Irish Math. Soc. Bulletin
Number 76, Winter 2015, 37–43
ISSN 0791-5578

3*D* PRINTING A ROOT SYSTEM.

PATRICK J. BROWNE

Abstract. In this short note we describe how a 3d printer was used to make a model of a root system.

1. Introduction

A 3D printer is a device used to make three dimensional objects. In 3D printing, additive processes are used, in which successive layers of material are laid down under computer control, as opposed to established techniques such as injection moulding. These objects can take on a wide variety of shapes not possible by traditional techniques, and are produced from a computer model. This allows for great accuracy and is very suited to producing objects with a mathematical origin. In recent years there has been interest and excitement about these machines, since they are now more affordable and easier than ever to use. In this short note, we describe how use was made of one of these printers to print a 3 dimensional mathematical object, in this case a root system. The choice of object was chosen since it is well known to the author. Pictures of root systems are clear, easy to understand, and can be used to great effect in the class room environment to engage a student in a rich new topic that may seem rather abstract at first glance. The author feels the contents of this note could make an excellent addition to an undergraduate student project. Other interesting shapes with mathematical origins that a potenial supervisor or student may wish to explore can be found in [4]. We will describe what a root system is, how to construct a virtual model of it on a computer and then how to export this model and print it. To assuage any curiosity the printed object can be seen in Figure 1.

2010 *Mathematics Subject Classification.* 17B22,97U60.
Key words and phrases. 3d printer, root systems.
Received on 18-6-2015; revised 15-12-2015.
The author wishes to acknowledge, Dr G. Brychkova in the Discipline of Botany and Plant Science at NUIG, for use of their 3d printer.

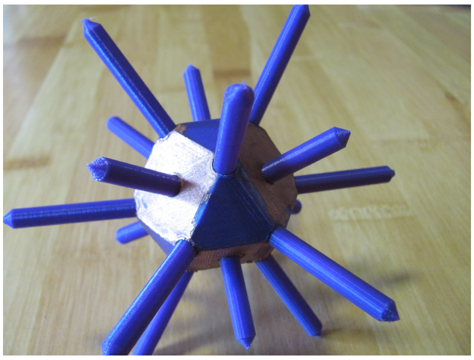

FIGURE 1. The finished 3d print.

2. Root systems

For the purposes of this note we will adopt the axiomatic approach to root systems. All basic facts and definitions can be found in Chapter 3 of [1] and Chapter 8 of [3]. Let $\mathbf{E}$ be a finite dimensional Euclidean vector space over $\mathbb{R}$, endowed with a positive definite symmetric bilinear form $(\cdot,\cdot)$. We define a reflection in $\mathbf{E}$ as an invertible linear transformation leaving pointwise fixed some hyperplane and sending any vector orthogonal to that hyperplane into its negative. Any non zero vector α in $\mathbf{E}$ defines reflection σ_α with reflecting hyperplane $P_\alpha = \{\beta \in \mathbf{E} | (\beta,\alpha) = 0\}$. An explicit formula for reflecting is given by,

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha.$$

The term $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ is often written as $\langle \beta,\alpha \rangle$. A subset Φ of $\mathbf{E}$ is called a *crystallographic root system* in $\mathbf{E}$ if the following axioms are satisfied:

(1) Φ is finite and spans E and does not contain 0.
(2) If $\alpha \in \Phi$ the only multiples of α in Φ are $\pm\alpha$.
(3) If $\alpha \in \Phi$ the reflection σ_α leaves Φ invariant.
(4) If $\alpha, \beta \in \Phi$ then $\langle \alpha,\beta \rangle \in \mathbb{Z}$.

Henceforth we will just refer to this as a *root system.* Axiom (4) limits the possible angles occurring between pairs of roots. To see this, recall that the cosine of the angle, θ, between two roots α and β is given by $\|\alpha\|\|\beta\| cos\theta = (\alpha,\beta)$. Therefore $\langle \beta,\alpha \rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} = 2\frac{\|\beta\|}{\|\alpha\|}cos\theta$. So that $\langle \beta,\alpha \rangle\langle \alpha,\beta \rangle = 4cos^2\theta$. This is a non negative integer, $0 \leq cos^2\theta \leq 1$, and $\langle \alpha,\beta \rangle, \langle \beta,\alpha \rangle$ have the same sign. If we also assume that $\alpha \neq \pm\beta$ and $\|\beta\| \geq \|\alpha\|$, enumeration of the possible values of $\langle \alpha,\beta \rangle\langle \beta,\alpha \rangle$, yields $\theta \in \{\frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{6}, \frac{5\pi}{6}\}$.

FIGURE 2. The A_1 root system.

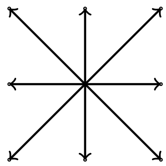

FIGURE 3. The B_2 root system.

We refer to the dimension of $\mathbf{E}$ as the *rank* of the root system. For example in rank 1, Axiom (2) means there is only one choice of root system, we represent this with Figure 2. The informed reader will of course recognise this as the root system in Lie theory that belongs to $\mathfrak{sl}(2, F)$. In rank 2 there are more choices, one of which is the root system B_2, shown in Figure 3. This root system consists of 2 spanning vectors and two distinct root lengths, where the ratio of these lengths is $\sqrt{2}$. The roots correspond to the vertices and midpoints of the edges of a square, the ones of greater length are denoted *long roots* and the others *short roots.*

The above drawings of root systems are clear and easy to understand. Unfortunately representations of rank 3 root systems by pictures can be crowded and difficult to see on the page. With some skill they can be drawn on the page in a pleasing manner, see Figures 8 and 9 in [5]. As an aside we also mention the excellent computer aided drawings seen after page 162 in [3]. Unfortunately these drawings are subject to a fixed point of view. For this reason we choose to construct a computer model of a rank 3 root system and then print it, allowing one to view the roots from any perspective. There are only three choices of irreducible root systems in rank 3, namely A_3, B_3 and C_3. The root systems of B and C are dual to each other and the root system of A has only one root length. For this reason we choose to model the B_3 root system since it captures most of the interesting details.

To construct the B_3 root system we follow the construction outlined in section 2.10 of [2] (alternatively section 12 of [1]). Take $\mathbb{R}^n$ with standard bases vectors $\{\epsilon_i\}$, define Φ to be the set of all vectors of squared length 1 or 2 in the standard lattice of $\mathbb{R}^n$. So Φ consists of the $2n$ short roots $\pm\epsilon_i$ and the $2n(n-1)$ long roots $\pm\epsilon_i \pm \epsilon_j$ where $i < j$.

For example in rank 2 there are 4 short roots $\{(1,0), (-1,0), (0,1), (0,-1)\}$, and 4 long roots $\{(1,1), (1,-1), (-1,1), (-1,-1)\}$, as shown in Figure 3. In rank 3 we have the a total of 18 roots summarised in Table 1. Examination of the vectors in Table 1 will show

Short roots	Long roots
$(\pm 1, 0, 0)$	$(\pm 1, \pm 1, 0)$
$(0, \pm 1, 0)$	$(0, \pm 1, \pm 1)$
$(0, 0, \pm 1)$	$(\pm 1, 0, \pm 1)$

TABLE 1. The 12 long and 6 short root vectors for B_3.

that the long roots correspond to the vertices of a regular cube-octahedron, that is the intersection of a cube and an octahedron, with the short roots being the centres of the square sides. One can best picture the cube-octahedron as a cube with its corners cut off. For a visual depiction of this please see Figure 9 in [5] and Figure 4.

3. Computer modeling

We now have our root system that we wish to print. To do this we need to construct it as a computer model and ready it for export. There is a wide choice of software available. This ranges from the very capable *Blender*[1] to the point and click web interfaces such as *Tinkercad*[2]. The former has the advantage of allowing precise manipulation, but the disadvantage of a very steep learning curve. While the latter is easy to pick up, it lacks the precision needed for our task.

The software we will use is OpenSCAD [6], a versatile package that allows precise control and is easy to pick up for anyone that is familiar with functions or scripting languages, ideal for an undergraduate student. To illustrate this, we present a sample of some code:

```
union()\\ form the union of two shapes
{cylinder(r = a,h=b,$fs=c);
\\ cylinder with radius
\\and fs is a measure of its smoothness
translate([x,y,z]){
\\[x,y,z] being the translation vector
```

[1]Blender is the free and open source 3D creation suite, https://www.blender.org/

[2]Tinkercad is a free, easy-to-learn online app anyone can use to create and print 3D models, https://www.tinkercad.com/

```
rotate(a=180,v=[d,e,f]){
\\[d,e,f] being the axis of rotation along with
\\angle a
cylinder(h=g,r1=a,r2=0,center=false);}
```

The root system of B_3 is easiest viewed as vectors extending from a cube-octahedron as seen in Figure 4. While we could construct the cube and octahedron ourselves, we opt to use the community resource of the *Thingiverse solids package*[7]. This is simply done as follows:

```
use <maths_geodesic.scad> // package to create
// platonic solids
include <test_platonic.scad>//
module cubeocto(){// a cubeoctahedron where the
// roots sit
intersection(){ // the intersection of two shapes
display_polyhedron(octahedron(90));
color("red")cube(90,center=true);}}
```

Once this package is loaded into our file we place the roots at the various points. Each root will simply be a cylinder joined to a cone. Since each root is also accompanied by its negative, we only need draw 9 roots and then use rotations to place their negatives. After some small effort the model in Figure 1 was constructed. At this point all that is left to do is to export the file to *.stl* format to ready it for printing.

4. Printing of the model

Once in possession of the .stl file, we check that it is physically possible to print. Any 3d printer will have software to do this, but there are also resources online such as *Willit 3D Print.*[3] When making a computer model it is best to try to avoid parts that are extended in free space, as the printer will need to print a removable support. Removing this support can sometimes damage delicate parts of your model. If possible try to design your models to avoid this. For this reason the author printed the cube octahedron with

[3]Willit 3D Print is the website using javascript and webgl, where you can analyse your 3D design (STL or AMF files) before you 3D print it, http://www.willit3dprint.com/

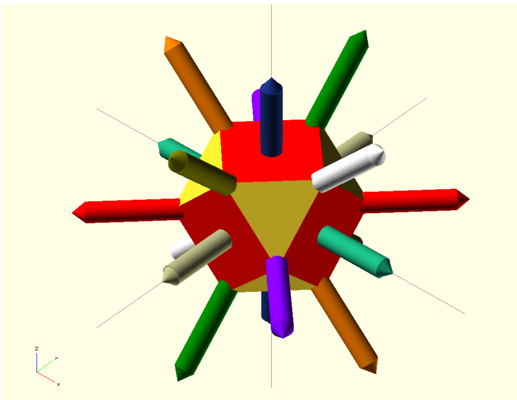

FIGURE 4. Computer model of the B_3 root system, with cube-octahedron.

holes for roots and the roots separately, see Figure 5. This was accomplished by taking the difference of the roots and the cube-octahedron:

```
 difference () {
cubeocto();\\ The order of the arguments
long_a3roots();\\determines how the difference
\\ is taken
short_b3roots();}
```

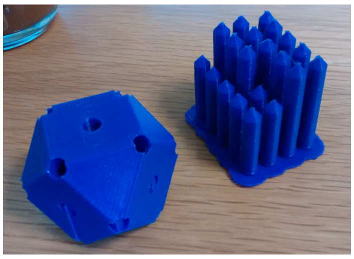

FIGURE 5. The printed model parts, with two spare roots.

This was purely to avoid having to remove the supports which may have led to damage. The printed model was approximately 12cm from one end to the other. The 3d printer that was used was the *maker bot replicator 2*. We were generously allowed use of the printer by the Discipline of Botany and Plant Science at NUIG and the author wishes to thank Dr G. Brychkova for her help.

REFERENCES

[1] J. E. Humphreys: *Introduction to Lie algebras and Representation Theory*, Graduate Texts in Mathematics, Springer, 1972.

[2] J. E. Humphreys: *Reflection Groups and Coxeter Groups*, Cambridge studies in advanced mathematics, Cambridge university press, 1990.
[3] B.C. Hall: *Lie Groups,Lie Algebras, and Representations* Graduate Texts in Mathematics, Springer, 2004.
[4] H & C. Ferguson: *Celebrating Mathematics in Stone and Bronze.* Notices of the AMS, page 840 Vol.57 Number 7, 2010.
[5] M. Ozols: *The classification of root systems*, Essay on the classification of root systems .
http://home.lu.lv/~sd20008/papers/essays/Root%20Systems%20%5Bpaper%5D.pdf(accessed 21-5-2015)
[6] OpenSCAD: *OpenSCAD is a software for creating solid 3D CAD objects.*, http://www.openscad.org/ (accessed 21-5-2015)
[7] Thingiverse: *MakerBot's Thingiverse is a thriving design community for discovering, making, and sharing 3D printable things.*, http://www.thingiverse.com/thing:10725/#files(accessed 28-5-2015)

Patrick Browne is currently employed at NUI Galway as a fixed term lecturer. His current research interests include, symmetric spaces and Lie theory.

(Patrick Browne) School of Mathematics, Statistics and Applied Mathematics. N.U.I. Galway.
E-mail address: patrick.browne@nuigalway.ie