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THE MAXIMUM FOR $\Delta\mathbf{u} + \mathbf{f}(\mathbf{u}) = \mathbf{0}$ ON AN ISOSCELES TRIANGLE

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Abstract. We use the moving planes method to prove that if u is a positive solution to the equation $\Delta u + f(u) = 0$, on an isosceles triangle T in R^2, with $u = 0$ on ∂T and f Lipschitz continuous and restoring, then u has a unique maximum value on the axis of symmetry of T. We conjecture that the location of the maximum is independent of f, and extend the result to a set in R^3.

1. Introduction

In recent years the use of moving planes and maximum principles for thin domains, have produced a number of interesting results concerning the solutions of elliptic partial differential equations on certain special classes of bounded domains in R^n . Most of these ideas can be found in the works of Berestycki and Nirenberg [1], Du [2], Fraenkel [3], and Gidas, Ni, and Nirenberg [4].

A recent paper by Cima and Derrick [5] explored the use of these methods on an isosceles triangle in the plane R^2 . In [5], we proved that if a bounded convex domain Ω in R^2 has two (or more) axes of symmetry, then the solution of the equation $\Delta u + f(u) = 0$, with $u|_{\partial\Omega} = 0$, where f is *restoring* ($f(u) > 0$ when $u > 0$), must have its maximum value where the axes of symmetry meet. Thus, for example, if Ω is an equilateral triangle, the maximum value of u will occur at the intersection of the angle bisectors, while for a circle it will be at the center. For an isosceles triangle, we were only able to prove that the maximum is achieved on a subset of its line of symmetry, since the use of moving planes shows that $u(x, y) = u(x, -y)$, implying that $u_y(x, 0) = 0$, $u_x(x, y) = u_x(x, -y)$, and $u_y(x, -y) = u_y(x, y)$. We were unable to show that the maximum was unique, nor were we

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able to find precise locations for maxima. However, we did provide numerical evidence that suggested two things: (1) the maximum was unique and (2) the maximum occurred at the same place regardless of the function $f(u)$. This suggests that it is possible that for convex domains Ω there may be a unique maximum whose location depends on Ω and not on $f(u)$.

In this paper we will prove that the maximum is unique for an isosceles triangle. We are still unable to show that the location of the maximum is independent of $f(u)$. In Section 2 we will give a short description of the moving plane method, and present some properties of analytic solutions of the problem described in the paragraph above. In particular we will show that the number of maximum points is finite. In Section 3 we will use eigenfunction theory to show that certain functions $f(u)$ have a single maximum on an isosceles triangle. This proof is based on work done by Payne [7] and Sperb [8]. In [6] Chanillo and Cabre have proved for smoothly bounded, strictly convex domains, that a unique critical point (maximum or minimum) exists for such problems. We produce an analogous result for isosceles triangles using continity and compactness tools. In Section 5 we extend these ideas to a triangular shaped region in R^3 .

2. Preliminaries

The partial differential equation that we shall consider in this paper is the elliptic equation of the form

$$\Delta u(x,y) + f(u(x,y)) = 0 \quad \text{on,} \quad \text{with} \quad u(x,y) = 0 \quad \text{on} \quad \partial\Omega, \quad (1)$$

where $u > 0$ in Ω, and f is Lipschitz continuous and restoring ($f(u) > 0$ when $u > 0$).

Definition. A bounded simply connected domain Ω in R^2 is *Symmetric-Convex* (S-C) when a pair of orthogonal straight lines m and n exist such that

(1) Ω is symmetric with respect to line m, and

(2) every straight line parallel to n (including n) that intersects Ω, must intersect Ω on a single open line segment. This property defines *n-convexity.*

For a given domain there may be many such pairs of orthogonal lines.

For the special case of the isosceles triangle, we will denote the domain Ω by T, symmetric in the x-axis and of height c, with base $|y| \leq b$ on y-axis; that is $T = \{(x,y) \in R^2 \mid \ |y| < b(1-(x/c)),\ 0 < x < c\}$. Observe that T is S-C. We shall denote by W_a the symmetric trapezoid with vertices at $(0, \pm b)$ and $(c, \pm a)$. Note that $W_0 = T$, and that most of the results we obtain for T are equally valid for W_a .

Note that a domain Ω is *convex* if for any two points a and b in Ω the line segment Λ joining a and b lies entirely in Ω.

We sketch the way the "moving planes" method is used in showing (as we did in [5]) that there is an interval $\Lambda = (x', x^*)$, along the x-axis, with $0 < x' < x^* \leq (1/2)\cdot c$ in which the maximum of u on T is attained. For $0 < a < b$, the horizontal line $L_{-a} = \{(x,y) : y = -a\}$ meets T and cuts a small triangular "cap" (an open subset of T), say $\Sigma(-a) = \{(x,y) \in T : y < -a\}$ from T. Reflecting the domain $\Sigma(-a)$ about L_{-a} we obtain the reflection $\Sigma(-a)^{\perp} \subset T$, each point $P \in \Sigma(-a)$ having a reflected point $P^{\perp} \in \Sigma(-a)^{\perp}$. Define

$$w(P; -a) \equiv u(P) - u(P^{\perp}), \tag{2}$$

for $P \in \bar{\Sigma}(-a)$. Since f is Lipschitz, w satisfies $\Delta w + (P; -a)w = 0$ on $\Sigma(-a)$ with

$$\gamma(P; -a) = \begin{cases} \dfrac{f(u(P)) - f(u(P^{\perp}))}{u(P) - u(P^{\perp})}, & P \in \Sigma(-a), \\ \\ 0, & P \in L_{-a}. \end{cases} \tag{3}$$

By the Maximum principle, it follows that $w \leq 0$ in $\bar{\Sigma}(-a)$, which can then be extended to $w < 0$ on $\Sigma(-a)$. Thus, $u(P) < u(P^{\perp})$ for all P in $\Sigma(-a)$, and hence that $u_y(P) > 0$ in $\Sigma(-a)$. Letting $-a \to 0$, it follows that $u_y > 0$ for $y < 0$ in T. A similar analysis shows that $u_y < 0$ for $y > 0$ in T. Thus $u_y(x, 0) = 0$.

The vertical line $L_k = \{(x,y) : x = k > (1/2)\cdot c\}$ cuts a triangular cap from T. Reflecting this cap about L_k we obtain in a similar manner as above, that $u(P) < u(P^{\perp})$ for P in the cap, and that $u_x < 0$ in this cap. Letting k decrease to $c/2$, it follows that $u_x < 0$ for $x > c/2$. Finally, bisecting the angle at $(0, b)$ in T, we show that points P below the line of bisection satisfy $u(P) < u(P^{\perp})$, where $P^{\perp}$ is the reflection of P in, and similarly for the angle of bisection at $(0, -b)$. Let x' be the point where these two lines of angle bisection

meet, then $u_x(x,0) > 0$ for all $0 < x \leq x'$. Hence, the maxima lie in the interval $\Lambda = (x', c/2)$.

Lemma 2.1. *If the solution u of problem (1) is real analytic, there are at most a finite number of maximum points of u on Λ.*

Proof. Assume there are infinitely many points in the open interval Λ , where u attains it's maximum M. Since $\bar{T}$ is compact, there must be an accumulation point in Λ. Let ξ be the value of the largest such accumulation point. Then there is an increasing sequence $x_j \to \xi \leq c/2$ for which $u(x_j, 0) = M$ for $j = 1, 2, 3, \ldots$. It follows by our assumption, and by considering the series expansion at ξ, that $u(x, 0) \equiv M$ in a neighborhood of ξ which is a contradiction. □

Remark. Notice that $u_y(x, 0) = 0$ on T for $0 < x < c$, so that for all partials of u, where one of the partials is in the y direction, such as $u_{x\ldots xy}(x, 0) = 0$ in $0 < x < c$. Thus, $u_{yx}(x, 0) = 0$ and $u_{xy}(x, 0) = 0$, so that the Hessian at any maximum point P, has the form

$$H(P) = \begin{bmatrix} u_{xx}(P) & 0 \\ 0 & u_{yy}(P) \end{bmatrix},$$

and has rank at least 1, because $\Delta u(P) = -f(u(P)) < 0$, with $u(P) > 0$ and f is restoring.

Lemma 2.2. *If the solution u to problem (1) is analytic, then the diagonal entries of the Hessian $H(P)$ at any maximum point P are nonpositive.*

Proof. Let x_* be the smallest positive value such that the point $P = (x_*, 0)$ satisfies $u(P) = M$, where M is the maximum of the solution u of problem (1) on T. From the remark above, at least one of terms $u_{xx}(P)$ and $u_{yy}(P)$ is negative. To reach a contradiction assume the lemma fails. That is, suppose to the contrary that $u_{xx}(P) > 0$, making $u_{yy}(P) < 0$. Let $(x, 0)$ be a point close to P. Expanding $u(x, 0)$ as a Taylor series centered at P, we get

$$u(x, 0) = M + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(P)(x - x_*)^2 + \ldots,$$

and for sufficiently small $|x - x_*|$, the second term dominates the rest of the Taylor series terms, so that $u(x, 0) > M$, a contradiction. Thus $u_{xx}(P)$ is nonpositive. If $u_{yy}(P) > 0$, then expand $u(x_*, y)$ as

a Taylor series centered at P:

$$u(x_*, y) = M + \frac{1}{2}\frac{\partial^2 u}{\partial y^2}(P)y^2 + ..., \tag{4}$$

and the second term will dominate the rest of the series again leading to a contradiction. Hence the diagonal terms of $H(P)$ are nonpositive, and at least one of them is negative. □

Remark. Observe that if $u_{yy}(P) = 0$ then no odd powered y-derivative of u can be the first nonzero such term in the Taylor series (4). The first such term will dominate further terms in a sufficiently small neighborhood of P. Hence, if $u_{yyy}(P) > 0$, the expansion of u,

$$u(x_*, y) = M + \frac{1}{3!}\frac{\partial^3 u}{\partial y^3}(P)y^3 + ...,$$

with $y > 0$, will contradict the maximality of M. Similarly, if $u_{yyy}(P) < 0$, $y < 0$ will yield a contradiction. Let $2k$ be the first nonzero even powered y-derivative. Then, by the proof in Lemma 2.2,

$$\frac{\partial^{2k} u}{\partial y^{2k}}(P) < 0.$$

Lemma 2.3. *Let P be a point on the x-axis in T, where the positive solution u of problem (1) has a (relative) maximum value. The level curves $u > K$, for $M - K > 0$ and small, are convex.*

Proof. Let $P = (x_*, 0)$ and consider the Taylor expansion of $u(x, y)$ in a neighborhood of $P = u(x_*, 0) = M$, $u_x(x_*, 0) = u_y(x_*, 0) = 0$. If the Hessian H has rank 2, then by Lemma 2.2, $u_{xx}(x_*, 0) = -\alpha$, $u_{yy}(x_*, 0) = -\beta$, with α, $\beta > 0$, so that the Taylor series has the form

$$u(x, y) = M - \alpha(x - x_*)^2 - \beta y^2 + O(|\epsilon|^3),$$

where $\epsilon^2 = (x - x_*)^2 + y^2$. If the Hessian has rank 1, the first nonzero terms in x and y will dominate the remaining terms in a small neighborhood, so the Taylor series is either

$$u(x, y) = M - \alpha(x - x_*)^2 - \beta y^{2j} + O(|\epsilon|^3),$$

or

$$u(x, y) = M - \alpha(x - x_*)^{2j} - \beta y^2 + O(|\epsilon|^3), \quad j \geq 2.$$

Letting $u(x,y) = k$, with $M - k > 0$ and small, then the level curve of height k through (x,y) has one of the following three forms:

$$\alpha(x - x_*)^2 + \beta y^2 + O(|\epsilon|^3) = M - K,$$
$$\alpha(x - x_*)^2 + \beta y^{2j} + O(|\epsilon|^3) = M - K,$$
$$\alpha(x - x_*)^{2j} - \beta y^2 + O(|\epsilon|^3) = M - K, \quad j \geq 2,$$

all convex.

A global result about convexity also holds: Since $u_y(x,-y) > 0 > u_y(x,y)$ for $y > 0$ in T, it follows trivially that the level curves of u are convex in the y-direction. □

Remark. Observe that the isosceles triangle T is contained in the rectangle $\mathbf{R} = \{(x,y)|\, 0 < x < c,\, |y| < b\}$, and contains the rectangle $\mathbf{R}_1 = \{(x,y)|\, 0 < x < c/2,\, |y| < b/2\}$. The positive eigenfunction for the problem $\Delta u + \lambda u = 0$, on $\mathbf{R}$, with $u = 0$ on $\partial\mathbf{R}$, is

$$u(x,y) = \sin\frac{\pi x}{c}\cos\frac{\pi y}{2b}, \quad \text{so that} \quad \lambda = \pi^2\left[\frac{1}{c^2} + \frac{1}{4b^2}\right].$$

Thus, by the Courant Nodal Theorem [9], since subdomains have larger eigenvalues, it follows that the first eigenvalue λ_1 of T satisfies

$$\lambda_{\mathbf{R}} = \pi^2\left[\frac{1}{c^2} + \frac{1}{4b^2}\right] < \lambda_1 < 4\pi^2\left[\frac{1}{c^2} + \frac{1}{4b^2}\right] = \lambda_{\mathbf{R}_1}.$$

Similar results hold for the problem $\Delta u + \lambda g(x,y)u = 0$ on Ω, with $g > 0$ on Ω, and $u = 0$ on $\partial\Omega$; subdomains have larger eigenvalues. We use this result in the next section to provide a proof for the existence of a unique maximum in T , for a certain class of functions $f(u)$.

3. Uniqueness of the Maximum

Assume we have a positive solution $u(x,y)$ to problem (1) on T. Consider the linear eigenvalue problem

$$\Delta v + \lambda f'(u)v = 0 \quad \text{on} \quad T, \qquad \text{with} \quad v = 0 \quad \text{on} \quad \partial T. \tag{5}$$

Theorem 3.1. *If f satisfies the following conditions:*
(a) $f > 0$,
(b) $f' > 0$,
(c) the first eigenvalue λ_1 of equation (5) satisfies $\lambda_1 > 1$,
then u has one critical (maximum) point on T.

Proof. Since our domain T is symmetric in y, the level lines $u_x = 0$ are symmetric, with $(0, b)$ and $(0, -b)$ as limit points, and must cross the x-axis at the critical points of u. We showed in [5] that along the x-axis in T, $u_x > 0$ for small x, and $u_x < 0$ for $x > c/2$, so the level curve $u_x = 0$ must cross the x-axis at some critical point of u. Suppose u has more than one critical point; let P and Q be two such points on the x-axis, so we have one of the following two situations:

(i) Either there is a closed loop ∂L passing through P and Q on which $u_x = 0$, properly bounding a subdomain L inside T, or

(ii) There are two nodal curves beginning at $(b, 0)$ and ending at $(-b, 0)$ (or viceversa), one passing through P, the other through Q, bounding a subdomain J in T, so that $u_x = 0$ on ∂J.

Taking the partial derivative of equation (1) with respect to x yields

$$\Delta u_x + \lambda f'(u) u_x = 0 \quad \text{on} \quad \partial T. \tag{6}$$

on T. In either case $u_x = 0$ on ∂L or ∂J, both curves bounding subdomains of T, implying by the Courant Nodal Theorem [9], $1 \geq \lambda_1$. But this contradicts (c). Hence, at most one maximum exists. □

Example. Let $f(u) = ku/(1 + u)$, so that $f'(u) = k/(1 + u)^2$, where $k = \lambda_{\mathbf{R}} < \lambda_1$. Then $f'(u) < k < \lambda_1$. Suppose a solution u exists to the problem $\Delta u + f(u) = 0$ on T , $u = 0$ on ∂T. Differentiating this equation with respect to x, we get $\Delta u_x + f'(u) u_x = 0$ on T. If P and Q are critical points of u, then by the proof above, at all points in the regions L or J, we get the eigenvalue $f'(u) < k < \lambda_1$, which is impossible since L and J are subdomains of T and their eigenvalues exceed λ_1 . Thus, a unique maximum exists.

For a smooth S-C domain Φ bisected symmetrically by the x-axis, define the derivative of the solution u in the direction θ, of the problem

$$\Delta u + f(u) = 0 \quad \text{on} \quad u = 0 \quad \text{on} \quad \partial\Phi$$

by

$$u_\theta(x, y) = u + u_x(x, y) \cos\theta + u_y(x, y) \sin\theta.$$

Cabre and Chanillo [6] show that there are two points on $\partial\Phi$ which are the boundary points of a nodal arc C_θ interior to Φ on which $u_\theta = 0$. Further, the nodal arc C_θ separates the domain Φ into two subdomains, one where $u_\theta < 0$ and one where $u_\theta > 0$. They prove

that all the nodal arcs C_θ meet at a single point $P = (x^*, 0)$ at which the solution u has a maximum $M = \max_\Phi u(x, y)$.

4. Construction of S-C domains $\mathbf{T_k}$

We now construct a set of nested smooth S-C domains T_k , each containing the isosceles triangle T in its interior. The construction will describe the boundary of each set T_k , and consist of an arc on each of three decreasingly small circles and an arc on each of three increasingly large circles.

Let T be the isosceles triangle with vertices at $(0, b)$, $(0, -b)$, $(c, 0)$, with $b, c > 0$. The boundary of T_k consists of an arc on the three (very) small circles

$$x^2+(y-b)^2 = 10^{-k}, \quad x^2+(y+b)^2 = 10^{-k}, \quad \text{and} \quad (x-c)^2+y^2 = 10^{-k}$$

determined by where they are intersected by the arcs of the (very) large circles of radii $r = 10^k(b + c) + 10^{-l}$, with centers lying on

(i) the positive x-axis at distance $10^k(b+c)$ from the points $(0, \pm b)$,

(ii) the line $(y - (b/2)) = (c/b)(x - (c/2))$ in the third quadrant at distance $10^k(b + c)$ from the points $(0, b)$ and $(c, 0)$, and

(iii) the line $(y+(b/2)) = -(c/b)(x+(c/2))$ in the second quadrant at distance $10^k(b + c)$ from the points $(0, -b)$ and $(c, 0)$.

Note that the large arcs meet the small circles tangentially. The point of tangency is where we switch from the large circle arcs to arcs on the small circles. Hence, the boundary of T_k is C^1 .

In each of the S-C domains T_k , let u^k be a solution to problem (1). Note that we can describe u^k by a Green's integral over T_k . Two nodal lines are of importance in each T_k: the nodal line u which is the part of the x-axis in T_k and meets the arc in (i) on the negative axis close to zero and on the small circle $(x - c)^2 + y^2 = 10^{-k}$, and the nodal line $u^k_{\pi/2}$ with boundary points on the circles $x^2 + (y \pm b)^2 = 10^{-k}$. These nodal lines correspond to the arcs in T_k on which $u^k_y = 0$ and $u^k_x = 0$, respectively. They meet at the point $(x^k_{\max}, 0)$ where u^k has its unique maximum M_k .

Now let $k \to \infty$. By Schauder's Theorem, the Green's integrals converge to a Green's integral that is the solution u to problem (1) on T. The nodal arcs $u^k_\theta = 0$ converge to nodal arcs u_θ on T, and even on $\bar{T}$, except for the three arcs u_0 and u_ψ , where $\tan\psi = \pm(b/c)$, which resemble a letter T.

Suppose that u does not have a unique maximum, but has two points $(x^*, 0)$ and $(x', 0)$ where $u = M$. Then there are distinct subsequences $\{k_i\}$ and $\{k_j\}$ both converging to ∞, so that the arcs

$$u_x^{k_i} = 0 \quad \text{and} \quad u_x^{k_j} = 0$$

meet the x-axis at x^* and x' , respectively. The limit points of these sequences form two arcs in T, one containing x^* and the other x' . The points in T between these arcs have u_x values that are both positive and negative, so in the limit all of these points have u_x value equal to 0. Hence the set where $u_x = 0$ contains an open set, which is impossible. Hence u has a unique maximum value at the limit point of the set $\{(x^k_{\max}, 0)\}$.

5. An extension to $\mathbf{R^3}$

Consider the 3-dimensional bounded solid domain F whose boundary ∂F consists of the four triangles T_1 , T_2 , T_3 , T_4 in R^3 passing through the following trios of vertices:

$$\begin{aligned} T_1 &= \{(1,0,0), (-1,0,0), (0,1,0)\}, \\ T_2 &= \{(1,0,0), (-1,0,0), (0,0,1)\}, \\ T_3 &= \{(1,0,0), (0,1,0), (0,0,1)\}, \\ T_4 &= \{(-1,0,0), (0,1,0), (0,0,1)\}. \end{aligned}$$

Assume a positive solution u to the problem

$$\Delta u + f(u) = 0 \quad \text{on} \quad F, \quad \text{with} \quad u = 0 \quad \text{on} \quad \partial F \tag{7}$$

exists. Using symmetry in x and the moving plane method, it is clear that $u_x > 0$ at points where $x < 0$, and $u_x < 0$ when $x > 0$. Thus, $u_x = 0$ on the isosceles triangle T_0 in F lying on the yz-plane. F is also symmetric about the plane $y = z$. Thus, the maximum value of u on F lies on the 45^o line in T_0 .

6. Conclusion

All the results we have proved in this paper extend to symmetric trapezoidal shaped domains with minor modifications. Our numerical calculations seem to indicate that the location of the maximum value does not depend on the function f, but we have been unable to prove this.

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