

**Cédric Villani: The Birth of a Theorem: A
Mathematical Adventure, The Bodley Head, London,
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REVIEWED BY DÓNAL HURLEY

The theorem referred to in the title of this book concerns *Landau damping* which is named after its discoverer, Lev Landau (1908 - 1968). Landau damping is the effect of damping (exponential decrease as a function of time) of longitudinal space-charge waves in plasmas. The starting point in the investigation of this is the Vlasov equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \left(\nabla W * \int f dv \right) \cdot \nabla_v f = 0,$$

which determines the statistical properties of plasmas. In 1946, Landau studied a linearised version of this model and concluded that the electrical forces weakened spontaneously over time without a corresponding increase in entropy. The problem which remained, and which Cédric Villani and Clément Mouhot decided to investigate in 2008, was whether or not Landau's results also applied to the nonlinear model.

Cédric Villani, born in 1973, is a French mathematician working primarily on partial differential equations, Riemannian geometry and mathematical physics. In 2010 he was awarded the Fields Medal for his work on Landau damping and the Boltzmann equation. He has also won many other awards including the European Mathematical Society Prize (2008) and the Henri Poincaré Prize (2009). Since 2013 he is a member of the French Academy of Sciences. In October 2014, he was the Hamilton Lecturer in Dublin where many of us had the pleasure of hearing him give a very stimulating and accessible talk. Clément Mouhot did his Ph.D under Villani's direction writing a thesis on the Boltzmann equation.

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In December 2006, while attending a conference at Oberwolfach, the legendary Institute for Mathematical Research in the Black Forest, Villani sat at a table with two mathematicians who discussed the Landau damping problem and heard them use words such as “weird” and “strange”. It meant nothing to him at the time but he did make a mental note and filed it away in a corner of his brain. At the beginning of 2008, Mouhot informed Villani what precisely was the difficulty with Landau’s work on Vlasov’s original model. Villani wondered if he and Mouhot could tackle it. He decided they should try, and so began their joint efforts over the next eighteen months.

The book consists of forty-four short chapters of dated reports on the progress of the two researchers. This helps the reader get a good sense of the progress of the proof; advances are slow on occasions and other times very rapid. Of course difficulties arise as errors are discovered. The proof required many detailed analytical arguments and these needed to be constantly checked/double checked. The work practice of the two collaborators required constant email exchanges. The text of many of these is included throughout the book. There are a number which are dated December 25, 2008, showing how absorbed both were in their quest. During the first six months of 2009, Villani was on leave in Princeton and, as well as the emails, there were several long phonecalls. The time difference between Paris and Princeton was exploited by them; Villani could work until midnight Princeton time and three hours later Mouhot was in his office in Paris to take up the task.

The two researchers also benefited from discussions with other researchers. Freddy Bouchet, on the occasion of a visit to the ENS de Lyon, suggested connections with work in modeling of galaxies. “When you model galaxies, you treat the stars as a fluid - as a gas of stars, in effect” he told Villani. On the occasion that Étienne Ghys, a colleague, noticed diagrams on the board in Villani’s office, he suggested a Kolmogorov-Arnold-Moser (KAM) theory connection. “KAM is found almost everywhere” he said. A year later, Villani did find a link between Landau damping and KAM theory although not exactly what Ghys suggested.

A heart-stopping moment is described when Villani received a manuscript, in his role as an Editor of a journal, titled “On the existence of exponentially decreasing solutions of the nonlinear Landau

damping problem". It looked like the authors had proved the result that he wished to prove by constructing solutions to the Vlasov equation that spontaneously relax toward the equilibrium. He wrote to the Editor-in-Chief saying that he faced a conflict of interest. However, on examining the manuscript more closely, he realised that the authors only proved that some damped solutions exist whereas what needed to be proved is that all solutions are damped.

At the beginning of 2009 while Villani was in Princeton, the outline of a proof began to emerge. Over the next few months, Villani gave several talks at Princeton and at other locations in the USA. During the question sessions at the end of his talks, two issues were raised as troubling to the experts in the audience; an analyticity assumption and the limit case in large time. At succeeding talks, he was able to improve the results and the expositions. When Villani returned to France at the end of June 2009, he and Mouhot felt the proof was in good shape. Their 180 page monster was ready to be submitted to *Acta Mathematica*.

On October 23, 2009, Villani received an email from *Acta Mathematica*. The six reports were very positive on the whole except, sure enough, they have misgivings on exactly the same two issues which were frequently raised at the end of his talks. The Editor was not persuaded that the results were definitive and also raised the issue of the length of the manuscript.

Another message on that day informed Villani that he'd just won the Fermat Prize which is awarded every two years to one or two mathematicians under the age of 45 who have made major contributions in one of those domains in which Pierre de Fermat worked (number theory, calculus of variations, probability theory). While this was a great consolation, it was not enough to get over the frustration of seeing their article turned down.

Four days later while in Ann Arbor, Michigan, to give a talk, he discussed his results with Jeff Rauch, a leading expert on partial differential equations. Rauch was also very concerned about the analyticity assumption. Villani found Rauch's reservations troubling so he decided to go through the article yet again, checking the soundness of the argument, step by step.

And there.... He had a moment of illumination when the light bulb went on in his head. Staring at the formulae, he realised it was not a question of fixing an error; it was a question of improving

the results. He and Mouhot took the whole thing apart and put it back together. They finally figured out how to lay to rest the two infernal objections, once and for all. They worked in feverish excitement revising, rechecking everything, improving everything. Their result was now stronger as they also solved a problem that had long intrigued specialists. On December 6, 2009, they resubmitted a new version of their “On Landau Damping” paper to *Acta Mathematica*. The covering letter of this submission is included in full in the book. Eventually, in November 2010, they got confirmation that their article was accepted. Their theorem had been born at last.

In between submission and acceptance of the paper, Villani was informed, February 16, 2010, that he had won the Fields Medal and he was presented with the medal at Hyderabad, India, by the President of India on August 19, 2010.

While most of the book is about the development of the theorem, there is much material of a general interest nature. He gives short biographical sketches of several well-known mathematicians, including Pierre de Fermat, John Nash, Grigori Perelman and Lev Landau. Nash was someone he greatly admired and he says that “next time our paths cross I shall dare to approach John Nash ... and ... will even tell him that he in my hero”. I hope he did have this opportunity as Nash and his wife were tragically killed when they were returning to Princeton from Oslo where, just 5 days earlier, he had been presented with the Abel Prize on May 18, 2015.

The book also includes much material illustrative of Villani’s personality: he obviously has a wonderful way with children and likes to tell his own offspring improvised tales which go on and on, the kind his daughter calls “imaginary stories” - her favourite kind. He is an incessant tea drinker and has an amusing description of the night he stole some tea bags from the Common Room of the School of Mathematics at Princeton when there were none in his own house. He enjoys very much listening to music and has very catholic tastes; he gives long lists of his favourite pieces.

This is indeed a charming book to read; for mathematicians, it reflects their experiences at doing research and for general readers there is much of general interest as well as illustrating the lifestyle of mathematicians. All readers will be absorbed in the roller-coaster excitement of the development of the theorem on Landau damping.

Dónal Hurley is retired from the School of Mathematical Sciences at University College Cork where he worked since 1973. His research interests are in the areas of dynamics of geodesic flows and mathematical physics.

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