha(x-x_*)^2+\beta y^{2j}+O(|\epsilon|^3)=M-K,$$
$$\alpha(x-x_*)^{2j}-\beta y^2+O(|\epsilon|^3)=M-K, \quad j\geq 2,$$

all convex.

A global result about convexity also holds: Since $u_y(x,-y)>0>u_y(x,y)$ for $y>0$ in T, it follows trivially that the level curves of u are convex in the y-direction. □

Remark. Observe that the isosceles triangle T is contained in the rectangle $\mathbf{R}=\{(x,y)|\,0<x<c,\,|y|<b\}$, and contains the rectangle $\mathbf{R}_1=\{(x,y)|\,0<x<c/2,\,|y|<b/2\}$. The positive eigenfunction for the problem $\Delta u+\lambda u=0$, on $\mathbf{R}$, with $u=0$ on $\partial\mathbf{R}$, is

$$u(x,y)=\sin\frac{\pi x}{c}\cos\frac{\pi y}{2b}, \quad \text{so that} \quad \lambda=\pi^2\left[\frac{1}{c^2}+\frac{1}{4b^2}\right].$$

Thus, by the Courant Nodal Theorem [9], since subdomains have larger eigenvalues, it follows that the first eigenvalue λ_1 of T satisfies

$$\lambda_{\mathbf{R}}=\pi^2\left[\frac{1}{c^2}+\frac{1}{4b^2}\right]<\lambda_1<4\pi^2\left[\frac{1}{c^2}+\frac{1}{4b^2}\right]=\lambda_{\mathbf{R}_1}.$$

Similar results hold for the problem $\Delta u+\lambda g(x,y)u=0$ on Ω, with $g>0$ on Ω, and $u=0$ on $\partial\Omega$; subdomains have larger eigenvalues. We use this result in the next section to provide a proof for the existence of a unique maximum in T , for a certain class of functions $f(u)$.

3. Uniqueness of the Maximum

Assume we have a positive solution $u(x,y)$ to problem (1) on T. Consider the linear eigenvalue problem

$$\Delta v+\lambda f'(u)v=0 \quad \text{on} \quad T, \qquad \text{with} \quad v=0 \quad \text{on} \quad \partial T. \tag{5}$$

Theorem 3.1. *If f satisfies the following conditions:*
(a) $f>0$,
(b) $f'>0$,
(c) the first eigenvalue λ_1 of equation (5) satisfies $\lambda_1>1$,
then u has one critical (maximum) point on T.

Proof. Since our domain T is symmetric in y, the level lines $u_x = 0$ are symmetric, with $(0, b)$ and $(0, -b)$ as limit points, and must cross the x-axis at the critical points of u. We showed in [5] that along the x-axis in T, $u_x > 0$ for small x, and $u_x < 0$ for $x > c/2$, so the level curve $u_x = 0$ must cross the x-axis at some critical point of u. Suppose u has more than one critical point; let P and Q be two such points on the x-axis, so we have one of the following two situations:

(i) Either there is a closed loop ∂L passing through P and Q on which $u_x = 0$, properly bounding a subdomain L inside T, or

(ii) There are two nodal curves beginning at $(b, 0)$ and ending at $(-b, 0)$ (or viceversa), one passing through P, the other through Q, bounding a subdomain J in T, so that $u_x = 0$ on ∂J.

Taking the partial derivative of equation (1) with respect to x yields

$$\Delta u_x + \lambda f'(u) u_x = 0 \quad \text{on} \quad \partial T. \tag{6}$$

on T. In either case $u_x = 0$ on ∂L or ∂J, both curves bounding subdomains of T, implying by the Courant Nodal Theorem [9], $1 \geq \lambda_1$. But this contradicts (c). Hence, at most one maximum exists. □

Example. Let $f(u) = ku/(1 + u)$, so that $f'(u) = k/(1 + u)^2$, where $k = \lambda_{\mathbf{R}} < \lambda_1$. Then $f'(u) < k < \lambda_1$. Suppose a solution u exists to the problem $\Delta u + f(u) = 0$ on T , $u = 0$ on ∂T. Differentiating this equation with respect to x, we get $\Delta u_x + f'(u) u_x = 0$ on T. If P and Q are critical points of u, then by the proof above, at all points in the regions L or J, we get the eigenvalue $f'(u) < k < \lambda_1$, which is impossible since L and J are subdomains of T and their eigenvalues exceed λ_1 . Thus, a unique maximum exists.

For a smooth S-C domain Φ bisected symmetrically by the x-axis, define the derivative of the solution u in the direction θ, of the problem

$$\Delta u + f(u) = 0 \quad \text{on} \quad u = 0 \quad \text{on} \quad \partial\Phi$$

by

$$u_\theta(x, y) = u + u_x(x, y)\cos\theta + u_y(x, y)\sin\theta.$$

Cabre and Chanillo [6] show that there are two points on $\partial\Phi$ which are the boundary points of a nodal arc C_θ interior to Φ on which $u_\theta = 0$. Further, the nodal arc C_θ separates the domain Φ into two subdomains, one where $u_\theta < 0$ and one where $u_\theta > 0$. They prove that all the nodal arcs C_θ meet at a single point $P = (x^*, 0)$ at which the solution u has a maximum $M = \max_\Phi u(x, y)$.

4. Construction of S-C domains $\mathbf{T_k}$

We now construct a set of nested smooth S-C domains T_k , each containing the isosceles triangle T in its interior. The construction will describe the boundary of each set T_k , and consist of an arc on each of three decreasingly small circles and an arc on each of three increasingly large circles.

Let T be the isosceles triangle with vertices at $(0, b)$, $(0, -b)$, $(c, 0)$, with $b, c > 0$. The boundary of T_k consists of an arc on the three (very) small circles

$$x^2+(y-b)^2 = 10^{-k}, \quad x^2+(y+b)^2 = 10^{-k}, \quad \text{and} \quad (x-c)^2+y^2 = 10^{-k}$$

determined by where they are intersected by the arcs of the (very) large circles of radii $r = 10^k(b + c) + 10^{-l}$, with centers lying on

(i) the positive x-axis at distance $10^k(b+c)$ from the points $(0, \pm b)$,

(ii) the line $(y - (b/2)) = (c/b)(x - (c/2))$ in the third quadrant at distance $10^k(b + c)$ from the points $(0, b)$ and $(c, 0)$, and

(iii) the line $(y+(b/2)) = -(c/b)(x+(c/2))$ in the second quadrant at distance $10^k(b + c)$ from the points $(0, -b)$ and $(c, 0)$.

Note that the large arcs meet the small circles tangentially. The point of tangency is where we switch from the large circle arcs to arcs on the small circles. Hence, the boundary of T_k is C^1 .

In each of the S-C domains T_k , let u^k be a solution to problem (1). Note that we can describe u^k by a Green's integral over T_k . Two nodal lines are of importance in each T_k: the nodal line u which is the part of the x-axis in T_k and meets the arc in (i) on the negative axis close to zero and on the small circle $(x - c)^2 + y^2 = 10^{-k}$, and the nodal line $u^k_{\pi/2}$ with boundary points on the circles $x^2 + (y \pm b)^2 = 10^{-k}$. These nodal lines correspond to the arcs in T_k on which $u^k_y = 0$ and $u^k_x = 0$, respectively. They meet at the point $(x^k_{\max}, 0)$ where u^k has its unique maximum M_k .

Now let $k \to \infty$. By Schauder's Theorem, the Green's integrals converge to a Green's integral that is the solution u to problem (1) on T. The nodal arcs $u^k_\theta = 0$ converge to nodal arcs u_θ on T, and even on $\bar{T}$, except for the three arcs u_0 and u_ψ , where $\tan \psi = \pm(b/c)$, which resemble a letter T.

Suppose that u does not have a unique maximum, but has two points $(x^*, 0)$ and $(x', 0)$ where $u = M$. Then there are distinct subsequences $\{k_i\}$ and $\{k_j\}$ both converging to ∞, so that the arcs

$$u_x^{k_i} = 0 \quad \text{and} \quad u_x^{k_j} = 0$$

meet the x-axis at x^* and x' , respectively. The limit points of these sequences form two arcs in T, one containing x^* and the other x' . The points in T between these arcs have u_x values that are both positive and negative, so in the limit all of these points have u_x value equal to 0. Hence the set where $u_x = 0$ contains an open set, which is impossible. Hence u has a unique maximum value at the limit point of the set $\{(x^k_{\max}, 0)\}$.

5. An extension to $\mathbf{R^3}$

Consider the 3-dimensional bounded solid domain F whose boundary ∂F consists of the four triangles T_1 , T_2 , T_3 , T_4 in R^3 passing through the following trios of vertices:

$$\begin{aligned} T_1 &= \{(1,0,0), (-1,0,0), (0,1,0)\}, \\ T_2 &= \{(1,0,0), (-1,0,0), (0,0,1)\}, \\ T_3 &= \{(1,0,0), (0,1,0), (0,0,1)\}, \\ T_4 &= \{(-1,0,0), (0,1,0), (0,0,1)\}. \end{aligned}$$

Assume a positive solution u to the problem

$$\Delta u + f(u) = 0 \quad \text{on} \quad F, \quad \text{with} \quad u = 0 \quad \text{on} \quad \partial F \tag{7}$$

exists. Using symmetry in x and the moving plane method, it is clear that $u_x > 0$ at points where $x < 0$, and $u_x < 0$ when $x > 0$. Thus, $u_x = 0$ on the isosceles triangle T_0 in F lying on the yz-plane. F is also symmetric about the plane $y = z$. Thus, the maximum value of u on F lies on the 45^o line in T_0 .

6. Conclusion

All the results we have proved in this paper extend to symmetric trapezoidal shaped domains with minor modifications. Our numerical calculations seem to indicate that the location of the maximum value does not depend on the function f, but we have been unable to prove this.

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A New Characterization of Boolean Rings with Identity

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ABSTRACT. We define the class of *nil-regular rings* and show that it coincides precisely with the class of *boolean rings*. We thus give a complete description of these rings.

1. INTRODUCTION

Throughout this note, let all rings considered be associative, not necessarily commutative a priory, containing an identity element 1. Since the present paper deals with boolean rings and their generalizations, the condition of having 1 is essential, because finite boolean rings always possess an identity, but this is not in all cases true for infinite boolean rings – e.g., just consider all finite subsets of a given infinite set under the operations of symmetric set difference and intersection.

Furthermore, almost all our notions and notations are standard and follow those from [5]. The other non-classical concepts and terminology will be explained below in detail. For example, a unit u is called *unipotent* if it is of the form $1+n$, where n is a nilpotent. Also, a ring R is called *abelian* if all its idempotents are central, that is, they commute with all elements of the ring.

For completeness, we first recall some classical definitions of ring theory. So, a ring R is said to be *regular* if, for each $a \in R$, there is $x \in R$ such that $a = axa$, and *strongly regular* if, for every $a \in R$, there is $x \in R$ such that $a = a^2x$. It is well known that the strongly regular rings are exactly the regular rings without nilpotent elements, and also that they are exactly the abelian regular rings (see, e.g., [4] and [7]). It is also a well-known fact that a ring R is strongly regular if and only if each element of R is a product of

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a unit and a central idempotent. Likewise, a ring R is said to be *unit-regular* if, for each $a \in R$, there exists a unit $u \in R$ such that $a = aua$. Moreover, it is known that a ring is unit-regular if and only if each of its elements is the product of a unit and an idempotent. Thus strongly regular rings are unit-regular; actually, the element x in the presentation $a = a^2x$ can be chosen to be a unit with $ax = xa$ (see, e.g., [3, Theorem 3]).

So, we are ready to state our main tool.

Definition. We shall say that an arbitrary ring R is *nil-regular* if, for every $r \in R$, there exists a nilpotent n with the property that $r = r(1+n)r = r^2 + rnr$. Such an element r is also said to be *nil-regular*.

If the element r can be written as $r = r^2(1+n) = r^2 + r^2n$, it is called *strongly nil-regular* and if this holds for each such r, the ring R is called *strongly nil-regular* as well.

The objective of the present paper is to characterize completely (strongly) nil-regular rings. Surprisingly, we shall show below that these rings do not possess non-trivial nilpotents; thus we ambiguously obtain that they coincide with the classical boolean rings in which each element is an idempotent. In fact, it is obviously seen that every boolean ring is (strongly) nil-regular by taking $n = 0$, but the eventual validity of the converse containment is definitely non-trivial. Before showing that it really holds, we will establish an equivalent property of nil-regular rings.

2. The main result

Proposition 2.1. *A ring R is nil-regular if and only if, for any $a \in R$, there exist an idempotent e and a nilpotent n such that $a = e(1+n)$.*

Proof. **Necessity**. Since by definition $a = a(1+n)a$, then we define $e = a(1+n)$ and therefore $a = ea = ea(1+n)(1+n)^{-1} = e(1+n)^{-1}$, where $(1+n)^{-1}$ is of the form $1+t$ for some nilpotent t. Moreover, $e^2 = e \cdot e = a(1+n)a(1+n) = a(1+n) = e$, as required.

Sufficiency. If $a = e(1+n)$, then $a(1+n)^{-1} = e$, so $a(1+n)^{-1}a(1+n)^{-1} = a(1+n)^{-1}$, and hence $a(1+n)^{-1}a = a$, where $(1+n)^{-1}$ is equal to $1+t$ for some nilpotent t. ∎

It is worthwhile noticing that since our basic definition is left-right symmetric, one plainly checks that the last statement may be written as $a = (1+m)f$ for any element $a \in R$ and for some existing nilpotent m and an idempotent f.

The next two technical lemmas are referred to in [1] and [2].

Lemma 2.2. *If R is a ring with unipotent units, then so is the corner ring eRe for any idempotent e.*

Proof. Letting $u \in eRe$ be a unit with inverse $v \in eRe$, it is routinely checked that $u+1-e$ is a unit in R with inverse $v+1-e$. Consequently, $u+1-e = 1+t$ for some nilpotent t, so that $t = u-e \in eRe$ is a nilpotent. We therefore have that $u = e+t$, which is obviously a unipotent, as desired. ∎

Lemma 2.3. *For any $n \geq 2$ and any non-zero ring R, the matrix ring $\mathbb{M}_n(R)$ cannot have unipotent units only.*

Proof. Since $\mathbb{M}_2(R)$ is isomorphic to a corner ring of $\mathbb{M}_n(R)$, in accordance with Lemma 2.2, it is enough to show that $\mathbb{M}_2(R)$ has units other than unipotent ones. To this aim, consider the matrix unit $M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, with inverse $\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$. Since $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}$ is clearly not a nilpotent, because it is again a unit with inverse $\begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix}$, we deduce that M is not a unipotent, as desired. ∎

We now have all the ingredients needed to prove the promised above new characterization of boolean rings.

Theorem 2.4. *Any nil-regular ring is boolean, and conversely. Even more, the following three conditions are equivalent:*
(a) R is nil-regular;
(b) R is unit-regular with unipotent units;
(c) R is boolean.

Proof. It is clear that the implications (c) implies (b) implies (a) hold. We will now show that (a) yields (c). To that goal, let (a) be satisfied and we will prove three things about the nil-regular ring R:

(1) *Each unit of R is unipotent.*

In fact, given an arbitrary unit u, we have $u = u(1+n)u = u$ for some nilpotent n, whence $u(1+n) = (1+n)u = 1$, implying that u is equal to $(1+n)^{-1} = 1+t$, where t is some nilpotent.

(2) *R has characteristic* 2.

In fact, as R is regular, its Jacobson radical $J(R)$ must be zero. But since 2 lies in $J(R)$, because $-1 = 1+n$ for some nilpotent n and hence $2 = n$, we deduce that $2 = 0$, as claimed.

(3) *R has no non-zero nilpotents.*

In fact, it is easily checked that nil-regular rings R being regular are semi-potent rings (i.e., each left ideal not contained in $J(R)$ contains a nonzero idempotent) with $J(R) = 0$, and knowing this let $n > 1$. Furthermore, a well-known result of [6] assures us that if $n \in \mathbb{N}$ is such that for any nonzero $a \in R$ with $a^n = 0$, we have $a^{n-1} \neq 0$, then RaR contains a system of n^2-matrix units; so RaR will contain a corner ring isomorphic to an n by n matrix ring. But, in view of Lemma 2.2, the property (1) goes down to corner rings, while, if $T \neq 0$ is any ring, then appealing to Lemma 2.3, the matrix $\mathbb{M}_n(T)$ cannot have the property of having only unipotent units, so that by (1) it is never nil-regular. This contradiction establishes our assertion. ∎

Using the argumentation above about strongly regular rings, and especially that they do not possess nontrivial nilpotent elements, one may deduce the following:

Proposition 2.5. *The following three conditions are equivalent:*
(i) R is strongly nil-regular;
(ii) R is strongly regular with trivial units;
(iii) R is boolean.

In particular, nil-regular rings are strongly nil-regular, and vice versa.

In conjunction with Theorem 2.4, concerning the element-wise description, we will illustrate now that there is a nil-regular element which is not an idempotent. In fact, in the matrix ring $\mathbb{M}_2(\mathbb{Z})$ one we have $E_{12} = E_{12}(1+E_{21})E_{12}$, where one obviously verifies that E_{21} is a nilpotent of index 2 and E_{12} is also a nilpotent of index 2 but not an idempotent, as expected. So, a question which immediately arises is whether or not any strongly nil-regular element is an idempotent. We conjecture that the answer is **no**, too.

The ideas presented above can be extended in the following way: In conjunction with [2] one may ask whether or not the rings R for which, for each $r \in R$, there exist an idempotent e and a nilpotent n such that $r(e+n)r = r$ are precisely the nil-clean rings.

We also call a ring R *π-nil-regular* if, for each $r \in R$, there exist an integer i and a nilpotent n such that $r^i = r^i(1+n)r^i = r^{2i} + r^i n r^i$; such an element r is also called *π-nil-regular*. Clearly, all π-nil-regular rings form a subclass of the class of π-regular rings. Also, one sees that a power of any unit in such a ring is a unipotent.

In this way, if for each $r \in R$ there are an integer i and a nilpotent n such that $r^i = r^{2i}(1+n) = r^{2i} + r^{2i}n$, the ring R is called *strongly π-nil-regular*; such an element r is also called *strongly π-nil-regular*. Evidently, these rings are strongly π-regular and thus unit-regular.

Recall also that a ring R is said to be *π-boolean* if, for every $r \in R$, there exits an integer i such that $r^{2i} = r^i$; such an element r is also called *π-boolean*. Apparently, π-boolean rings are (strongly) π-nil-regular.

So, in accordance with Theorem 2.4, we close with the following question:

Problem 1. Does it follow that (strongly) π-nil-regular rings do not contain non-trivial nilpotent elements, that is, are (strongly) π-nil-regular rings exactly the π-boolean rings?

If the existing nilpotent element in the definition of (strongly) π-nil-regular rings is unique, we then call such rings *uniquely (strongly) π-nil-regular*. So, the next query arises quite naturally:

Problem 2. Characterize uniquely (strongly) π-nil-regular rings. Are they just the abelian (strongly) π-nil-regular ones?

Problem 3. Describe the element-wise relationships between π-nil-regular, strongly π-nil-regular and π-boolean elements. Is it true that for any element r of a ring R it is strongly π-nil-regular if and only if there exists an integer $i > 0$ such that r^i is strongly nil-regular? Likewise, is the record of a strongly nil-regular element left-right symmetric?

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Ireland's Participation in the 56th International Mathematical Olympiad

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This document contains a report on Ireland's participation in the 56^{th} International Mathematical Olympiad (IMO). Any reported facts are accurate to the best of my knowledge, while any opinions expressed are entirely my own.

The 56^{th} International Mathematical Olympiad (IMO) took place in Chiang Mai, Thailand, from 4–16 July 2015. A total of 577 students (52 of whom were girls) participated from 104 countries. This record number of participating countries was achieved only once before in IMO history, at the 50^{th} IMO in Germany in 2009 (the number of participating students was higher this year than in 2009).

The Irish delegation consisted of six students (see Table 1). Accompanying the students to the competition in Chiang Mai were the Team Leader, Mark Flanagan (UCD) and the Deputy Leader, Gordon Lessells (UL).

1. Team selection and preparation

The team detailed in Table 1 consisted of those six students (in order) who scored highest in the Irish Mathematical Olympiad (IrMO),

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Name	School	Year
Luke Gardiner	Gonzaga College, Dublin 6	6^{th}
Paul Clarke	St. Paul's College, Dublin 5	6^{th}
Oisín Flynn-Connolly	Home-schooled, Co Cavan	5^{th}
Anna Mustaţa	Bishopstown Community School	3^{rd}
Ioana Grigoras	Mount Mercy College, Cork	5^{th}
Robert Sparkes	Wesley College, Dublin 16	5^{th}

Table 1. The Irish contestants at the 56^{th} IMO

which was held for the 28^{th} time on Saturday, 25^{th} April, 2015. The IrMO contest consists of two 3-hour papers on one day with five problems on each paper. The students who participated in the IrMO sat the exam simultaneously in one of five *Mathematics Enrichment Centres* (UCC, UCD, NUIG, UL and MU). This year, a total of 107 students took part in the IrMO. The top performer is awarded the *Fergus Gaines cup*; congratulations to Luke Gardiner, who achieved this honour in IrMO 2015.

The students who participate in the IrMO usually attend extra-curricular Mathematics Enrichment classes, which are offered at the five Mathematics Enrichment Centres listed in the previous paragraph. These classes run each year from January until April and are offered by volunteer academic mathematicians from these universities or nearby third-level institutions. More information on the organisation of these classes, as well as links to the individual maths enrichment centres, can be found at the Irish Maths Enrichment/IrMO website `http://www.irmo.ie/`.

Each year in November, the Irish Mathematical Olympiad starts with *IrMO Round 1*, a contest that is held in schools during a regular class period. In 2014, more than 12,000 students, mostly in their senior cycle, from about 260 second level schools participated in Round 1. Teachers were encouraged to hand out invitations to their best performing students to attend the mathematics enrichment classes in their nearest maths enrichment centre.

Some of the top performing students in the IrMO are not in their final year of school, and therefore have a chance to compete for a place on the Irish IMO team again the next year. It is important that such 'returning' students are kept mathematically engaged and challenged. To this end, an Irish Maths Olympiad "Squad" was formed, building on a very successful experience in 2013. The 19 students who were among the top performers in IrMO 2014 and who were also eligible to participate in IMO 2015 were invited to participate in two training camps (one in June and one at the end of August) as well as a centrally organised "remote training" programme.

The remote training programme, which was initiated in 2013, works as follows. At the beginning of each month from September to December inclusive, two sets of three problems are emailed to the participating students. They return their (complete or incomplete) solutions, by email or by post, to the proposer of the problems before the end of the month. The problem proposer then provides feedback on their work, as well as full solutions. The rationale is that the problem-solving abilities of talented students is enhanced most when they work on problems *themselves*, and the provision of worked solutions is also most effective after the students have worked hard on the problems in question. The eight trainers involved in the remote training in 2014 were Mark Flanagan, Eugene Gath, Norbert Hoffmann, Gordon Lessells, Maria Losada, John Murray, Anca Mustaţa and Andrei Mustaţa.

Each of the five maths enrichment centres listed above hosts a local contest for the students, which takes place in February or March (this contest is specific to that centre). In addition, this year a number of students from Ireland was invited to participate in the British Mathematical Olympiad Round 1 (28 November 2014) and Round 2 (29 January 2015). This is a great opportunity for talented students as they get to experience challenging problem solving in a real olympiad-style environment. I would like to thank UKMT, and in particular Geoff Smith, for giving our students this opportunity.

Three training camps were organised at various locations for the Irish Maths Olympiad Squad; during these mathematically intense 3–5 day events, students have the opportunity to socialise with their enthusiastic peers and to increase motivation for their work throughout the year. A kick-start camp for the remote training was organised in UCC for the wider squad from 20–23 August 2014. A training camp for the top performing students in IrMO 2015 was held at Mary Immaculate College, Limerick, from 3–5 June 2015, featuring an IMO-style exam in which $3\frac{1}{2}$ hours were given to solve 3 problems. A training camp for the six members of the Irish IMO team was held at the University of Limerick from 23 to 25 June 2015. The camps were organised by Anca Mustaţa, Bernd Kreussler and Gordon Lessells. The sessions with the students at these camps were directed by Mark Burke, Mark Flanagan, Eugene Gath, Finbarr Holland, Bernd Kreussler, Jim Leahy, Gordon Lessells, Anca Mustaţa, Andrei Mustaţa, and special guest Maria Losada (the IMO Team Leader of Colombia).

A final joint training camp was held immediately before the IMO in collaboration with the team from Trinidad and Tobago. This camp, which builds on the success of a similar collaboration last year, was held at the Holiday Garden Hotel in Chiang Mai. The sessions were conducted by the two Deputy Leaders Gordon Lessells and Jagdesh Ramnanan. As per last year, the students enjoyed this opportunity to train with a team having a comparable level of problem-solving ability.

2. The days in Chiang Mai

The team (including Leader and Deputy Leader) arrived around 10:30pm on Tuesday, the 3rd of July. We were pleasantly surprised to find that an IMO delegation was there to help us. They had been deployed to help us to secure a visa-on-arrival which was needed for one of our team members, Ioana Grigoras. However, as our connecting flight was delayed and arrived later than expected, there was no connecting flight to be had to Chiang Mai and we had to stay the night in Bangkok. Fortunately, the airline put us up in a rather fabulous 5-star hotel which was located very near to Bangkok International Airport (plus a welcome meal before going to sleep!).

The next morning we awoke refreshed after so many hours travelling, and took an early flight to Chiang Mai. There we were met by an offical welcome party who garlanded us with flowers. We all proceeded to the Holiday Garden Hotel, at which the team would carry out their intensive pre-IMO training camp in collaboration with the Trinidad and Tobago team. A meeting room in the hotel proved inexpensive to book and served this purpose adequately. Led by the Irish Deputy Leader Gordon Lessells and the TTO Deputy Leader Jagdesh Ramnanan, this consisted of a lot of intensive problem-solving sessions. It also allowed our students to acclimatise to the hot weather and the time difference. On the day after our arrival, I left Gordon and the students at the training camp and travelled to the Jury site.

The Jury of the IMO, which is composed of the Team Leaders of the participating countries and a Chairperson who is appointed by the organisers, is the prime decision making body for all IMO matters. Its most important task is choosing the six contest problems out of a shortlist of 30 problems provided by the IMO Problem Selection Committee, also appointed by the host country. This year's Chairperson of the Jury was Associate Professor Dr. Soontorn Oraintara. He proved to be a very efficient and effective Chairman of the Jury.

The Jury meetings involved much intense discussion and debate around choosing the 6 problems for the IMO paper. Some points were noteworthy this year:

- One of the first pieces of business for the Jury was to vote on the adoption of a problem selection protocol, used already in IMO 2013 and IMO 2014, which would ensure that one problem from each of the four areas (algebra, combinatorics, geometry and number theory) would be included in problems 1, 2, 4 and 5. This protocol has the principal advantage of ensuring a balance between the four areas among the less difficult problems in the contest (of course, in principle problems 3 and 6 are possible for any student to solve, but in practice these problems are often extremely difficult). However, the protocol also has the disadvantage that many problems are eliminated before a holistic view of the paper is taken.
- One item which attracted quite some attention this year was the absence of candidate 'easy' problems on the shortlist, i.e., candidate Problems 1 and 4. This is not a new phenomenon but has been observed in the last few IMOs. For example, the algebra problem on the shortlist which ranked as the easiest (as judged by the Problem Selection Committee) was eventually classed by the Jury as having a 'medium' level of difficulty, and not an 'easy' one. The easiest problem in the number theory category was N1, but this was excluded due to its similarity to a Chinese Olympiad problem from 2010. Certainly, a range of options for 'easy' level problems is important for the Jury to be able to choose a good IMO paper.
- This year, quite a lot of time was available for devising and debating the marking schemes for the problems, which in my opinion was a very good addition. As a result, the marking schemes were fair and well thought out, with consideration given to the many diverse ways in which students might make progress on a particular problem. Another positive aspect of the marking schemes was that care was taken that students would not lose a mark on a problem due to some very minor arithmetic slip, or the non-inclusion of some very minor calculation which could have been done mentally. In the past, such deductions have caused frustration to some students,

as a very minor error can cause the denial of a medal or an Honourable Mention at the IMO.

- An unexpected complication arose this year, in that some Deputy Leaders were accidentally given access to the Second Day contest paper on the morning of the First Day of the contest. The Jury dealt with this irregularity as follows: on the evening of the First Day of the contest, they selected in a short time a replacement Second Day contest paper from a number of alternatives (the Problem Selection Committee had efficiently prepared these alternatives based on the Jury's preferences as expressed during their previous days of deliberations). Thus the Jury and Problem Selection Committee dealt with this irregular situation in a highly efficient and professional manner, and the IMO 2015 contest papers were both of a high standard.

The opening ceremony of IMO 2015, which took place on the 9$^{\text{th}}$ of July, was presided over by Her Royal Highness Princess Galyani Vadhana Krom Luang Naradhiwas Rajanagarindra. Certainly, it was a mark of the official recognition given to the status of the IMO contest that a member of the host nation's royal family was in attendance for this ceremony.

The two exams took place on the 10$^{\text{th}}$ and 11$^{\text{th}}$ of July, starting at 9 o'clock each morning. On each day, $4\frac{1}{2}$ hours were available to solve three problems. During the first 30 minutes, the students were allowed to ask questions if they had difficulties in understanding the formulation of a contest problem.

Usually at the IMO, soon after the students have finished the contest, the Leaders join the students and Deputy Leader at the IMO site. This year was different, however, in that the Leaders remained at the Jury site for the entire IMO duration; instead, the Deputies were moved to the Jury site, arriving in the evening of July 11. Fortunately, the excellent organisation and staffing of the IMO 2015 contest site ensured that sufficient supervision was given to the students at all times.

Upon the arrival of the Deputies, Gordon and I went into the detailed study of our student's scripts. Our team had some nice solutions to some difficult problems. The team performed best on Problem 4, which was a Geometry problem. Paul and Anna gave excellent solutions both of which were worth full marks, and Ioana gained significant partial credit on this problem by successfully reducing the problem to an easier one. Oisín managed to completely solve Problem 1, which was a great achievement not least because the problem had two distinct parts, meaning that two fundamental insights were needed to completely solve the problem. I noticed that as expected, Problems 2 and 5, which had a rather technical nature and thus favoured the better trained and more experienced students, were somewhat more difficult for our students than is previous years. It was obvious that a higher level of training and preparation could have been useful for our students in dealing with these problems.

On one of the coordination days, the students were entertained with an excursion to a traditional Chiang Mai umbrella production center, the popular Sankampaeng hot springs, as well as a visit to the spectacular temple "Wat Doi Suthep". This provided some welcome relaxation after the intensive concentration of the contest days.

The final Jury meeting, at which the medal cut-offs were decided, took place on Tuesday 14^{th} July. The closing ceremony was held on the following day, followed by a Farewell Banquet that evening. Gordon and I accompanied the team back to Ireland on Thursday 16^{th} July.

3. The problems

The two exams took place on the 10^{th} and 11^{th} of July, starting at 9 o'clock each morning. On each day, $4\frac{1}{2}$ hours were available to solve three problems. Since the second day of the contest was a Saturday, candidates who were unable to sit the paper on this day for religious reasons were permitted to enter quarantine during that day, and to sit Paper 2 after sunset.

First Day. Problem 1.

We say that a finite set $\mathcal{S}$ of points in the plane is *balanced* if, for any two different points A and B in $\mathcal{S}$, there is a point C in $\mathcal{S}$ such that $AC = BC$. We say that $\mathcal{S}$ is *centre-free* if for any three different points A, B and C in $\mathcal{S}$, there is no point P in $\mathcal{S}$ such that $PA = PB = PC$.

(a) Show that for all integers $n \geqslant 3$, there exists a balanced set consisting of n points.

(b) Determine all integers $n \geqslant 3$ for which there exists a balanced centre-free set consisting of n points.

(The Netherlands)

Problem 2.

Determine all triples (a, b, c) of positive integers such that each of the numbers

$$ab - c, \quad bc - a, \quad ca - b$$

is a power of 2.

(*A power of* 2 *is an integer of the form* 2^n*, where* n *is a non-negative integer.*)

(Serbia)

Problem 3.

Let ABC be an acute triangle with $AB > AC$. Let Γ be its circumcircle, H its orthocentre, and F the foot of the altitude from A. Let M be the midpoint of BC. Let Q be the point on Γ such that $\angle HQA = 90^\circ$, and let K be the point on Γ such that $\angle HKQ = 90^\circ$. Assume that the points A, B, C, K and Q are all different, and lie on Γ in this order.

Prove that the circumcircles of triangles KQH and FKM are tangent to each other.

(Ukraine)

Second Day. Problem 4.

Triangle ABC has circumcircle Ω and circumcentre O. A circle Γ with centre A intersects the segment BC at points D and E, such that B, D, E and C are all different and lie on line BC in this order. Let F and G be the points of intersection of Γ and Ω, such that A, F, B, C and G lie on Ω in this order. Let K be the second point of intersection of the circumcircle of triangle BDF and the segment AB. Let L be the second point of intersection of the circumcircle of triangle CGE and the segment CA.

Suppose that the lines FK and GL are different and intersect at the point X. Prove that X lies on the line AO.

(Greece)

Problem 5.

Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f\big(x + f(x+y)\big) + f(xy) \;=\; x + f(x+y) + yf(x)$$

for all real numbers x and y.

(Albania)

Problem 6.

The sequence $a_1, a_2, \ldots$ of integers satisfies the following conditions:

(i) $1 \leqslant a_j \leqslant 2015$ for all $j \geqslant 1$;

(ii) $k + a_k \neq \ell + a_\ell$ for all $1 \leqslant k < \ell$.

Prove that there exist two positive integers b and N such that

$$\left| \sum_{j=m+1}^{n} (a_j - b) \right| \leqslant 1007^2$$

for all integers m and n satisfying $n > m \geqslant N$.

(Australia)

4. The results

Table 2 shows the scores achieved by all contestants on the 6 problems. The Jury tries to choose the problems such that Problems 1 and 4 are the most accessible, while Problems 2 and 5 are more challenging. Problems 3 and 6 are usually the most difficult problems, whose existence on the paper is justified in posing a sizeable challenge even to the top students in the IMO competition. It may be seen from Table 2 that this gradient of difficulty was generally maintained this year also.

The medal cut-offs were as follows: 26 points needed for a Gold medal (39 students), 19 for Silver (100 students) and 14 for Bronze (143 students). A further 126 students were awarded an Honourable Mention (an Honourable Mention is awarded to any student who did not win a medal, but achieved 7 points out of 7 on at least one problem). Overall, only 30.9 % of the possible points were scored by the contestants, compared to the figure of 38.2 % last year. The low level of the medal cutoffs, together with the low number of points scored by the contestants, is a testament to the extreme difficulty of this year's IMO. It is noteworthy that the average scores for problems 1 and 4 were significantly lower than last year.

	P1	P2	P3	P4	P5	P6
0	75	240	479	24	301	514
1	23	32	43	103	60	7
2	14	25	1	28	83	7
3	22	17	2	16	10	11
4	15	14	3	5	8	0
5	18	39	0	3	3	5
6	23	71	4	3	11	1
7	370	122	28	378	84	15
average	5.348	2.971	0.505	5.189	1.709	0.296

TABLE 2. The number of contestants achieving each possible number of points on Problems 1–6.

Table 3 shows the results of the Irish contestants. In light of the extreme difficulty of this year's IMO, the three Honourable Mentions awarded to the Irish contestants represents a very good achievement. Indeed, 2015 represents the fourth year in a row with at least two Honourable Mentions for the Irish team.

Name	P1	P2	P3	P4	P5	P6	total	ranking
Paul Clarke	3	1	0	7	1	0	12	307
Oisín Flynn-Connolly	7	0	0	0	2	0	9	365
Anna Mustaţa	0	0	0	7	1	0	8	394
Ioana Grigoras	1	0	0	3	0	0	4	465
Luke Gardiner	1	1	0	1	0	0	3	480
Robert Sparkes	0	0	0	1	0	0	1	532

TABLE 3. The results of the Irish contestants

The figures in Table 4 have the following meaning. The first figure after the problem number indicates the percentage of all points scored out of the maximum possible. The second number is the same for the Irish team and the last column indicates the Irish average score as a percentage of the overall average.

The Irish students' performance on Problem 4 (geometry), as given in Table 4, shows that Irish students are becoming internationally competitive in this subject area. Note that a similar relative performance was seen in a geometry Problem 4 last year (73.9%), which suggests that our students' improvement in this subject area is somewhat sustained in the last few years.

Problem	topic	all countries	Ireland	relative
1	combinatorics	61.5	28.6	46.4
2	number theory	19.4	4.8	24.5
3	geometry	9.3	0.0	0.0
4	geometry	68.5	45.2	66.1
5	algebra	21.6	9.5	44.1
6	algebra	5.1	0.0	0.0
all		30.9	14.7	47.5

TABLE 4. Relative results of the Irish team for each problem

It is noteworthy that two of the Irish contestants this year won awards in Mathematical Olympiads other than IMO. Luke Gardiner was chosen also for the team representing jointly the UK and Republic of Ireland at the Balkan Maths Olympiad, and won a Bronze medal at this competition. Anna Mustaţa represented Ireland at the European Girls' Mathematical Olympiad (EGMO) in Minsk, Belarus, in April 2015, and Anna Mustaţa won a Silver medal at this competition. Congratulations to Luke and to Anna on these great achievements.

It is also notable that a student on the Trinidad and Tobago team, Prasanna Ramakrisnan, achieved a silver medal. This is the second ever silver medal for Trinidad and Tobago. Congratulations to Prasanna on this great achievement.

Although the IMO is a competition for individuals only, it is interesting to compare the total scores of the participating countries. This year's top teams were USA (185 points), China (181 points) and the Republic of Korea (161 points). Ireland, with 37 points in total, shared the 77$^{\text{th}}$ place with Albania.

This year, only one student achieved a "perfect score" (42 points). This was Alex Song of Canada, who now moves into the leading position in the IMO Hall of Fame, having achieved 5 Gold medals and one Bronze medal in 6 participations at the IMO.

The detailed results can be found on the official IMO website, which is located at `http://www.imo-official.org`.

5. Outlook

The next countries to host the IMO will be

2016	Hong Kong	6–16 July
2017	Brazil	
2018	Romania	
2019	United Kingdom	

6. Conclusions

This year's Irish IMO team performed to a level consistent with that achieved on average over the last few years. The number of points scored was lower than last year, but this fact must be set against the fact that this year's IMO was at an extremely difficult level.

Since Ireland's first participation in 1988, the Irish teams have won 8 medals and 34 Honourable Mentions. 13 of these 34 Honourable Mentions were achieved in the last three years. This is evidence that while there are fluctuations in performance year on year, a generally sustained team-level improvement can be detected within the last few years. The extra effort being invested in training activities in the last few years correlates well with this improvement.

However, there are still some key challenges which in my opinion, will be key to improving Ireland's IMO performance in the longer term:

(1) Students who achieve excellent results at the IMO are invariably students who immerse themselves in problem-solving activities. While our current training activities are very beneficial to students in that they provide a "way in" to this kind of activity, students must reach a level of independence where they can work on their own. This can involve working on problems with relatively little guidance required from trainers, as well as finding their own problems as well as their own training materials. Challenges for trainers in this context include (a) how to help students to develop this level of independence, and (b) how to keep such students motivated and enthusiastic while having a less intensive level of contact with the students. In my opinion, the remote training is an excellent step in this direction as it demands a higher level of independence of the students, while maintaining a structured form of support from trainers.

(2) In looking at the general performance of students at the IMO, and in particular on an international scale, it may be seen

that students who get involved in problem-solving activities at an earlier age have a much enhanced probability to succeed at a high level. This point is not independent of Point 1 above, since students who grasp problem-solving at an earlier age will naturally reach independence at an earlier stage, and then have more years to hone their problem-solving abilities in an independent manner. Some important work has begun in the last few years which aims to address this gap in early problem-solving opportunities. For example, the maths enrichment centre at UCC now runs Junior Maths Enrichment Classes for students in second and third year; this initiative is a by-product of the "Maths Circles" initiative which was set up for Junior Cycle students in second level schools in the Cork area in 2013. It would be extremely good if such early-stage regional activities became more widespread, and if teachers can be motivated to support problem-solving activities at a local level. It is worthwhile to note that engagement in such problem-solving activities greatly enhances the general mathematical education of school-level students. It is worth mentioning two mathematics contests in this context. The first is the PRISM (Problem Solving for Post-Primary Schools) competition, which is a multiple-choice contest designed to involve of the majority of pupils in mathematical problem solving; it has a paper for Junior Cycle students and one for Senior Cycle students. This contest is organised since 2006 by mathematicians from NUI Galway, and takes place in October every year during Maths Week. It normally attracts about 2 500 participants. The second is the Kangaroo Mathematics contest, an international mathematics contest which introduces school students aged 7-19 years to Mathematics challenges in a fun and enjoyable way, thus inspiring their further interest and advancement in Mathematics. The Kangaroo contest is undertaken by students in over 60 countries, and over 6 million students took part in 2015. This contest was organised in Ireland this year by Michael Cotter and Mark Flanagan, and attracted about 600 participants. This number is extremely low compared to that of other countries around the world, and strongly indicates that to be successful in Ireland, the Kangaroo contest crucially requires the support and involvement of enthusiastic Irish maths teachers.

(3) While our students are well equipped to solve problems at the level of the Irish Maths Olympiad, they have less experience is attempting problems at IMO level. This can be disheartening for students who, at the IMO contest, find themselves unable to comfortably deal with the difficulty level as well as aspects of time management within the exam. Students from other countries have more experience in sitting exams of the difficulty level and duration of the IMO; this is an experience we need to build into our training programmes. However, this must be done carefully, as placing students into an IMO-style examination prematurely could destroy their confidence. This point has a relationship with Point 2 above, as it is envisaged that this difficulty will be easier to address when students have a longer time to reach the point on the training trajectory where they are ready to engage with IMO-level problems.

(4) The delivery of these new initiatives, as well as the running of training camps and the sending of a full team of six students, together with Leader and Deputy Leader, to the IMO contest requires sustained funding, and to increase the level of provision of training requires increased funding. Efforts to secure funding for these activities should be increased.

7. Acknowledgements

Ireland could not participate in the International Mathematical Olympiad without the continued financial support of the Department of Education and Skills (DES), which is gratefully acknowledged. Thanks to its Minister, Jan O'Sullivan TD, and the members of her department, especially Mary Whelan, for their continuing help and support. Also, thanks to the Royal Irish Academy, its officers, the Committee for Mathematical Sciences, and especially Rebecca Farrell, for support in obtaining funding. Also instrumental to funding the Irish IMO participation this year was the generous donation received by the Irish Mathematical Trust from Eoghan Flanagan, who was himself a member of the Irish IMO team in 1993 and 1994. This sponsorship, together with the funding by the DES, enabled Ireland to send a full team of six students with leader and deputy leader to the IMO. In particular, the pre-IMO training camp in Chiang Mai was only possible because of Eoghan's generous sponsorship.

The foundation for the success of the contestants is the work with the students done in the enrichment programmes at the five universities. This work is carried out for free by volunteers in their spare time. Thanks go to this year's trainers at the five Irish centres:

- At UCC:
  Kieran Cooney, Gleb Dzus, David Goulding, Finbarr Holland, Desmond MacHale, Ben McKay, Joseph Manning, Anca Mustaţa, Andrei Mustaţa, Keegan O'Mahoney and Jonathan Peters.
- At UCD:
  Gary McGuire, Kevin Hutchinson, Tom Laffey, Mark Flanagan, Eugene Kashdan, Helena Smigoc, Rupert Levene, Elena Arabini, Masha Vlashenko, Mary Hanley, Marius Ghergu, Nina Snigireva, Chris Boyd and Anthony Cronin.
- At NUIG:
  Daron Anderson, James Cruickshank, Graham Ellis, Kevin Jennings, Niall Madden, Götz Pfeiffer, James McTigue, Rachel Quinlan, Jerome Sheahan and James Ward.
- At UL:
  Mark Burke, Ronan Flatley, Eugene Gath, Norbert Hoffmann, Bernd Kreussler, Jim Leahy and Gordon Lessells.
- At MU:
  Stefan Bechtluft-Sachs, Stephen Buckley, Peter Clifford, David Malone, Ollie Mason, John Murray, Anthony G. O' Farrell, Lars Pforte, Adam Ralph and David Redmond.

Thanks also to the above named universities for permitting the use of their facilities in the delivery of the enrichment programme, and especially to University College Cork, to Mary Immaculate College, Limerick, and to the University of Limerick, for their continued support and hosting of the pre-olympiad training camps.

Finally, thanks to the hosts for organising this year's IMO in Thailand and especially to the team guide in Chiang Mai, Sornram Permmee.

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Boolean Rings are Definitely Commutative!

DESMOND MACHALE

Abstract. A ring $\{R, +, .\}$ is called Boolean if $r^2 = r$ for all $r \in R$. We present four proofs that a Boolean ring is commutative.

A ring $\{R, +, .\}$ is called Boolean if $r^2 = r$ for all $r \in R$. In this bicentenary year of Boole's birth we present four proofs that a Boolean ring is commutative. Our first proof is the standard one found in many textbooks.

Proof 1. For all $r \in R$ we have $r = r^2 = (-r)^2 = -r$, so $r + r = 0$. Next, for all x and y in R, $x+y = (x+y)^2 = x^2+xy+yx+y^2$, so by cancellation in the group $\{R, +\}$, we have $xy+yx = 0 = xy+xy$, by the above. Again by cancellation we have $xy = yx$, as required. □

Proof 2. As in Proof 1, $xy + yx = 0$, for all x and y in R. Since for all $r \in R$, $0.r = 0 = r.0$ we have $(xy + yx)x = x(xy + yx)$ or $xyx + y.x^2 = x^2.y + xyx$. Cancelling xyx and remembering that $x^2 = x$, we get $xy = yx$, as required. □

Proof 3. Since for all r, $r^2 = r$ it follows that if $r^2 = 0$ then $r = 0$. Now for all x and y in R we have $(xy - xyx)^2 = xyxy + xyxxyx - xyxyx - xyxxy = xyxy + xyxyx - xyxyx - xyxy = 0$. So $xy - xyx = 0$ and $xy = xyx$. Then $(yx - xyx)^2 = yxyx + xyxxyx - yxxyx - xyxyx = yxyx + xyxyx - yxyx - xyxyx = 0$. So $yx - xyx = 0$ and $yx = xyx$. Thus $xy = yx$ as required. □

Proof 4. For $a, b \in R$ if $ab = 0$, then $ba = (ba)^2 = b(ab)a = 0$. Now, $0 = xy - xy = xy - x^2y = x(y - xy)$, so $0 = (y - xy)x = yx - xyx$. Also, $0 = yx - yx = yx - yx^2 = (y - yx)x$, so $0 = x(y - yx) = xy - xyx$. Thus $xy = yx$ for all x and y in R. □

We note it is immediate in all four proofs that $xy = yx = xyx = yxy$, for all x and y.

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D. Rosenthal, D. Rosenthal and P. Rosenthal: A Readable Introduction to Real Mathematics, Springer, 2014. ISBN:978-3-319-05653-1, ebook GBP 27.99, hardcover GBP 35.99, 161+xii pp.

REVIEWED BY ROBIN HARTE

This book comes with a public health warning: very hard to put down. Contrary to the impression given by its possibly over-loud title - probably deriving from the excitement of the bridging course in Toronto - this book is basically a new and refreshing introduction to number theory. In outline, in twelve chapters, it progresses from the natural numbers and induction through modular arithmetic, the "fundamental theorem of arithmetic" and the Euclidean algorithm to the RSA method of public key encryption. After a look at complex numbers and the cardinality of infinite sets, it goes on to discuss "plane geometry", ruler-and-compass constructibility, and surds. But the heart of it is the number theory: prime factorization, the ϕ function, the Euclidean algorithm, Fermat's and Wilson's theorem, here put to work in the service of RSA and public key encryption. This reviewer has always been a little afraid of number theory: all that stuff about phi functions and prime number density has seemed arcane and irrelevant, very far away from "real mathematics". Now. after immersing himself in this little book, number theory begins to make sense.

"Uncle Petros", fronting for Apostolos Doxiadis [1], tells his nephew that "addition is natural, but multiplication is artificial". To see what he is getting at just, on day two of your introductory course, write out the first few natural numbers in factorized form:

$$1, 2, 3, 2^2, 5, 2 \cdot 3, 7, 2^3, 3^2, 2 \cdot 5, 11, 2^2 3, 13, 2 \cdot 7, 3 \cdot 5, 2^4, 17, 2 \cdot 3^2, 19 \ :$$

is there any discernible pattern? Number theory is born out of the attempt to answer that question.

Peter Rosenthal - who with Heydar Radjavi wrote the book on invariant subspaces - has with his extended family come down to

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earth with a vengeance and struck bedrock. All about natural numbers and prime factorization, a conversational style of writing conceals some very serious mathematics. The fundamental role of the Euclidean algorithm permeates the account: things are sometimes proved twice, first without and then with its help. There is a succinct account of Public Key Cryptography and RSA, which with our obsession with internet security has made number theory cool, introduced [2] to an Irish audience by our own mathematical family of Sarah Flannery and Dave; the arcane Fermat and Wilson theorems are put to work reach "encryptors" and "decryptors", either with or without the help of the Euclidean algorithm. Modular arithmetic leads to the "Chinese remainder theorem" which is buried in an INTEL chip, and to tests for divisibility by 9, 11 and indeed 7: for example the serial number of an Irish euro note, beginning with the letter T , is always equal to 6 mod 9, and the serial number T39484135244 would therefore suggest forgery.

For serendipity it will be hard to beat the identification of ruler-and-compass constructible numbers with "surds", here given a rather careful definition. This reviewer however wonders whether these surds should more properly be called "quadratic surds", in that it not immediately obvious whether or not the cube root of two is such a thing. He is also reassured to see that the constructions are carried out in "numerical space" $\mathbb{R}^2$ rather than some "Euclidean geometry" of whose foundations he would be unsure. The authors are careful to frighten nobody: but they could afford to incorporate a carefully sealed-off appendix listing axioms for the real number system $\mathbb{R}$, and at one point to offer a simple statement of the terrifying Gelfond-Schneider theorem. They could also tell us that the "natural numbers" are in a sense *defined by induction*: Bertrand Russell explained [3] that from his prison cell as a conscientious objector.

Having taken by the scruff of its neck the half-defined word "surd" and given it a specific meaning of their own, the Rosenthals could also do the same for the honorary real number ∞, which to us represents the smallest infinite cardinal, coinciding with the first infinite ordinal:

$$\mathbb{N} = \{1, 2, 3, \ldots\} \subseteq \infty = \{0, 1, 2, \ldots\} \subseteq \infty + 1 = \infty \cup \{\infty\};$$

only now do we introduce the inscrutable "function" $\aleph : n \mapsto \aleph_n$ from ordinals to cardinals, for which

$$\aleph_0 = \infty < \aleph_1 \leq |\mathbb{R}| = 2^\infty = \aleph_k \ ,$$

where the status of the ordinal k is one of the great mysteries of mathematics.

There are just one or two dropped stitches: for example the product

$$100,000,559 = 53 \cdot 223 \cdot 8461$$

is offered as an example of a prime number, and technology has now caught up with their $3 + 2^{3,000,005}$. It is perhaps not entirely clear what the intended audience would make of this book if left to read it on their own: but to this rather jaded ex operator theorist it begins to make sense of all that "number theory" he never could get to grips with.

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[1] Apostolos Doxiadis, Uncle Petros and Goldbach's conjecture, 1992.
[2] Sarah Flannery, In code: a mathematical journey, 2001.
[3] Bertrand Russell, The Principles of mathematics, Norton, 1903.

Robin Harte Still loosely attached to TCD, Robin is the author of *Spectral mapping theorems, a bluffer's guide* (Springer briefs in mathematics, 2014), and also of *Invertibility and Singularity* (Dekker 1988), now shortly to be re-released by Dover Books.

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Cédric Villani: The Birth of a Theorem: A Mathematical Adventure, The Bodley Head, London, 2015. ISBN:978-1-84792-252-6, GBP 18.99, 269 pp.

REVIEWED BY DÓNAL HURLEY

The theorem referred to in the title of this book concerns *Landau damping* which is named after its discoverer, Lev Landau (1908 - 1968). Landau damping is the effect of damping (exponential decrease as a function of time) of longitudinal space-charge waves in plasmas. The starting point in the investigation of this is the Vlasov equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \left(\nabla W * \int f \, dv \right) \cdot \nabla_v f = 0,$$

which determines the statistical properties of plasmas. In 1946, Laudau studied a linearised version of this model and concluded that the electrical forces weakened spontaneously over time without a corresponding increase in entropy. The problem which remained, and which Cédric Villani and Clément Mouhot decided to investigate in 2008, was whether or not Landau's results also applied to the nonlinear model.

Cédric Villani, born in 1973, is a French mathematician working primarily on partial differential equations, Riemannian geometry and mathematical physics. In 2010 he was awarded the Fields Medal for his work on Landau damping and the Boltzmann equation. He has also won many other awards including the European Mathematical Society Prize (2008) and the Henri Poincaré Prize (2009). Since 2013 he is a member of the French Academy of Sciences. In October 2014, he was the Hamilton Lecturer in Dublin where many of us had the pleasure of hearing him give a very stimulating and accessible talk. Clément Mouhot did his Ph.D under Villani's direction writing a thesis on the Boltzmann equation.

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In December 2006, while attending a conference at Oberwolfach, the legendary Institute for Mathematical Research in the Black Forest, Villani sat at a table with two mathematicians who discussed the Laudau damping problem and heard them use words such as "weird" and "strange". It meant nothing to him at the time but he did make a mental note and filed it away in a corner of his brain. At the beginning of 2008, Mouhot informed Villani what precisely was the difficulty with Landau's work on Vlasov's original model. Villani wondered if he and Mouhot could tackle it. He decided they should try, and so began their joint efforts over the next eighteen months.

The book consists of forty-four short chapters of dated reports on the progress of the two researchers. This helps the reader get a good sense of the progress of the proof; advances are slow on occasions and other times very rapid. Of course difficulties arise as errors are discovered. The proof required many detailed analytical arguments and these needed to be constantly checked/double checked. The work practice of the two collaborators required constant email exchanges. The text of many of these is included throughout the book. There are a number which are dated December 25, 2008, showing how absorbed both were in their quest. During the first six months of 2009, Villani was on leave in Princeton and, as well as the emails, there were several long phonecalls. The time difference between Paris and Princeton was exploited by them; Villani could work until midnight Princeton time and three hours later Mouhot was in his office in Paris to take up the task.

The two researchers also benefited from discussions with other researchers. Freddy Bouchet, on the occasion of a visit to the ENS de Lyon, suggested connections with work in modeling of galaxies. "When you model galaxies, you treat the stars as a fluid - as a gas of stars, in effect" he told Villani. On the occasion that Étienne Ghys, a colleague, noticed diagrams on the board in Villani's office, he suggested a Kolmogorov-Arnold-Moser (KAM) theory connection. "KAM is found almost everywhere" he said. A year later, Villani did find a link between Landau damping and KAM theory although not exactly what Ghys suggested.

A heart-stopping moment is described when Villani received a manuscript, in his role as an Editor of a journal, titled "On the existence of exponentially decreasing solutions of the nonlinear Landau damping problem". It looked like the authors had proved the result that he wished to prove by constructing solutions to the Vlasov equation that spontaneously relax toward the equilibrium. He wrote to the Editon-in-Chief saying that he faced a conflict of interest. However, on examining the manuscript more closely, he realised that the authors only proved that some damped solutions exist whereas what needed to be proved is that all solutions are damped.

At the beginning of 2009 while Villani was in Princeton, the outline of a proof began to emerge. Over the next few months, Villani gave several talks at Princeton and at other locations in the USA. During the question sessions at the end of his talks, two issues were raised as troubling to the experts in the audience; an analyticity assumption and the limit case in large time. At succeeding talks, he was able to improve the results and the expositions. When Villani returned to France at the end of June 2009, he and Mouhot felt the proof was in good shape . Their 180 page monster was ready to be submitted to *Acta Mathematica*.

On October 23, 2009, Villani received an email from *Acta Mathematica*. The six reports were very positive on the whole except, sure enough, they have misgivings on exactly the same two issues which were frequently raised at the end of his talks. The Editor was not persuaded that the results were definitive and also raised the issue of the length of the manustript.

Another message on that day informed Villani that he'd just won the Fermat Prize which is awarded every two years to one or two mathematicians under the age of 45 who have made major contributions in one of those domains in which Pierre de Fermat worked (number theory, calculus of variations, probability theory). While this was a great consolation, it was not enough to get over the frustration of seeing their article turned down.

Four days later while in Ann Arbor, Michigan, to give a talk, he discussed his results with Jeff Rauch, a leading expert on partial differential equations. Rauch was also very concerned about the analyticity assumption. Villani found Rauch's reservations troubling so he decided to go through the article yet again, checking the soundness of the argument, step by step.

And there.... He had a moment of illumination when the light bulb went on in his head. Staring at the formulae, he realised it was not a question of fixing an error; it was a question of improving the results. He and Mouhot took the whole thing apart and put it back together. They finally figured out how to lay to rest the two infernal objections, once and for all. They worked in feverish excitement revising, rechecking everything, improving everything. Their result was now stronger as they also solved a problem that had long intrigued specialists. On December 6, 2009, they resubmitted a new version of their "On Landau Damping" paper to *Acta Mathematica*. The covering letter of this submission is included in full in the book. Eventually, in November 2010, they got confirmation that their article was accepted. Their theorem had been born at last.

In between submission and acceptance of the paper, Villani was informed, February 16, 2010, that he had won the Fields Medal and he was presented with the medal at Hyderabad, India, by the President of India on August 19, 2010.

While most of the book is about the development of the theorem, there is much material of a general interest nature. He gives short biographical sketches of several well-known mathematicians, including Pierre de Fermat, John Nash, Grigori Perelman and Lev Landau. Nash was someone he greatly admired and he says that "next time our paths cross I shall dare to approach John Nash ... and ... will even tell him that he in my hero". I hope he did have this opportunity as Nash and his wife were tragically killed when they were returning to Princeton from Oslo where, just 5 days earlier, he had been presented with the Abel Prize on May 18, 2015.

The book also includes much material illustrative of Villani's personality: he obviously has a wonderful way with children and likes to tell his own offspring improvised tales which go on and on, the kind his daughter calls "imaginary stories" - her favourite kind. He is an incessant tea drinker and has an amusing description of the night he stole some tea bags from the Common Room of the School of Mathematics at Princeton when there were none in his own house. He enjoys very much listening to music and has very catholic tastes; he gives long lists of his favourite pieces.

This is indeed a charming book to read; for mathematicians, it reflects their experiences at doing research and for general readers there is much of general interest as well as illustrating the lifestyle of mathematicians. All readers will be absorbed in the roller-coaster excitement of the development of the theorem on Landau damping.

Dónal Hurley is retired from the School of Mathematical Sciences at University College Cork where he worked since 1973. His research interests are in the areas of dynamics of geodesic flows and mathematical physics.

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David Attis: Mathematics and the Making of Modern Ireland, Trinity College Dublin from Cromwell to the Celtic Tiger, Docent Press, 2014. ISBN:9780988744981, USD 17.99, 484 pp.

REVIEWED BY DAVID SPEARMAN

Mathematics in Trinity College Dublin has been a core discipline and area of activity since the college's foundation at the end of the sixteenth century up to the present day. Its history is well documented and has been a subject of interest to historians of science; it forms the central strand of this book. Mathematics is interpreted in a broad sense, reflecting the historical grouping in Trinity of the chairs of mathematics and of natural philosophy. And, despite the book's title, Irish non-Trinity scientists such as Boyle, Kelvin, Stokes, Larmor and Boole are included in the narrative. The author's interest in Irish mathematics reaches back for over twenty years. As an undergraduate in Chicago his enthusiasm for physics was channelled towards history of science which led him to Cambridge and then to Princeton and a doctoral thesis which formed the basis from which this book has developed. Over the years he has made many visits to Ireland where he has consulted widely and studied source material. It is important that the material and information that he has gleaned and a listing of the references from which he has drawn, together with his own commentary and assessment, should be made easily available as it now is with the publication of this book. David Attis insists however that the history as he presents it in this book is different from what might normally be expected of a history of mathematics. "It explores" he says "the contested and contingent aspects of mathematical activity as a means to understand not the logical development of mathematical truth but the social and political development of modern Ireland". I have some doubts about this approach and the constraints which it imposes on the narrative as he presents it. I agree that an account of the mathematical achievements of a relatively small scientific community is not of itself an adequate basis for understanding the logical

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development of mathematical truth, but to suggest that it can provides a means to understand the social and political development of modern Ireland seems to me to be unrealistic. And pursuing that objective can have the further disadvantage of neglecting other strands of influence and of deflecting attention from the diversity of individual motivation and outlook. Examine, by all means, the social and political contexts within which these individuals lived and the extent to which their work was influenced by them; that, as well as the impact their achievements may have had both in the social and scientific contexts, is a proper part of the historical narrative. The characters in this story form a varied bunch, differing in background, in attitude and interests, and also in political outlook. The Trinity Fellows may have shared a common commitment to the Union but many of them were proud to count themselves Irishmen. They were Protestants, most of them ordained clergy, but typically not religious zealots. An important part of their Protestantism was their resistance to authority; this manifested itself in their readiness to adopt the new learning and to replace the Aristotelian world view with Bacon, Copernicus and Galileo. Fellows were required to be celibate while holding their Fellowships but they could resign if they wanted to marry, and for those in holy orders there was the prospect of moving to a country rectory in one of the livings owned by the College, where some at least managed to continue their scholarly investigations. They came from different and often relatively modest social backgrounds. Hamilton's father was a land agent; the family's strained financial circumstances were at least partially the reason that he was sent to live with and be educated by his uncle who ran the diocesan school in Trim. Brinkley, whom Hamilton succeeded as Professor of Astronomy, was born to a single mother and was supported through school and at Cambridge by teachers and clergy who recognised his talent. Incidentally, Brinkley would probably not have resigned to become Bishop of Cloyne if the professorial salary had been closer to that of the bishopric. And the world would have been the poorer! James MacCullagh was the son of a small farmer in County Tyrone. He lived modestly in College, allowing him to accumulate his salary over many years so that when the Cross of Cong came on the market he was able to buy it and present it to the Academy, expressing his distress at the prevailing lack of interest in our national heritage and the hope that his intervention might encourage others to do likewise. It did, and led to further acquisitions and the formation of the collection of Irish antiquities now housed in the National Museum. They kept in touch with their English colleagues. They travelled to London and maintained contact through regular correspondence. MacCullagh's friend Charles Babbage arranged for him to become a member of the Athenaeum, giving him a base where he could stay when he was in London. And they also corresponded with French, German, Russian and other colleagues. They regularly attended the annual meetings of the British Association after its establishment in 1831. Quite a few of them were Fellows of the Royal Society: Boyle and Petty were Founder Fellows. The College was extremely fortunate in its library. Thanks to Luke Challoner, one of the original Fellows, and James Ussher, one of the initial scholars who became a Fellow, then a bishop and then Archbishop of Armagh, both avid book collectors, members of the College had access from the very early days to a larger and broader collection than was then available in Cambridge. Trinity mathematicians, were to the fore in establishing the Dublin Philosophical Society in 1684. These included William Molyneux, who had translated Gallileo and Descartes into English and did important work on optics, who was a friend of John Locke and had Locke's Essay concerning human understanding introduced into the College curriculum. Another was St George Ashe, who was the second Donegall Lecturer in Mathematics, succeeding Myles Symner, and later became Provost; during his Provostship the College celebrated its first centenary during which both the continuity of the traditional religious learning and a commitment to the new interest in science were emphasised. If one seeks to find some strands of continuity in Trinity mathematics perhaps geometry might spring to mind. Hugh Hamilton, Fellow and Professor of Natural Philosophy, who was also the great-great-great-grandfather of J L Synge, wrote a treatise on Conic Sections in 1758. Bartholomew Lloyd's Analytic Geometry, published 60 years later, was the book that was to awaken decisively Hamilton's interest in mathematics. MacCullagh was also a student of Lloyd, and he too was a fine geometer, passing this on to his student George Salmon whose five books on geometry an the theory of invariants went through many editions and were translated into as many as five languages. J L Synge inherited this tradition and incorporated his geometric outlook in his enormously influential books on Special and General Relativity. But perhaps the strongest thread linking these scholars is their shared spirit of enquiry and their enthusiastic pursuit of knowledge and understanding for its own sake. This includes Miles Symner (sometimes wrongly categorised as a mere Cromwellian soldier whose only role was to teach the rudiments of surveying), Molyneux, Hamilton, MacCullagh, the two Lloyds, Salmon and Synge, and still continues in the string theorists, with their geometries of the world, as well as those who use lattice models to obtain results in Quantum Field Theory and are led from this to develop high performance computing. It is that same spirit and outlook that motivated de Valera to establish the Institute for Advanced Studies and to bring Schrodinger to Dublin, and that led Ernest Walton to explore the atomic nucleus. Unfortunately, and despite the host of ancillary practical results delivered as by-products, this pursuit of fundamental understanding seems to receive scant encouragement today.

David Attis has brought together a great deal of useful information which he presents in an accessible and readable way. He has adopted a particular perspective which he pursues with determination. His book is interesting and thought-provoking. It raises many questions, waiting temptingly to be pursued. And if some of his readers are stimulated to broaden the panorama and to interpose their own individual alternative narratives then that surely is no bad thing.

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PROBLEMS

IAN SHORT

Problems

We begin with three integrals.

Problem 76.1.

$$\text{(a)} \int_0^\infty \sin(x^2)\,\mathrm{d}x \qquad \text{(b)} \int_0^1 \frac{x-1}{\log x}\,\mathrm{d}x \qquad \text{(c)} \int_{-1}^1 \frac{\cos x}{e^{1/x}+1}\,\mathrm{d}x$$

I learnt the next problem from a popular-mathematics lecture given by Vicky Neale of the University of Oxford.

Problem 76.2. For each point z on the unit circle, let ℓ_z denote the closed line segment from z to z^2. Consider the collection of those points in the closed unit disc that each lie at the intersection of two distinct line segments ℓ_z and ℓ_w. What shape is the complement in the unit disc of this collection of points?

We finish with a problem proposed by Wenchang Chu of Universitá del Salento, Italy.

Problem 76.3. Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}\tan\left(\frac{x}{2^n}\right)$.

Solutions

Here are solutions to the problems from *Bulletin* Number 74.

The first problem was solved by the North Kildare Mathematics Problem Club, and it is their solution that we present. The solution was also known to Grahame Erskine of the Open University, who suggested the problem.

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Problem 74.1. Given a positive integer A, let B be the number obtained by reversing the digits in the base n expansion of A. The integer A is called a *reverse divisor* in base n if it is a divisor of B that is not equal to B.

For example, using decimal expansions, if we reverse the digits of the integer 15, then we obtain 51. Since 15 is not a divisor of 51, the integer 15 is not a reverse divisor in base 10.

For which of the positive integers n between 2 and 16, inclusive, is there a two-digit reverse divisor in base n?

You may also wish to attempt the more difficult problem of classifying those positive integers n for which there is a two-digit reverse divisor in base n.

Solution 74.1. Given a positive integer n and two integers a and b with $1 \leqslant a, b < n$, let $[ab]_n = an + b$. The number $[ab]_n$ is defined to be a *two-digit reverse divisor* in base n if $[ba]_n$ is divisible by, but not equal to, $[ab]_n$.

By inspection, there are no two-digit reverse divisors in bases 1, 2 or 3. For $n > 3$, we claim that there is a two-digit reverse divisor in base n if and only if $n + 1$ is composite. To see this, suppose first that $n + 1$ is composite. Choose positive integers a and k with $n + 1 = (a + 1)(k + 1)$, and define $b = n - a - 1$. Then $bn + a = k(an + b)$, so we have a two-digit reverse divisor, unless $k = 1$. In the case $k = 1$, we define $a' = 1$ and $b' = n - 2$ to give $b'n + a' = a(a'n + b')$. This time we certainly have a two-digit reverse divisor, because $a = (n - 1)/2 > 1$.

Conversely, suppose that $n + 1$ is equal to a prime p, and suppose that $[ab]_n$ is a two-digit reverse divisor in base n. Let $k = (bn + a)/(an + b)$. One calculates that

$$p(b - ak) = (b - a)(k + 1).$$

Thus one of $b - a$ or $k + 1$ is divisible by p. But $1 \leqslant b - a < p$ and

$$1 < k + 1 < b + 2 \leqslant n + 1 = p,$$

so we have a contradiction. Hence, contrary to our earlier assumption, there does not exist a two-digit reverse divisor in base n. □

The second problem was solved by Henry Ricardo (New York Math Circle, New York, USA), the North Kildare Mathematics Problem Club, and the proposer, Ángel Plaza (Universidad de Las Palmas de Gran Canaria, Spain). All solutions were similar. The solution we present is that of Henry Ricardo. *Nesbitt's inequality* features in the solution, which says that for positive real numbers a, b, and c, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geqslant \frac{3}{2}.$$

This inequality can be proven by using two applications of the rearrangement inequality with the triples (a,b,c) and $(1/(a+b), 1/(b+c), 1/(c+a))$.

Problem 74.2. Let $f:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ be an increasing, convex function with $f(1)=1$, and let x, y, and z be positive real numbers. Prove that for any positive integer n,

$$\left(f\left(\frac{2x}{y+z}\right)\right)^n+\left(f\left(\frac{2y}{z+x}\right)\right)^n+\left(f\left(\frac{2z}{x+y}\right)\right)^n\geqslant 3.$$

Solution 74.2. By Nesbitt's inequality,

$$\frac{2x}{y+z}+\frac{2y}{z+x}+\frac{2z}{x+y}\geqslant 3.$$

Then, as the function $g(x)=f(x)^n$ is convex and increasing with $g(1)=1$, we can use Jensen's inequality to write

$$\begin{aligned}g\left(\frac{2x}{y+z}\right)+g\left(\frac{2y}{z+x}\right)+g\left(\frac{2z}{x+y}\right)&\geqslant 3g\left(\frac{\frac{2x}{y+z}+\frac{2y}{z+x}+\frac{2z}{x+y}}{3}\right)\\&\geqslant 3g(1)=3,\end{aligned}$$

as required. □

The third problem was solved by the North Kildare Mathematics Problem Club, Ángel Plaza, and the proposer Finbarr Holland (University College Cork). Finbarr has pointed out that in fact the result in the problem can quickly be derived using Bernstein polynomials. Bernstein polynomials are useful for approximating continuous functions; they provide one way of proving the Weierstrass approximation theorem. To solve the problem, simply calculate the nth Bernstein polynomial of the function $f(x)=x^j$ and use the value $x=1/2$. Nonetheless, we present a complete, more elementary, solution here, which does not require Bernstein polynomials. The solution is essentially the same as that of the North Kildare Mathematics Problem Club and Ángel Plaza.

In this problem, we use the standard notation

$$f(n)\sim g(n)\quad\text{as}\quad n\to\infty,$$

where f and g are positive functions, to mean that

$$\frac{f(n)}{g(n)} \to 1 \quad \text{as} \quad n \to \infty.$$

Problem 74.3. Prove that for $j = 0, 1, 2, \ldots,$

$$\sum_{k=0}^{n} k^j \binom{n}{k} \sim n^j 2^{n-j} \quad \text{as} \quad n \to \infty.$$

Solution 74.3. Let $f_0(x) = (1+x)^n$, and for $j = 1, 2, \ldots$ let

$$f_j(x) = x \frac{\mathrm{d}\, f_{j-1}(x)}{\mathrm{d}\, x}.$$

We shall prove by induction that for $j = 0, 1, 2, \ldots,$

$$f_j(x) = n^j x^j (1+x)^{n-j} + p_j(x, n),$$

where p_j is a polynomial in x and n, and its degree as a polynomial in n is less than j. This equation holds when $j = 0$ (p_0 is the zero polynomial, which we assume has degree -1 in each variable). Suppose that it holds when $j = m - 1$. Then one can check that

$$f_m(x) = n^m x^m (1+x)^{n-m} + q(x, n),$$

where

$$q(x, n) = (m-1) n^{m-1} x^{m-1} (1+x)^{n-m} + x \frac{\partial}{\partial x} p_{m-1}(x, n).$$

The degree of q as a polynomial in n is less than m, so the inductive proof is complete. We deduce that, for $x \geqslant 1$,

$$f_j(x) \sim n^j x^j (1+x)^{n-j}, \quad j = 0, 1, 2, \ldots .$$

Now,

$$f_0(x) = (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k,$$

so

$$\sum_{k=0}^{n} k^j \binom{n}{k} = f_j(1) \sim n^j 2^{n-j}, \quad j = 0, 1, 2, \ldots .$$

□

Finally, we return to a problem of Niall Ryan (University of Limerick) from issue 72. Niall had a solution to this problem, but it was quite involved. The solution we include here, provided by Finbarr Holland, deals with the 'odd case' only.

Problem 72.1. For each integer $n \geqslant 0$, let

$$S_n = \sum_{m=0}^{\infty} \frac{K_{2m}}{2m+n+1} + \sum_{\substack{m=0 \\ 2m \neq n}}^{\infty} \frac{K_{2m}}{2m-n},$$

where

$$K_{2m} = \left[\frac{(2m)!}{2^{2m}(m!)^2}\right]^2.$$

Prove that

$$S_n = \begin{cases} 0, & n \text{ odd}, \\ 2K_n \left(\log 2 - \displaystyle\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}\right), & n \text{ even}. \end{cases}$$

Solution 72.1. Let

$$a_n = \frac{\Gamma(\frac{1}{2}+n)}{\Gamma\left(\frac{1}{2}\right)\Gamma(n+1)} = \frac{(2n)!}{2^{2n}(n!)^2} = \sqrt{K_{2n}}.$$

We aim to prove that for $n = 0, 1, 2, \ldots,$

$$S_{2n+1} = \sum_{m=0}^{\infty} a_m^2 \left(\frac{1}{2m+2n+2} + \frac{1}{2m-2n-1}\right) = 0.$$

Clearly, it is enough to prove that $T_n = 0$ for $n = 0, 1, 2, \ldots,$ where

$$T_n = \sum_{j=0}^{n} (-1)^j \binom{n}{j} S_{2j+1} = 0.$$

To establish this, notice that

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{1}{2z+2j+2} = \frac{n!}{2\prod_{j=0}^{n}(z+j+1)} = \frac{n!\Gamma(z+1)}{2\Gamma(z+n+2)},$$

and

$$\begin{aligned} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{1}{2z-2j-1} &= \frac{(-1)^n n! 2^n}{\prod_{j=0}^{n}(2z-2j-1)} \\ &= \frac{(-1)^n n! \Gamma(z - \frac{2n+1}{2})}{2\Gamma(z+\frac{1}{2})}. \end{aligned}$$

Next we appeal to a well-known result about hypergeometric series, which can be found, for example, in Section 10.31 of *An introduction to the theory of functions of a complex variable* by E.T. Copson

(OUP, 1970). Recall that the hypergeometric series $F(a,b;c;z)$ is defined for $|z|<1$ by the equation

$$\sum_{m=0}^{\infty}\frac{\Gamma(a+m)\Gamma(b+m)}{\Gamma(c+m)}\frac{z^m}{m!}=\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}F(a,b;c;z),$$

and it is defined for $|z|\geqslant 1$ by analytic continuation. In fact, by Abel's continuity theorem, the series above can also be used to define $F(a,b;c;1)$. The result we need, from page 251 of Copson's text, is that

$$F(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

if $\mathrm{Re}[c-a-b]>0$. Using this observation, we see that

$$\begin{aligned}
T_n &= \frac{n!}{2}\sum_{m=0}^{\infty}a_m^2\left(\frac{\Gamma(m+1)}{\Gamma(n+2+m)}+(-1)^n\frac{\Gamma\left(-\frac{2n+1}{2}+m\right)}{\Gamma\left(\frac{1}{2}+m\right)}\right)\\
&= \frac{n!}{2\pi}\sum_{m=0}^{\infty}\left(\frac{\Gamma\left(\frac{1}{2}+m\right)\Gamma\left(\frac{1}{2}+m\right)}{m!\Gamma(n+2+m)}+(-1)^n\frac{\Gamma\left(\frac{1}{2}+m\right)\Gamma\left(-\frac{2n+1}{2}+m\right)}{m!\Gamma(1+m)}\right)\\
&= \frac{n!}{2\pi}\left(\frac{\Gamma\left(\frac{1}{2}\right)^2F\left(\frac{1}{2},\frac{1}{2};n+2;1\right)}{\Gamma(n+2)}+(-1)^n\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(-\frac{2n+1}{2}\right)F\left(\frac{1}{2},-\frac{2n+1}{2};1;1\right)}{\Gamma(1)}\right)\\
&= \frac{n!}{2\pi}\left(\frac{\pi\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)^2}+(-1)^n\frac{\Gamma\left(-\frac{2n+1}{2}\right)\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)}\right)\\
&= \frac{n!}{2\pi}\left(\frac{\pi\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)^2}+(-1)^n\frac{\pi\Gamma(n+1)}{\sin\left(-\left(n+\frac{1}{2}\right)\pi\right)\Gamma\left(n+\frac{3}{2}\right)^2}\right)\\
&= 0.
\end{aligned}$$

The well-known functional equation $\Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z)$ is used in the second-last line. □

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com` in any format (we prefer Latex). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

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