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EDITORIAL

We have agreed a new exchange, with the Jaen Journal of Approximation. Please send expressions of interest in housing the exchange copies to me at the address below. The IMS Committee will then decide on the allocation.

This issue of the Bulletin has an account by Fiacre Ó Cairbre that documents the growing awareness of Hamilton among the general public, as well as research articles on algebra and number theory, a few book reviews, and the problem page. It is a pleasure to acknowledge the work of Ian Short on this ever-popular page.

Members are reminded that the next Annual Scientific Meeting of the Society will take place at Queen's University, Belfast. Details will be found at `http://ims2014.martinmathieu.net/`

The Newsletter of the EMS (and more) may be accessed online at

`http://www.ems-ph.org/journals/journal.php?jrn=news`

AOF. Department of Mathematics and Statistics, NUI, Maynooth, Co. Kildare

E-mail address: `ims.bulletin@gmail.com`

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MATHEMATICS EDUCATION AND THE PUBLIC'S INTERACTION WITH THE HAMILTON STORY

FIACRE Ó CAIRBRE

Abstract. The life and works of William Rowan Hamilton have produced a lot of public interest over many years. Consequently, Hamilton's story has contributed greatly to mathematics education in relation to the general public. This paper illustrates the wide diversity of ways that the general public has interacted with the Hamilton story.

1. Introduction

This paper discusses some of the rich diversity of public interactions with the Hamilton story and also illustrates the wide appeal of the Hamilton story in the general public. The main motivation for this article comes from the fact that many people, mainly from outside mathematics, frequently contact me wishing to know about Hamilton and then many of them want to create something related to Hamilton. Consequently, the list of art works, media events, activities and more, related to Hamilton has been growing a lot over the years. I felt it was time to write about this abundance of items related to Hamilton and also to write about some of the background stories. In writing this article, I decided to go back the whole way rather than just cover what was done during my time. As you will see, many people have been inspired by the Hamilton story to create works of art. Such people include poets, sculptors, painters, novelists, a songwriter and many more. Hamilton's life and works have been written about in many places. For those who wish to read more about Hamilton, see [1], [2], [3], [4], [5], [7]. In particular, the story about his creation of quaternions along the banks of the Royal Canal on October 16, 1843, appears in many of the examples of the public's interaction with Hamilton below.

I also believe that many of the items in the list below are connected to mathematics education in relation to the general public. From

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my experience in promoting mathematics in the general public, I find that the Hamilton story (and other stories from the history of mathematics) often changes people's perception of mathematics for the better. I believe this change of perception can be crucial in any form of positive mathematics education because after the change in perception, people are then more likely to enhance their understanding, awareness and appreciation of mathematics. I see this happening frequently when I promote mathematics in the general public. Also, the general public plays a significant part in mathematics education in second and third level because parents, policy makers and the media are all members of the general public and can exert great influence on the attitude of young people and society at large towards mathematics. Jack Gannon, from Cabra, put it well when he said, "On account of the walk, Hamilton is in the folk consciousness of the local people".

I find that the "big picture" approach is very successful in changing people's perception of mathematics for the better [6]. The Hamilton story (combined with the annual Hamilton walk) is a great example of something that has all the big picture items. Some of the big picture items are, in no particular order: history of mathematics, stories, human element and famous characters, beauty, drama, motivation, practical power and applications, freedom, creativity, Irish connection, outdoor activity, research, word origins, humour, deductive reasoning, abstraction, cultural connections, tricks/magic and puzzles.

The fact that about half of the items in the list below relate to the last ten years shows that the Hamilton story is alive and very vibrant today. Also, it's a great example of something from the history of mathematics that continues to inspire people (many of whom are outside mathematics) to respond to the story. The response can take different forms including the creation of works of art.

The criterion for inclusion in the list below is that (a) the item was created by somebody outside mathematics or (b) the item (or the main part of the item) is easily accessible to the general public or (c) there is a good chance that the general public will interact with the item. The reason for this criterion is that I am looking at the Hamilton story in relation to the general public. Consequently, there are many important and interesting mathematical items related to

Hamilton that are not on the list below. I also decided to omit many fine books that contain chapters or sections about Hamilton even if they are accessible to the general public, because there are just too many such books.

2. Public interactions with the Hamilton story

1. 1830 – Thomas Kirk sculpted a marble bust of Hamilton. It was acquired by the National Gallery of Ireland in 1982. It's interesting to note that Hamilton was only 25 in 1830. The curator of the National Gallery of Ireland, Brendan Rooney, told me that the bust is currently in storage. Typically, only about a quarter of the gallery's pieces are on public display and the rest are in storage. However, the bust was part of a public exhibition in the National Gallery recently.

2. 1833 – Terence Farrell executed a bust of Hamilton. A picture of this bust appears in the frontispiece of [1].

3. 1842 – There was a portrait of Hamilton in the Dublin University Magazine.

4. 1865 – Shortly before his death, Hamilton was elected as the First Foreign Associate of the newly established National Academy of Sciences of the US. This meant that the Academy deemed Hamilton to be the greatest living scientist.

5. 1867 – John Henry Foley sculpted a bust of Hamilton. This bust is on display in the Long Room in Trinity College.

6. 1882 – 1891 Robert Graves wrote a biography of Hamilton in three volumes [1].

7. 1894 – The Royal Irish Academy acquired a portrait of Hamilton. It was an oil painting by Sarah Purser and was donated by Hamilton's descendant, John O'Regan. Hamilton was President of the RIA from 1837 to 1846. The RIA librarian, Petra Schnabel, also told me that there is a bust of Hamilton in the RIA. However, it's not known who sculpted the bust or when it was done.

8. 1910 – A statue of Hamilton was finished in the Royal College of Science in order to be ready for the College's inauguration in 1911. The statue is now in a niche on the facade of Government Buildings on Merrion St. If you stand on Merrion St. and look through the

gates, you will see Hamilton on the left in the distance. The statue was designed by Oliver Sheppard and executed, at least in part, by an anonymous Bavarian sculptor.

9. 1924 – Éamon De Valera scratched Hamilton's quaternion formulas on the wall of his prison cell when he was in Kilmainham jail. He was paying homage to Hamilton who scratched the quaternion formulas on the bridge in Cabra on October 16, 1843. I suppose one could say that Hamilton's action was, "One small scratch for a man, one giant leap for mathematics", because his quaternions liberated algebra from the shackles of arithmetic. Hamilton opened up a whole new landscape in mathematics. Also, there are many applications of quaternions today, including computer animation, computer games, special effects in movies, space navigation and much more. There is no trace of De Valera's formulas. However, in 1966, when he visited Kilmainham jail, he wrote the quaternion formulas on an envelope. This envelope is now on display in the museum in Kilmainham jail. De Valera was a mathematician and greatly admired Hamilton. He lectured in mathematics in Maynooth during 1912–14. Stephen Buckley told me that, during the war of independence in 1920–21, some of the general public would say that De Valera was one of the few people in the world who understood what a quarter of onions were!

10. 1939 – On July 6, while introducing the Bill to establish the Dublin Institute of Advanced Studies, Taoiseach Éamon De Valera said in the Dáil, "This is the country of Hamilton, a country of great mathematicians".

11. 1939 – Hamilton's quaternions were mentioned in James Joyce's Finnegans Wake. The line is, "wondering was it hebrew set to himmeltones or the quicksilversong of qwaternions; his troubles may be over but his doubles have still to come". Hamilton was mentioned in at least two other places in Finnegans Wake. Sam Slote told me that Joyce is conflating Hamilton with two other Hamiltons in this line. James Archibald Hamilton was the first astronomer at Armagh Observatory and he observed the transit of Mercury, i.e. quicksilver. James Hamilton was a Scottish clergyman who published a book of psalms.

12. 1943 – Two stamps were produced to celebrate the centenary of Hamilton's creation of quaternions. Both the $2\frac{1}{2}$p and $\frac{1}{2}$p stamps

show the same picture of Hamilton and were designed by Sean O'Sullivan.

13. 1943 – Seán Keating painted a portrait of Hamilton. Éimear O'Connor, who has written a book on Keating, told me that it is unknown where this painting is. Éimear has never seen this portrait.

14. 1943 – Flann O'Brien (aka Myles na Gopaleen) commemorated the centenary of Hamilton's creation of quaternions with several references in his Cruiskeen Lawn piece in the Irish Times on June 18. In the references below, the house in Merrion Square is the Dublin Institute of Advanced Studies. The first mention of quaternions is in, "And what's the book about, says I. It's about quateernyuns, says the brother. That's a quare one". Two references appear in, "Of course all the brother's sums isn't done in the digs. He does be inside in a house in Merrion Square doin sums as well. If anybody calls, says the brother, tell them I'm above in Merrion Square workin at the quateernyuns, says he, and take any message. There does be other lads in the same house doin sums with the brother. The brother does be teachin them sums. He does be puttin them right about the sums and the quateernyuns". Another reference is in, "I do believe the brother's makin a good thing out of the sums and the quateernyuns. Your men couldn't offer him less than five bob an hour and I'm certain sure he gets his tea thrown in". A final reference appears in, "Begob the sums and the quateernyuns is quickly shoved aside when the alarm for grub is sounded and all hands piped to the table. The brother thinks there's a time for everything".

15. 1946 – Seán Keating painted a picture of Hamilton scratching the quaternion formulas on the bridge at Broombridge in Cabra. This painting is now in Dunsink Observatory. The painting was commissioned by Felix Hackett.

16. 1958 – On November 13, Taoiseach Eamon De Valera, unveiled a plaque at Broombridge to commemorate Hamilton's creation of quaternions. The unveiling received substantial coverage in the newspapers the following day. It appeared with a photograph on the front page of the Irish Times and was also prominently featured in the Irish Independent and Irish Press. De Valera started his speech with, "I am glad, as head of the Government, to be able to honour the memory of a great scientist and a great Irishman, It is a

great personal satisfaction for me to be present, because it was well over fifty years ago since I had heard the story of the bridge and the birth of quaternions". De Valera also said, "On many occasions since I first heard this story I have come to this place as a holy place". See [4] for the rest of what De Valera said. Michael Biggs designed the plaque.

17. 1958 – The name of the bridge over the Royal Canal at Broombridge, where Hamilton scratched his quaternion formulas, was officially changed to Hamilton Bridge. See [4] for more details on this.

18. 1962 – Gerry Brady submitted a poem about Hamilton to the Irish Times and it was published on June 13.

19. 1963 – A crater on the moon was named after Hamilton.

20. 1970 – Annraoi de Paor told me that he named his son, Liam Ruán, after Hamilton.

21. 1974 – The local Fianna Fáil Cumann in Cabra West was called after Hamilton. Tom Breen told me that the idea for the name came from Vivion De Valera who was a local TD at the time. Vivion was a son of Éamon De Valera.

22. 1980 – Thomas Hankins wrote a biography of Hamilton [2].

23. 1983 – A stamp was produced as part of the Europa series which celebrated the great works of the human genius. It was a 29p stamp and showed Hamilton's quaternion formulas. The stamp was designed by Peter Wildbur.

24. 1983 – Sean O'Donnell wrote a biography of Hamilton [7].

25. 1988 – Mike O'Regan set up the Hamilton Trust in England. Mike is a great–great grandson of Hamilton. The Hamilton Trust develops mathematics resources for teachers and students.

26. 1990 – Anthony G. O'Farrell initiated an annual walk to commemorate Hamilton's creation of quaternions. The walk takes place on October 16 and retraces Hamilton's steps by starting at Dunsink Observatory, where he lived, and ending up at Broombridge, where he created quaternions. The walk takes about forty–five minutes. I organise the walk now and it typically attracts about 200 people from a wide diversity of backgrounds, including many from the general public and second level schools. The large number of

participants from the general public indicates that there is a substantial public interest in Hamilton and the walk. Furthermore, I receive many calls from the media (television, radio and newspaper) and other bodies every year expressing an interest in doing a piece on Hamilton and the walk. Consequently, Hamilton's story and the walk have appeared many times on a variety of television and radio programmes and in lots of newspaper articles. See [4] for more on the walk.

27. 1991 – Henry O'Brien asked James J. Cox to paint a portrait of Hamilton. The portrait is now on display at Broombridge during the annual Hamilton walk.

28. 1991 – David Simms coined the term, Broomsday, for October 16 because Hamilton created quaternions at Broombridge on that day. Coincidentally, Broomsday and Bloomsday are on the sixteenth (of different months).

29. 1992 – The Hamilton building opened in Trinity College and it contained the School of Mathematics.

30. 1995 – There was a reference to Hamilton's creation of quaternions in Act One of Sebastian Barry's play, The Only True History of Lizzie Finn. Sebastian e-mailed me to say that he saw the plaque at Broombridge while he was walking his dogs around 1990.He went on to write that there was something quite intoxicating about the plaque to him, especially marooned in the very altered landscape of that part of the northside – it put Hamilton on the path beside him, and he himself was forever scratching bits of speeches on scraps of paper when they occurred to him, stimulated by the walk, but usually without a handy pencil. The reference in the play appears when Robert says, "Can I tell you a very strange thing? William Rowan Hamilton, the mathematician, was walking along the Royal Canal near Finglas, puzzling the secret of quaternion multiplication, when suddenly a kingfisher came firing out of the green bank, like a blue bullet, a blue revelation. And in that moment of strange blue fire, jolted by it, it came to him: $i^2 = j^2 = k^2 = ijk = -1$. And he scratched it hurriedly on the stone of the bridge nearby. Blue fire, Lizzie Finn, blue fire".

31. 1990s – Anthony O'Farrell put up a portrait of Hamilton in the Department of Mathematics at NUI, Maynooth. The portrait is a photograph of Sarah Purser's painting in the RIA above.

32. 1997 – A new housing estate in Trim was named after Hamilton because he spent his early youth in Trim. The estate is called Hamilton Court. I suppose one could call Hamilton a meathamatician because of his Meath connections!

33. 1999 – LMFM (Louth Meath FM) radio asked me to do a show on Hamilton.

34. 1999 – Éigse na Mí (Meath heritage festival weekend) invited me to give a talk on Hamilton in Trim.

35. Annually since the 1990s – The Irish Times produces a piece that promotes the Hamilton walk. Dick Ahlstrom typically writes the piece based on my press release. Sometimes, other journalists write substantial pieces on Hamilton based on my interaction with them. For example, Arminta Wallace wrote a piece for An Irishwoman's Diary in the Irish Times in 2011. Other newspapers cover the walk from time to time. I also promote the walk on my annual radio show about mathematics on LMFM.

36. 2000 – Séamus MacGabhann of Ríocht na Midhe (Meath Archaeological and Historical Society) invited me to write a paper on Hamilton. The Society was interested in Hamilton because he spent his early youth in Trim, Co. Meath. The paper appeared in the annual journal [5].

37. 2001 – The Hamilton Institute at NUI, Maynooth was founded under the leadership of Robert Shorten and Doug Leith.

38. 2001 – Slane Historical Society invited me to give a talk on Hamilton.

39. 2001 – Gary McGuire and I wrote a paper about the walk and Hamilton [3].

40. 2002 – The Royal Irish Academy awarded an annual Hamilton Prize to mathematics students in nine of the Higher Education Institutions in Ireland. Each mathematics department is invited to nominate its best student in the penultimate year of undergradute study. The awards are presented to all the students as part of the RIA's Hamilton Day activities on or around October 16.

41. 2002 – The Royal Irish Academy initiated an annual public Hamilton lecture which occurs as part of its Hamilton Day activities on or around October 16. The lectures are given by internationally

renowned speakers. Fields Medallists and Nobel Prize winners have been among those that have given the RIA Hamilton lecture. Luke Drury and Rebecca Farrell also usually bring the speaker on the Hamilton walk.

42. 2003 – Jack Gannon was inspired by the Hamilton walk to write a song called the Ballad of Rowan Hamilton. The song has been played many times on programmes about Hamilton and the walk on radio and television since then. See [4] for more details on the song.

43. 2003 - The Hamilton Mathematics Institute was founded at Trinity College.

44. 2003 – A television crew covered the Hamilton walk and the walk appeared on the Six One news on RTÉ 1 television that evening.

45. 2004 – A plaque was erected on the house in Trim where Hamilton spent his early youth. The house is called St. Mary's Abbey and is beautifully situated on the banks of the Boyne across from the spectacular ruins of Trim Castle. Peter Higgins now lives in the house and he put the plaque up after it was proposed by the National Committee for Science and Engineering Commemorative Plaques.

46. 2005 – The Government designated the year as Hamilton Year – Celebrating Irish Science, because it was the bicentenary of Hamilton's birth. Many events were held all over Ireland to celebrate Hamilton year, some of which are listed below.

47. 2005 – On August 4 the Cabra Community Council celebrated Hamilton's birthday by organising a huge party and a barge trip along the Royal Canal.

48. 2005 – June Robinson wrote a poem about Hamilton called the Benefactor. See [4] for more on this.

49. 2005 – Léargas produced a thirty minute television documentary on Hamilton and it was shown on RTÉ 1 on November 14.

50. 2005 – A housing area in Cabra was named Rowan Hamilton Court. Hugh Flanagan and Liam O'Neill lobbied Dublin City Council to have the housing area named after Hamilton and they were successful.

51. 2005 – Cieran Perry and Henry O'Brien purchased, on behalf of the Cabra Community, a large banner in celebration of Hamilton.

The idea for the banner was proposed by the Cabra Community Council. It creates a very festive atmosphere at the end of the Hamilton walk where it is draped across the bridge.

52. 2005 – The Central Bank of Ireland launched a new 10 euro collector's coin to celebrate the bicentenary of Hamilton's birth. The coin was designed by Michael Guilfoyle.

53. 2005 – An Post produced a Hamilton stamp to celebrate the bicentenary of Hamilton's birth. It was designed by Ger Garland.

54. 2005 – St. Declan's College, in Cabra, named one of its classes after Hamilton. It also awarded the new Hamilton coins to the students who obtained the best results in the Junior and Leaving Cert exams.

55. 2006 – Eoin Gill and Sheila Donegan of Calmast in Waterford IT founded Maths Week Ireland. The key day of the week is October 16 when Hamilton created quaternions. Consequently, Maths Week is always the week including October 16. The aim of Maths Week is to enhance the awareness, appreciation and understanding of mathematics among schoolchildren and the general public. Maths Week Ireland is now the most successful Maths Week in the world with over 200,000 participating directly in 2013.

56. 2006 – Mary Mulvihill's radio crew covered the Hamilton walk and it appeared on her radio show, Quantum Leap, on RTÉ 1 later in the week.

57. 2006 – Mick and Maureen Kelly founded a company called Science Heritage Ireland. Mick had participated in the 2005 Hamilton walk and he wrote, "That walk had a profound effect on me. Hearing not only a Nobel laureate and a professor of mathematics sing Hamilton's praises, but also local poets, school children, balladeers and the Cabra Community Council, spurred me to turn my desire to celebrate Ireland's Science Heritage into action. That action turned out to be a family run business called Science Heritage Ireland selling placemats and coasters celebrating Hamilton".

58. 2006 – Neville Henderson painted a picture of Hamilton scratching the quaternion formulas on the bridge and it was used in a brochure for the above Science Heritage Ireland company. Susan Waine also produced a design concept for the Hamilton placemat and coaster.

59. 2007 and 2008 – Mick and Maureen Kelly set up a stall at Broombridge at the end of the Hamilton walk. They sold their Hamilton placemats and coasters mentioned above.

60. 2008 – Joey Burns created a bog oak sculpture in Trim with some of Hamilton's equations on it. Trim town council had commissioned a piece of art for Castle St. in the town. Joey told me that he was inspired by the Hamilton television documentary in 2005 above to create something related to Hamilton. His sculpture shows the salmon of knowledge rising from the water and reaching for the hazelnuts. Hamilton's equations lie in the waves near the base of the sculpture. In Celtic Mythology, the salmon gains all the world's knowledge when it eats the hazelnuts from the trees surrounding the well at the source of the River Boyne. The well is in Cairbre (aka Carbury in Co. Kildare) and the Boyne flows through Trim near the sculpture.

61. 2008 – Conn Ó Muineachain asked me to contribute a piece on Hamilton for a show on Raidió na Gaeltachta.

62. 2008 – James J. Cox painted his second portrait of Hamilton. This is also on display at Broombridge during the annual Hamilton walk.

63. 2009 – Joey Burns created several oak log benches along the banks of the Boyne in Trim. There are some lines from Hamilton's poetry on the benches. Trim Town Council had commissioned a piece of art for the banks of the river. Hamilton was also a poet and won the Chancellor's Prize twice in Trinity College and published his literary work in journals and magazines. Hamilton's motivation for doing mathematics was the quest for beauty. He once wrote, "Mathematics is an aesthetic creation, akin to poetry, with its own mysteries and moments of profound revelations". See [5] for more on Hamilton's poetry.

64. 2009 – Philip Bromwell asked me to appear on the Capital D programme on RTÉ 1 television. The show went out in early October in order to promote the annual walk and the Hamilton story.

65. 2009 – Eleanor Burnhill invited me to contribute to a radio piece about Hamilton and the walk. The piece appeared on Morning Ireland on RTÉ 1 on the morning of the walk.

66. 2009 – There was a display related to Hamilton in the National Wax Museum which opened in Dublin.

67. 2009 – The South Fingal Heritage Society invited me to give a talk on Hamilton because of his Fingal connections.

68. 2009 – Elmgreen Golf Club created the Rowan Hamilton Singles Matchplay Trophy. The golf club is adjacent to Dunsink Observatory where Hamilton lived. Richard Wilson proposed that the trophy be named after Hamilton.

69. 2009 – Mick and Maureen Kelly produced T-shirts with the Quaternion formulas and a Hamilton slogan.

70. 2010 – In order to mark the twentieth anniversary of the Hamilton walk, I wrote a paper called Twenty Years of the Hamilton Walk [4]

71. 2010 – I was invited to appear on the Seán Moncrieff show on Newstalk radio to talk about Hamilton and to promote the Hamilton walk.

72. 2010 – A radio crew from Phoenix FM covered the walk and the show went out later in the week.

73. 2011 – Mary Mulvihill produced a one hour audio walking tour that was recorded during the Hamilton walk.

74. 2012 – Daniel Doyle created a giant sand sculpture that showed Hamilton scratching his quaternion formulas on the bridge in Cabra. The sculpture was in the courtyard of Dublin Castle as part of the annual exhibition of sand sculptures. The theme for the exhibition was Bright Sparks to coincide with the fact that Dublin was the European City of Science. Daniel chose Hamilton because his parents were from Cabra and they used to tell him the Hamilton story when he was young. Daniel is part of a trio of artists called Duthain Dealbh which means Fleeting Sculpture. They create ephemeral works of art for events all over the world using sand, ice and snow.

75. 2012 – Simone Corr was a student in the Dún Laoghaire Institute of Art, Design and Technology and she wanted to produce a work of art related to Hamilton as part of her student project. Simone contacted me and we collaborated on a video installation about Hamilton and mathematics. The piece was selected to appear

in an exhibition called The Flaneur at the Olivier Cornet Gallery in Temple Bar in Dublin.

76. 2012 – Kathryn Maguire was creating a piece of art for the Five Lamps Festival in Dublin. She wanted to do something related to the Royal Canal. This led her to Hamilton. She contacted me and we discussed various ideas. Subsequently, Kathryn and her team produced some grafitti art which was on display along the Royal Canal at Croke Park during the festival. I thought graffiti art would be appropriate because Hamilton did his own graffiti when he scratched his quaternion formulas on the bridge. Hamilton's piece of mathematical vandalism would change the world of mathematics forever. Maurice O'Reilly also gave a talk along the canal as part of the festival.

77. 2012 – Des Gunning invited me to give a talk on Hamilton for Phizzfest which is the Phibsborough Community Arts Festival. They were interested in Hamilton because of his connection with the Royal Canal which passes through Phibsborough.

78. 2012 – John Coll sculpted a bust of Hamilton which is now in Ballinaclough House near Nenagh. Ballinaclough house is connected to Hamilton's wife, Helen Bayly, who was from the Nenagh area. Helen lived in another house nearby called Bayly Farm. Desmond Bayly, who lives in Bayly Farm, told me that Bayly Farm is the house where Hamilton and Helen spent their honeymoon. Desmond is related to Helen and the Bayly Farm Country House Accommodation website mentions the Hamilton connection.

79. 2012 – The sixteenth hole at Elmgreen Golf Club was renamed as the Rowan Hamilton Corner. Richard Wilson came up with this idea. Richard told me that this hole is unique in the world of golf because it's the only hole named after a mathematician! The golf club is adjacent to Dunsink Observatory where Hamilton lived.

80. 2012 – Eoin Gill and Sheila Donegan produced T–shirts with Hamilton's Icosian Game on them.

81. 2013 - There was an interview with me about the Hamilton walk on the Six One news on RTÉ 1 television, on the Nuacht on TG4 and on News Today on RTÉ 2 television.

82. 2013 – Iggy McGovern wrote a book of poetry called A Mystic Dream of 4. The book comprises sixty four sonnets about the life

and times of Hamilton. The sonnets are mainly in the voices of relatives, friends and colleagues of Hamilton.

83. 2013 – The Director of Poetry Ireland, Joe Woods, read a piece on Hamilton on the RTÉ 1 radio show, Sunday Miscellany. Joe had come on the walk previously.

84. 2014 – John Coll sculpted a bronze bust of Hamilton for the School of Mathematics, Statistics and Applied Mathematics in NUI, Galway. Aisling McCluskey told me that the school wanted to have a gallery of art and they commissoned John Coll to do the bust.

85. 2014 – Seán Duke invited me to contribute to an RTÉ radio 1 show called What's it All About. It was an hour long show about mathematics and Hamilton featured prominently.

3. Some current items related to Hamilton

There always seems to be a variety of ongoing items related to Hamilton. Here are four current examples:

(a) Liam O'Neill and Jack Gannon have submitted proposals for the inclusion of Hamilton's name on the new Luas station at Broombridge. I plan to get more involved in the attempt to put Hamilton's name on the new station.

(b) The Kavaleer animation company contacted me because they were interested in producing an animated movie about Hamilton. They were aware that Hamilton's quaternions are important in computer animation and they wanted to kind of reverse that and do some computer animation on Hamilton himself!

(c) Dick Ahlstrom is writing a fictional novel based on Hamilton's life.

(d) Clare Tuffy from Newgrange has invited me to give a talk on Hamilton as part of Heritage Week later this year.

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Fiacre Ó Cairbre is a Senior Lecturer in Mathematics at NUI, Maynooth.

Department of Mathematics and Statistics, NUI, Maynooth
E-mail address: fiacre.ocairbre@nuim.ie

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TWO TRIGONOMETRIC IDENTITIES

HORST ALZER AND WENCHANG CHU

Abstract. We show that the trigonometric identities

$$\prod_{k=1}^{n-1}\Big\{1-\cos\frac{2k\pi}{n}\Big\}^{(n-k)\ell+m}=2^{(1-n)(\ell n/2+m)}\,n^{\ell n+2m}$$

and

$$\prod_{k=1}^{n-1}\Big\{\cos(2\theta)-\cos\frac{2k\pi}{n}\Big\}^{(n-k)\ell+m}=2^{(1-n)(\ell n/2+m)}\Big\{\frac{\sin(n\theta)}{\sin\theta}\Big\}^{\ell n+2m}$$

are valid for all $\ell, m\in\mathbf{Z}$ and $2\le n\in\mathbf{N}$. They extend the results due to Baica and Gregorac, who proved the identities for the special case $\ell=1$, $m=-1$. Moreover, we determine all ℓ, m, n such that the first trigonometric product just displayed is an integer.

In 1986, Baica [1] applied methods from cyclotomic fields to provide a rather long and complicated proof for the following interesting trigonometric identity:

$$\prod_{k=1}^{n-1}\Big\{1-\cos\frac{2k\pi}{n}\Big\}^{n-k-1}=2^{(1-n)(n/2-1)}\,n^{n-2} \tag{1}$$

where $n=2,3,4,\cdots$. Baica also remarked that "any proof avoiding cyclotomic fields could be very difficult, if not insoluble" [1, P. 705].

In 1989, Gregorac [3] used properties of Chebyshev polynomials to present a new proof of (1). Actually, he proved the identity

$$\prod_{k=1}^{n-1}\Big\{\cos(2\theta)-\cos\frac{2k\pi}{n}\Big\}^{n-k-1}=2^{(1-n)(n/2-1)}\Big\{\frac{\sin(n\theta)}{\sin\theta}\Big\}^{n-2} \tag{2}$$

for $n=2,3,4,\cdots$, which, letting θ tend to 0, leads to (1).

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Here, we extend (1) and (2). First, we offer an elementary short and simple proof of a generalization of Baica's identity. In order to verify our result we only make use of three well-known properties of sine and cosine,

$$1-\cos(2\theta) = 2\,\sin^2\theta, \tag{3}$$

$$\sin(\pi-\theta) = \sin\theta, \tag{4}$$

$$\prod_{k=1}^{n-1}\sin\frac{k\pi}{n} = 2^{1-n}\,n. \tag{5}$$

Formula (5) as well as many related formulas involving trigonometric functions can be found in [2, Eq. 4.14].

We have the following extension of identity (1).

Theorem 1. *Let ℓ, m be integers and let $n \geq 2$ be a natural number. Then,*

$$\prod_{k=1}^{n-1}\Big\{1-\cos\frac{2k\pi}{n}\Big\}^{(n-k)\ell+m} = 2^{(1-n)(\ell n/2+m)}\,n^{\ell n+2m}. \tag{6}$$

Proof. Applying (3) yields

$$\prod_{k=1}^{n-1}\Big\{1-\cos\frac{2k\pi}{n}\Big\}^{(n-k)\ell+m} = 2^{(n-1)(\ell n/2+m)}\prod_{k=1}^{n-1}\Big\{\sin\frac{k\pi}{n}\Big\}^{2[(n-k)\ell+m]}. \tag{7}$$

From (4) we conclude that

$$\prod_{k=1}^{n-1}\Big\{\sin\frac{k\pi}{n}\Big\}^{(n-k)\ell+m} = \prod_{k=1}^{n-1}\Big\{\sin\frac{(n-k)\pi}{n}\Big\}^{k\ell+m} = \prod_{k=1}^{n-1}\Big\{\sin\frac{k\pi}{n}\Big\}^{k\ell+m}. \tag{8}$$

Using (8) and (5) gives

$$\begin{aligned}\prod_{k=1}^{n-1}\Big\{\sin\frac{k\pi}{n}\Big\}^{2[(n-k)\ell+m]} &=\prod_{k=1}^{n-1}\Big\{\sin\frac{k\pi}{n}\Big\}^{(n-k)\ell+m}\prod_{k=1}^{n-1}\Big\{\sin\frac{k\pi}{n}\Big\}^{(n-k)\ell+m}\\ &=\prod_{k=1}^{n-1}\Big\{\sin\frac{k\pi}{n}\Big\}^{(n-k)\ell+m}\prod_{k=1}^{n-1}\Big\{\sin\frac{k\pi}{n}\Big\}^{k\ell+m}\\ &=\prod_{k=1}^{n-1}\Big\{\sin\frac{k\pi}{n}\Big\}^{\ell n+2m}=\big(2^{1-n}\,n\big)^{\ell n+2m}. \qquad (9)\end{aligned}$$

Combining (7) and (9) leads to (6). □

Next, we extend Gregorac's identity (2). We need the following formulas:

$$\sin\Big(\frac{\pi}{2}-\theta\Big)=\cos\theta, \tag{10}$$

$$\sin(2\theta)=2\sin\theta\cos\theta, \tag{11}$$

$$\cos y-\cos x=2\sin\frac{x+y}{2}\sin\frac{x-y}{2}, \tag{12}$$

$$\frac{\sin(n\theta)}{\sin\theta}=2^{n-1}\prod_{k=1}^{n-1}\Big\{\cos\theta-\cos\frac{k\pi}{n}\Big\}. \tag{13}$$

Identity (13) is the well-known product representation for the Chebyshev polynomials of the second kind.

Theorem 2. *Let ℓ, m be integers and let $n\geq 2$ be a natural number. Then, for $\theta\in\mathbf{R}$,*

$$\prod_{k=1}^{n-1}\Big\{\cos(2\theta)-\cos\frac{2k\pi}{n}\Big\}^{(n-k)\ell+m}=2^{(1-n)(\ell n/2+m)}\Big\{\frac{\sin(n\theta)}{\sin\theta}\Big\}^{\ell n+2m}. \tag{14}$$

Proof. Using (10) gives

$$\prod_{k=1}^{n-1}\sin\Big(\frac{k\pi}{2n}-\frac{\theta}{2}\Big)=\prod_{k=1}^{n-1}\sin\Big(\frac{(n-k)\pi}{2n}-\frac{\theta}{2}\Big)=\prod_{k=1}^{n-1}\cos\Big(\frac{k\pi}{2n}+\frac{\theta}{2}\Big). \tag{15}$$

Now, we apply (12), (15) and (11). Then we have

$$\begin{aligned}\prod_{k=1}^{n-1}\Big\{\cos\theta-\cos\frac{k\pi}{n}\Big\}=&\prod_{k=1}^{n-1}\Big\{2\sin\Big(\frac{k\pi}{2n}+\frac{\theta}{2}\Big)\sin\Big(\frac{k\pi}{2n}-\frac{\theta}{2}\Big)\Big\}\\=&\prod_{k=1}^{n-1}\Big\{2\sin\Big(\frac{k\pi}{2n}+\frac{\theta}{2}\Big)\cos\Big(\frac{k\pi}{2n}+\frac{\theta}{2}\Big)\Big\}\\=&\prod_{k=1}^{n-1}\sin\Big(\frac{k\pi}{n}+\theta\Big).\end{aligned}\tag{16}$$

From (4) and (12) we obtain

$$\begin{aligned}\prod_{k=1}^{n-1}\sin^2\Big(\frac{k\pi}{n}+\theta\Big)=&\prod_{k=1}^{n-1}\Big\{\sin\Big(\frac{k\pi}{n}+\theta\Big)\sin\Big(\frac{(n-k)\pi}{n}-\theta\Big)\Big\}\\=&\prod_{k=1}^{n-1}\Big\{\sin\Big(\frac{k\pi}{n}+\theta\Big)\sin\Big(\frac{k\pi}{n}-\theta\Big)\Big\}\\=&2^{1-n}\prod_{k=1}^{n-1}\Big\{\cos(2\theta)-\cos\frac{2k\pi}{n}\Big\}.\end{aligned}\tag{17}$$

Applying (13), (16) and (17) yields

$$\begin{aligned}\Big\{\frac{\sin(n\theta)}{\sin\theta}\Big\}^{2m}=&2^{2m(n-1)}\prod_{k=1}^{n-1}\Big\{\cos\theta-\cos\frac{k\pi}{n}\Big\}^{2m}\\=&2^{2m(n-1)}\prod_{k=1}^{n-1}\sin^{2m}\Big(\frac{k\pi}{n}+\theta\Big)\\=&2^{m(n-1)}\prod_{k=1}^{n-1}\Big\{\cos(2\theta)-\cos\frac{2k\pi}{n}\Big\}^{m}.\end{aligned}\tag{18}$$

From (4) and (12) we get

$$\begin{aligned}\prod_{k=1}^{n-1} \sin^n\Big(\frac{k\pi}{n}+\theta\Big) &= \prod_{k=1}^{n-1}\Big\{\sin^{n-k}\Big(\frac{k\pi}{n}+\theta\Big)\sin^k\Big(\frac{k\pi}{n}+\theta\Big)\Big\}\\ &= \prod_{k=1}^{n-1}\Big\{\sin^{k}\Big(\frac{(n-k)\pi}{n}+\theta\Big)\sin^k\Big(\frac{k\pi}{n}+\theta\Big)\Big\}\\ &= \prod_{k=1}^{n-1}\Big\{\sin^{k}\Big(\frac{k\pi}{n}-\theta\Big)\sin^k\Big(\frac{k\pi}{n}+\theta\Big)\Big\}\\ &= 2^{(1-n)n/2}\prod_{k=1}^{n-1}\Big\{\cos(2\theta)-\cos\frac{2k\pi}{n}\Big\}^k. \end{aligned} \tag{19}$$

Combining (13), (16) and (19) gives

$$\Big\{\frac{\sin(n\theta)}{\sin\theta}\Big\}^n = 2^{(n-1)n/2}\prod_{k=1}^{n-1}\Big\{\cos(2\theta)-\cos\frac{2k\pi}{n}\Big\}^k. \tag{20}$$

Finally, (18) and (20) lead to

$$\begin{aligned}\Big\{\frac{\sin(n\theta)}{\sin\theta}\Big\}^{2m+\ell n} &= 2^{(n-1)(\ell n/2+m)}\prod_{k=1}^{n-1}\Big\{\cos(2\theta)-\cos\frac{2k\pi}{n}\Big\}^{m+k\ell}\\ &= 2^{(n-1)(\ell n/2+m)}\prod_{k=1}^{n-1}\Big\{\cos(2\theta)-\cos\frac{2k\pi}{n}\Big\}^{(n-k)\ell+m}.\end{aligned}$$

This is equivalent to (14). □

Remark 1. Setting $\ell = 1$ and $m = -1$ in (6) and (14), respectively, gives (1) and (2).

Remark 2. Let $\ell, m \in \mathbf{Z}$ and $2 \le n \in \mathbf{N}$ with $\ell n + 2m > 0$. Applying (14) and the well-known inequality

$$\Big|\frac{\sin(n\theta)}{n\sin\theta}\Big| \le 1 \quad (n = 1, 2, 3, \ldots)$$

we obtain for all $\theta \in \mathbf{R}$:

$$\prod_{k=1}^{n-1}\Big\{\cos(2\theta)-\cos\frac{2k\pi}{n}\Big\}^{(n-k)\ell+m} \le 2^{(1-n)(\ell n/2+m)} n^{\ell n+2m}. \tag{21}$$

Setting $\theta = 0$ we conclude from (6) that the given upper bound is sharp. If $\ell n + 2m < 0$, then (21) holds with "$\geq$" instead of "$\leq$".

The representation (6) reveals that if $\ell, m \in \mathbf{Z}$, $2 \leq n \in \mathbf{N}$, then the product

$$P_n(\ell, m) = \prod_{k=1}^{n-1} \Big\{1 - \cos\frac{2k\pi}{n}\Big\}^{(n-k)\ell+m}$$

is a rational number. In view of this result it is natural to ask whether there exist numbers ℓ, m, n such that $P_n(\ell, m)$ is an integer. The next theorem answers this question.

Theorem 3. *Let ℓ, m be integers and $n \geq 2$ a natural number. The product $P_n(\ell, m)$ is an integer if and only if $\ell n + 2m = 0$*

$$\text{or} \quad \ell n + 2m > 0 \quad \text{with} \quad n = 2^r \quad (r = 1, 2); \tag{22}$$

$$\text{or} \quad \ell n + 2m < 0 \quad \text{with} \quad n = 2^r \quad (3 \leq r \in \mathbf{N}). \tag{23}$$

Proof. Using (6) we obtain:
if $\ell n + 2m = 0$, then $P_n(\ell, m) = 1$;
if $n = 2^r$ $(r = 1, 2)$ and $\ell n + 2m > 0$, then

$$P_n(\ell, m) = 2^{(\ell n + 2m)/2} \in \mathbf{Z};$$

if $n = 2^r$ $(r \geq 3)$ and $\ell n + 2m < 0$, then

$$P_n(\ell, m) = 2^{-(2^r - 2r - 1)(\ell n + 2m)/2} \in \mathbf{Z}.$$

Now, let $P_n(\ell, m) \in \mathbf{Z}$. We asssume (for a contradiction) that none of (22), (23) and $\ell n + 2m = 0$ is satisfied. We have

$$\begin{aligned} P_2(\ell, m) &= 2^{\ell+m}, \\ P_3(\ell, m) &= \Big\{\frac{3}{2}\Big\}^{3\ell+2m}, \\ P_4(\ell, m) &= 2^{2\ell+m}, \\ P_5(\ell, m) &= \Big\{\frac{5}{4}\Big\}^{5\ell+2m}. \end{aligned}$$

Case 1. $\ell n + 2m > 0$.
Then, $P_3(\ell, m) \notin \mathbf{Z}$ and $P_5(\ell, m) \notin \mathbf{Z}$. Let $n \geq 6$. From

$$2^{(n-1)(\ell n + 2m)/2} \cdot K = n^{\ell n + 2m} \quad (K \in \mathbf{N}) \tag{24}$$

we conclude that 2 divides $n^{\ell n+2m}$. This implies that n is even. Let $n = 2^r q$, where $r \geq 1$ and q is odd. Then, (24) leads to

$$2^{((n-1)/2-r)(\ell n+2m)} \cdot K = q^{\ell n+2m}.$$

Since q is odd, we obtain

$$\frac{n-1}{2} - r \leq 0. \tag{25}$$

Hence,

$$2^r \leq 2^r q = n \leq 2r + 1.$$

If follows that $r = 1$ or $r = 2$. However, this contradicts (25), since $n \geq 6$.

Case 2. $\ell n + 2m < 0$.
Then, $P_n(\ell, m) \notin \mathbf{Z}$ for $n = 2, 3, 4, 5$. Let $n \geq 6$. From (24) we obtain

$$n^{-(\ell n+2m)} \cdot K = 2^{-(n-1)(\ell n+2m)/2}.$$

This yields $n = 2^r$ with $r \geq 3$. A contradiction. The proof is complete. □

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Horst Alzer received his doctoral degree in mathematics from the University of Bonn and his habilitation in mathematics from the University of urzburg. His main interest of research are inequalities and special functions.

Wenchang Chu received, in July 1987, his PhD in Combinatorial mathematics under the supervision of Prof. L. C. Hsu at Dalian University of Technology (P.R.China). Since then he has been employed in Academia Sinica (Beijing), Università di Roma La Sapienza, Freie Universität Berlin, INRIA (Paris), and

currently in Salento University (Italy). His research interests cover broadly the interaction among classical analysis, combinatorics and special functions.

(Alzer) Morsbacher Str. 10, 51545 Waldbröl, Germany

E-mail address, Alzer: H.Alzer@gmx.de

(Chu) Dipartimento di Matematica e Fisica "Ennio De Giorgi", Università del Salento, Lecce-Arnesano, P. O. Box 193, 73100 Lecce, Italy

E-mail address, Chu: chu.wenchang@unisalento.it

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INFINITELY MANY POSITIVE INTEGER SOLUTIONS OF THE QUADRATIC DIOPHANTINE EQUATIONS $x^2 - 8B_nxy - 2y^2 = \pm 2^r$

OLCAY KARAATLI, REFİK KESKİN, AND HUILIN ZHU

Abstract. In this study, we consider the quadratic Diophantine equations given in the title and determine when these equations have positive integer solutions. Moreover, we find all positive integer solutions of them in terms of Balancing numbers B_n, Pell and Pell-Lucas numbers, and the terms of the sequence $\{v_n\}$, where $\{v_n\}$ is defined by $v_0 = 2$, $v_1 = 6$, and $v_{n+1} = 6v_n - v_{n-1}$ for $n \geq 1$.

1. Introduction

A Diophantine equation is an equation in which only integer solutions are allowed. The name "Diophantine" comes from Diophantos, an Alexandrian mathematician of the third century A. D., who proposed many Diophantine problems; but such equations have a very long history, extending back to ancient Egypt, Babylonia, and Greece. In general, a quadratic Diophantine equation is an equation in the form

$$ax^2 + bxy + cy^2 + dx + ey + f = 0, \tag{1}$$

where $a, b, c, d, e,$ and f are fixed integers. There has been much interest in determining all integer solutions to Diophantine equations among mathematicians. In particular, several papers [30, 6, 29, 2, 3, 4, 17, 33, 7, 12, 10] deal with such equations. The principal question when studying a given Diophantine equation is whether a solution exists; and in the case they exist, how many solutions there are and whether there is a general form for the solutions. For more details on Diophantine equations, see [21, 31, 23, 8, 14, 32, 20].

In [11], Keskin and Yosma considered the Diophantine equations

$$x^2 - L_nxy + (-1)^n y^2 = \pm 5^r, \tag{2}$$

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where $n > 0$, $r > 1$, and L_n denotes the n^{th} Lucas number. The authors determined when (2) have positive integer solutions. Later, applying some properties of Fibonacci and Lucas numbers, they gave all positive integer solutions of (2) in terms of Fibonacci and Lucas numbers. In [13], Keskin, Karaatlı, and Şiar determined when the equations

$$x^2 - 5F_nxy - 5(-1)^n y^2 = \pm 5^r, \tag{3}$$

where F_n denotes the n^{th} Fibonacci number, have positive integer solutions under some assumptions that $n \geq 1$, $r \geq 0$, using some basic properties of Fibonacci and Lucas sequences and also some cases in which Fibonacci and Lucas sequences have square terms. Then the authors found all positive integer solutions of (3). In this study, we are interested in determining explicitly all positive integer solutions (x, y) of the equations

$$x^2 - 8B_nxy - 2y^2 = \pm 2^r, \tag{4}$$

where B_n denotes the n^{th} balancing number, in terms of balancing numbers, Pell and Pell-Lucas numbers, and the terms of the sequence $\{v_n\}$.

2. Close Relations Between Balancing Sequence, Pell and Pell-Lucas Sequences, and the sequence $\{v_n\}$

Before we can explain about the sequences mentioned in the title above, we need to recall the generalized Fibonacci and Lucas sequences.

Let P and Q be non-zero integers. We consider the generalized Fibonacci sequence $\{U_n\}$

$$U_0 = 0,\ U_1 = 1,\ U_{n+1} = PU_n - QU_{n-1} \text{ for } n \geq 1 \tag{5}$$

and the generalized Lucas sequence $\{V_n\}$

$$V_0 = 2,\ V_1 = P,\ V_{n+1} = PV_n - QV_{n-1} \text{ for } n \geq 1. \tag{6}$$

The numbers U_n and V_n are called the n^{th} generalized Fibonacci and Lucas numbers, respectively. Moreover, generalized Fibonacci and Lucas numbers for negative subscripts are defined as

$$U_{-n} = \frac{-U_n}{Q^n} \text{ and } V_{-n} = \frac{V_n}{Q^n} \tag{7}$$

with $n \geq 1$. If $\alpha = (P + \sqrt{P^2 - 4Q})/2$ and $\beta = (P - \sqrt{P^2 - 4Q})/2$, assuming $P^2 - 4Q \neq 0$, are zeros of $x^2 - Px + Q$, then we have the well known Binet formulas

$$U_n = (\alpha^n - \beta^n)/(\alpha - \beta) \text{ and } V_n = \alpha^n + \beta^n \tag{8}$$

for all $n \in \mathbb{Z}$. When $P = 1$ and $Q = -1$, $\{U_n\} = \{F_n\}$ and $\{V_n\} = \{L_n\}$ are the familiar sequences of Fibonacci and Lucas numbers, respectively. For $P = 2$ and $Q = -1$, $\{U_n\}$ and $\{V_n\}$ are the familiar Pell sequence $\{P_n\}$ and Pell-Lucas sequence $\{Q_n\}$, respectively. Furthermore, when $Q = 1$, we represent $\{U_n\}$ and $\{V_n\}$ by $\{u_n\}$ and $\{v_n\}$. It clearly follows from (3) that

$$u_{-n} = -u_n \text{ and } v_{-n} = v_n \tag{9}$$

for all $n \geq 1$. For further details on generalized Fibonacci and Lucas sequences, we refer the reader to [9, 22, 25, 26].

The terms of a sequence $\{U_n\}$ may be partitioned into disjoint classes by means of the following equivalence relation:

$U_m \sim U_n$ if and only if there exist non-zero integers x and y satisfying $x^2 U_m = y^2 U_n$, or equivalently $U_m U_n$ is a square. If $U_m \sim U_n$, then U_m and U_n are said to be in the same square-class. A square-class containing more than one term of the sequence is called non-trivial. Similarly, we can define the square-class of $\{V_n\}$.

Balancing numbers were introduced by Behera and Panda [1], by considering natural numbers b and r satisfying the equation

$$1 + 2 + ... + (b - 1) = (b + 1) + (b + 2) + ... + (b + r). \tag{10}$$

Here, r is the *balancer* corresponding to the *balancing number* b. The n^{th} balancing number is denoted by B_n and the balancing numbers B_n for $n \geq 2$ are obtained from the recurrence relation

$$B_0 = 0,\ B_1 = 1,\ B_{n+1} = 6B_n - B_{n-1} \text{ for } n \geq 1. \tag{11}$$

Actually, substituting $P = 6$ and $Q = 1$ into (5) and (6) gives that $u_n = B_n$ and the sequence $\{v_n\}$, which is mentioned in the title of this section. This means that both balancing sequence and the sequence $\{v_n\}$ are special cases of the generalized Fibonacci and Lucas sequences for the case when $P = 6$ and $Q = 1$. Now we state some well known definition, theorems, and identities regarding the sequences $\{P_n\}, \{Q_n\}, \{B_n\}$, and $\{v_n\}$ that will be needed later.

Definition 2.1. Let a and b be integers, at least one of which is not zero. The greatest common divisor of a and b, denoted by (a, b), is the largest integer which divides both a and b.

Theorem 2.2. *Let γ and δ be the roots of the equation $x^2 - 2x - 1 = 0$. Then we have $P_n = \dfrac{\gamma^n - \delta^n}{2\sqrt{2}}$ and $Q_n = \gamma^n + \delta^n$ for all $n \in \mathbb{Z}$.*

Theorem 2.3. *Let α and β be roots of the characteristic equation $x^2 - 6x + 1 = 0$. Then $u_n = \dfrac{\alpha^n - \beta^n}{4\sqrt{2}}$ and $v_n = \alpha^n + \beta^n$ for all $n \in \mathbb{Z}$.*

From the theorems above, it is easily seen that $B_n = P_{2n}/2$ and $v_n = Q_{2n}$ for $n \geq 0$. We assume from this point on that $P = 6$ for the sequence $\{v_n\}$.

Most of the properties below for $\{P_n\}, \{Q_n\}, \{B_n\}$, and $\{v_n\}$ are well known (see, for example [28]). Properties (17) to (22) can be easily obtained by using Binet's formulas. Properties (23) to (26) can be found in [26, 27]. Properties (31) and (32) were proved in [18]. The proofs of the others are easy and will be omitted.

$$Q_n^2 - 8P_n^2 = 4(-1)^n, \tag{12}$$

$$v_n^2 - 32u_n^2 = 4, \tag{13}$$

$$P_{2n} = P_n Q_n \text{ and } B_{2n} = B_n v_n, \tag{14}$$

$$v_n = B_{n+1} - B_{n-1}, \tag{15}$$

$$Q_{2n} = Q_n^2 - 2(-1)^n, \tag{16}$$

$$v_m v_n + 32 B_m B_n = 2 v_{m+n}, \tag{17}$$

$$v_m v_n - 32 B_m B_n = 2 v_{m-n}, \tag{18}$$

$$B_m v_n + B_n v_m = 2 B_{m+n}, \tag{19}$$

$$B_m v_n - B_n v_m = 2 B_{m-n}, \tag{20}$$

$$v_{m+n}^2 - 32 B_m B_n v_{n+m} - 32 B_m^2 = v_n^2, \tag{21}$$

$$32 B_{m+n}^2 - 32 B_n B_{m+n} v_m - v_m^2 = -v_n^2, \tag{22}$$

$$(B_n, v_n) = 1 \text{ or } 2, \tag{23}$$

$$B_m | B_n \Leftrightarrow m | n, \tag{24}$$

$$v_m | v_n \Leftrightarrow m | n \text{ and } n/m \text{ is odd}, \tag{25}$$

$$v_m | B_n \Leftrightarrow m | n \text{ and } n/m \text{ is even}, \tag{26}$$

$$2 | B_n \Leftrightarrow 2 | n \Leftrightarrow 2 | P_n, \tag{27}$$

$$2 \nmid B_n \Leftrightarrow 2 \nmid n \Leftrightarrow 2 \nmid P_n, \tag{28}$$

$$2 | Q_n \text{ and } 2 | v_n. \tag{29}$$

Moreover, from (12) and (13), it is clear that

$$4 \nmid Q_n \text{ and } 4 \nmid v_n, \tag{30}$$

respectively.

If $d = (m, n)$, then

$$\begin{cases} (P_m, Q_n) = Q_d \text{ if } m/d \text{ is even} \\ (P_m, Q_n) = 1 \text{ otherwise.} \end{cases} \tag{31}$$

Let $m = 2^a m'$, $n = 2^b n'$, m', n' are odd, $a, b \geq 0$, and $d = (m, n)$. Then

$$(V_m, V_n) = \begin{cases} V_d \text{ if } a = b \\ 1 \text{ or } 2 \text{ if } a \neq b. \end{cases} \tag{32}$$

3. Some Theorems and Lemmas

In this section, we shall need some new theorems, lemmas, and corollaries. The following theorem gives us some information about the sum of the squares of balancing numbers. Since it is readily proved by using Binet's formulas, we omit the details.

Theorem 3.1. *Let B_k denotes the k^{th} balancing number. Then*

$$\sum_{k=1}^{n} B_k^2 = \frac{1}{32}(B_{2n+1} - (2n+1)) \tag{33}$$

Hence, we have the following immediate corollary.

Corollary 3.2. *Let n be an odd positive integer. Then*

$$B_n \equiv n (\text{mod} 32). \tag{34}$$

Since the proof of the following lemma is straightfoward induction, we omit the details.

Lemma 3.3. *Let n be an even positive integer. Then*

$$B_n \equiv 3n (\text{mod} 32). \tag{35}$$

Now we can give the similar property for v_n as a result of Corollary 3.2, Lemma 3.3, and the identity (15).

Corollary 3.4. *Let n be a nonnegative integer. Then*

$$v_n \equiv \begin{cases} 2 (\text{mod } 32) \text{ if } n \text{ is even} \\ 6 (\text{mod } 32) \text{ if } n \text{ is odd} \end{cases}. \tag{36}$$

In the equations $x^2 - 8B_n xy - y^2 = \pm 2^r$, replacing x by x^2 and y by y^2, we come across the square terms of balancing sequence, Pell and Pell-Lucas sequences, and the sequence $\{v_n\}$. So, we must state some theorems concerning the square terms of these sequences.

The following theorem is given by Ljunggren [15] and also by Cohn [5].

Theorem 3.5. *If $n \geq 1$, then the equation $P_n = x^2$ has positive solutions $(n, x) = (1, 1)$ or $(7, 13)$.*

We state the following theorem from [24].

Theorem 3.6. *Let $P > 0$ and $Q = -1$. If $U_n = wx^2$ with $w \in \{1, 2, 3, 6\}$, then $n \leq 2$ except when $(P, n, w) = (2, 4, 3)$, $(2, 7, 1)$, $(4, 4, 2)$, $(1, 12, 1)$, $(1, 3, 2)$, $(1, 4, 3)$, $(1, 6, 2)$, or $(24, 4, 3)$.*

The following theorem is given by [19].

Theorem 3.7. *Let $P > 2$ and $Q = 1$. If $u_n = cx^2$ with $c \in \{1, 2, 3, 6\}$ and $n > 3$, then $(P, n, c) = (338, 4, 1)$ or $(3, 6, 1)$.*

The proof of the following theorem can be obtained from Theorem 3.7, but we here give a different proof.

Theorem 3.8. *Let n be a positive integer. There is no balancing number except 1 satisfying the equation $B_n = x^2$.*

Proof. Assume that $B_n = x^2$ for some $x > 0$. Suppose n is even. Then $n = 2k$ for some $k > 0$. By (14), it follows that

$$B_n = B_{2k} = B_k v_k = x^2. \tag{37}$$

Firstly, let k be odd. Then by Corollaries 3.2 and 3.4, it is seen that $B_k \equiv k (\text{mod } 32)$ and $v_k \equiv 6 (\text{mod } 32)$. Substituting these into (37) gives $x^2 \equiv 6k (\text{mod } 32)$, implying that $x^2 \equiv 6k (\text{mod } 8)$. Since k is odd, $k \equiv 1, 3, 5, 7 (\text{mod } 8)$. Hence, we immediately have

$$x^2 \equiv 6k \equiv 2, 6 (\text{mod } 8), \tag{38}$$

which is impossible since $x^2 \equiv 0, 1, 4 (\text{mod } 8)$. Secondly, let k be even. Then by (27), (29), and (23), it is clear that $(B_k, v_k) = 2$. Thus, x is even. Taking $B_k = 2a$ and $v_k = 2b$ with $(a, b) = 1$, we get $x^2 = B_k v_k = 4ab$, implying that $ab = (x/2)^2$. Then $a = u^2$ and $b = v^2$ for some $u, v > 0$. Hence, we have $B_k = 2a^2$. Using the fact that $B_k = P_{2k}/2$ gives $P_{2k} = (2u)^2$. By Theorem 3.5, we obtain $2k = 1$ or 7. But both of them are impossible in integers. Now suppose n is odd. Since $B_n = P_{2n}/2$, we have $P_{2n} = 2x^2$. By (14), it is clear that $P_n Q_n = 2x^2$. Furthermore, by the help of (12), (28), and (29), it can be seen that $(P_n, Q_n) = 1$. Then either

$$P_n = u^2,\ Q_n = 2v^2 \tag{39}$$

or

$$P_n = 2u^2,\ Q_n = v^2 \tag{40}$$

for some $u, v > 0$.

If (39) holds, then by Theorem 3.5, we obtain $n = 1$ or 7. When $n = 1$, $B_1 = 1 = x^2$ and therefore $x = 1$ is a solution. When $n = 7$, there is no integer x such that $B_7 = 40391 = x^2$. If (40) holds, then from (29), we see that v is even. This implies that $4|Q_n$, which is impossible by (30). This completes the proof. □

Theorem 3.9. *There is no positive integer x such that $v_n = x^2$.*

Proof. Assume that $v_n = x^2$ for some $x > 0$. By Corollary 3.4, it follows that $v_n \equiv 2, 6 (\text{mod } 8)$. Hence, $x^2 \equiv 2, 6 (\text{mod } 8)$, which is impossible. This completes the proof. □

Theorem 3.10. *If $n \geq 0$ and $x > 0$ are integers such that $v_n = 2x^2$, then $(n, x) = (0, 1)$.*

Proof. Assume that $v_n = 2x^2$ for some $x > 0$. Clearly, n is not odd, if it were then by Corollary 3.4, we get $2x^2 \equiv 6 (\text{mod } 8)$, which is impossible. So, n is even. Also, by Theorems 2.2 and 2.3, we see that $v_n = Q_{2n}$ and by (16), $Q_{2n} = Q_n^2 - 2$. Hence, we get $v_n = Q_n^2 - 2$. On the other hand, by (12), since $Q_n^2 - 8P_n^2 = 4$, we immediately have $v_n = 8P_n^2 + 2 = 2x^2$, implying that $4P_n^2 + 1 = x^2$. This shows that $x^2 - (2P_n)^2 = 1$. Solving this equation gives $x = 1$. Thus, $n = 0$. This completes the proof. □

Theorem 3.11. *There is no positive integer x such that $B_n = v_m x^2$.*

Proof. Assume that $B_n = v_m x^2$ for some $x > 0$. Since $v_m | B_n$, it follows from (26) that $m|n$ and $n = 2km$ for some $k > 0$. This implies by (14) that

$$B_n = B_{2km} = B_{km} v_{km} = v_m x^2. \tag{41}$$

Let k be odd. Then $B_{km} \frac{v_{km}}{v_m} = x^2$. Clearly, from (23), $d = \left(B_{km}, \frac{v_{km}}{v_m}\right) = 1$ or 2. If $d = 1$, then $B_{km} = a^2$, $v_{km} = v_m b^2$ for some $a, b > 0$. By Theorem 3.8, we have $km = 1$. This yields that $k = 1$, $m = 1$, and therefore $n = 2$. Hence, we conclude that $B_2 = v_1 x^2$, i.e., $6 = 2x^2$, which is impossible in integers. If $d = 2$, then $B_{km} = 2a^2$, $v_{km} = 2v_m b^2$ for some $a, b > 0$. From (29) and (30), since v_m is even and $4 \nmid v_m$, it is seen that $v_{km} = 2v_m b^2$ is impossible.

Now let k be even. Then from (41), we have $\frac{B_{km}}{v_m} v_{km} = x^2$. Using the fact that $\left(\frac{B_{km}}{v_m}, v_{km}\right) = 1$ or 2, we get

$$B_{km} = v_m a^2, \ v_{km} = b^2 \tag{42}$$

or

$$B_{km} = 2v_m a^2, \ v_{km} = 2b^2 \tag{43}$$

for some $a, b > 0$. Assume (42) is satisfied. Since k is even, it follows from Corollary 3.4 that $v_{km} = b^2 \equiv 2 \pmod 8$, which is impossible. Assume (43) is satisfied. Then by Theorem 3.10, we get $km = 0$, implying that $n = 0$, which is impossible. This completes the proof. □

Theorem 3.12. *There is no positive integer x such that $B_n = 2v_m x^2$.*

Proof. Assume that $B_n = 2v_m x^2$ for some $x > 0$. Since $v_m | B_n$, it follows from (26) that $m|n$ and $n = 2km$ for some $k > 0$. This implies from (14) that

$$B_n = B_{2km} = B_{km} v_{km} = 2v_m x^2. \tag{44}$$

Let k be even. Then $\frac{B_{km}}{v_m} v_{km} = 2x^2$. Clearly, from (25), $d = \left(\frac{B_{km}}{v_m}, v_{km}\right) = 1$ or 2. If $d = 1$, then either

$$B_{km} = v_m a^2, \ v_{km} = 2b^2 \tag{45}$$

or

$$B_{km} = 2v_m a^2, \ v_{km} = b^2 \tag{46}$$

for some $a, b > 0$. It is obvious by Theorems 3.11 and 3.9 that both (45) and (46) are impossible. If $d = 2$, then either

$$B_{km} = 2v_m a^2, \ v_{km} = (2b)^2 \tag{47}$$

or

$$B_{km} = v_m (2a)^2, \ v_{km} = 2b^2 \tag{48}$$

for some $a, b > 0$. From (30), since $4 \nmid v_m$, it is seen that (47) is impossible. It is obvious by Theorem 3.11 that (48) is impossible.

Now let k be odd. Then from (44), we have $B_{km}\frac{v_{km}}{v_m} = 2x^2$. Clearly, from (25), $d = \left(B_{km}, \frac{v_{km}}{v_m}\right) = 1$ or 2. If $d = 1$, then

$$B_{km} = a^2,\ v_{km} = 2v_m b^2 \tag{49}$$

or

$$B_{km} = 2a^2,\ v_{km} = v_m b^2 \tag{50}$$

for some $a, b > 0$. If (49) holds, then by Theorem 3.8, it follows that $km = 1$, i.e., $k = 1$ and $m = 1$. This implies that $v_1 = 2v_1 b^2$, which is impossible in integers. It is clear from Theorem 3.7 that (50) is impossible.

If $d = 2$, then

$$B_{km} = 2a^2,\ v_{km} = v_m(2b)^2 \tag{51}$$

or

$$B_{km} = (2a)^2,\ v_{km} = 2v_m b^2 \tag{52}$$

for some $a, b > 0$. (51) is impossible by Theorem 3.7. If (52) holds, then by Theorem 3.8, it follows that $km = 1$, i.e., $B_1 = 1 = (2a)^2$, which is impossible. This completes the proof. $\square$

Theorem 3.13. *(Theorem 2 of* [16]*) The generalized Lucas sequence has at most one non-trivial square-class. Furthermore, if* $P \equiv 2 \pmod 4$, *then we have not non-trivial square-class except* $(v_1, v_2) = (338, 114242)$ *when* $P = 338$, $Q = 1$. *If* $P \equiv 0 \pmod 4$, *then we have not non-trivial square-classes when* $2 \nmid mn$ *or* $2|(m, n)$.

4. Positive Integer Solutions of the Equations $x^2 - 8B_nxy - 2y^2 = \pm 2^r$ in Terms of Balancing Numbers, Pell and Pell-Lucas Numbers, and the Terms of the Sequence $\{v_n\}$

In this section, we determine when the equations $x^2 - 8B_nxy - 2y^2 = \pm 2^r$, $x^2 - 8B_nxy^2 - 2y^4 = \pm 2^r$, and $x^4 - 8B_nx^2y - 2y^2 = \pm 2^r$ have positive integer solutions under the assumptions that $n, r \geq 0$. Moreover, we give all positive integer solutions of the equations above.

We omit the proof of the following theorem, as it is based a straightforward induction.

Theorem 4.1. *Let $k \geq 0$ be an integer. Then all nonnegative integer solutions of the equation $u^2 - 2v^2 = 2^k$ are given by*

$$(u, v) = \begin{cases} (2^{\frac{k-2}{2}} v_m,\ 2^{\frac{k+2}{2}} B_m) \text{ if } k \text{ is even} \\ (2^{\frac{k+1}{2}} P_{2m+1},\ 2^{\frac{k-3}{2}} Q_{2m+1}) \text{ if } k \text{ is odd} \end{cases}$$

with $m \geq 0$ and all nonnegative integer solutions of the equation $u^2 - 2v^2 = -2^k$ are given by

$$(u, v) = \begin{cases} (2^{\frac{k-2}{2}} Q_{2m+1},\ 2^{\frac{k}{2}} P_{2m+1}) \text{ if } k \text{ is even} \\ (2^{\frac{k+3}{2}} B_m,\ 2^{\frac{k-3}{2}} v_m) \text{ if } k \text{ is odd} \end{cases}$$

with $m \geq 0$.

Theorem 4.2. *If k is even, then all positive integer solutions of the equation $x^2 - 8B_n xy - 2y^2 = 2^k$ are given by $(x, y) = (2^{\frac{k}{2}} \frac{v_{n+m}}{v_n}, 2^{\frac{k+4}{2}} \frac{B_m}{v_n})$ with $m \geq 1$, $n|m$ and $m = 2rn$ for some $r > 0$. If k is odd, then the equation $x^2 - 8B_n xy - 2y^2 = 2^k$ has positive integer solutions only when $n = 0$ and the solutions are given by $(x, y) = (2^{\frac{k+1}{2}} P_{2m+1}, 2^{\frac{k-3}{2}} Q_{2m+1})$ with $m \geq 0$.*

Proof. Assume that $x^2 - 8B_n xy - 2y^2 = 2^k$ for some $x, y > 0$. Multiplying both sides of the equation by 4 and completing the square give $(2x - 8B_n y)^2 - (64B_n^2 + 8)y^2 = 2^{k+2}$. It is clear from (13) that $64B_n^2 + 8 = 2v_n^2$. Hence, the preceding equation becomes $(2x - 8B_n y)^2 - 2(v_n y)^2 = 2^{k+2}$. Let k be even. Then by Theorem 4.1, we obtain $|2x - 8B_n y| = 2^{\frac{k}{2}} v_m$ and $v_n y = 2^{\frac{k+4}{2}} B_m$. Since $4 \nmid v_n$ and v_n is even, it follows that $(4, v_n) = 2$ and therefore $\frac{v_n}{2} y = 2^{\frac{k+2}{2}} B_m$. It can be easily seen that $(\frac{v_n}{2}, 2^{\frac{k+2}{2}}) = 1$. Thus, we get $\frac{v_n}{2} | B_m$, that is $\frac{v_n}{2} | 2\frac{B_m}{2}$ for even m. Since $(\frac{v_n}{2}, 2) = 1$, $\frac{v_n}{2} | \frac{B_m}{2}$, implying that $v_n | B_m$. Therefore, we get from (26) that $n|m$ and $m = 2rn$ for some $r > 0$. Hence, we conclude that $y = 2^{\frac{k+4}{2}} \frac{B_m}{v_n}$. Suppose that $2x - 8B_n y = 2^{\frac{k}{2}} v_m$. Substituting the value of y into the preceding equation gives $x = 2^{\frac{k}{2}} \frac{v_m v_n + 32 B_m B_n}{2v_n}$. This implies from (17) that $x = 2^{\frac{k}{2}} \frac{v_{m+n}}{v_n}$. Now suppose that $2x - 8B_n y = -2^{\frac{k}{2}} v_m$. In a similar manner, we readily obtain $x = 2^{\frac{k}{2}} \frac{32 B_m B_n - v_m v_n}{2v_n}$. This gives from (18) $x = -2^{\frac{k}{2}} \frac{v_{m-n}}{v_n}$. But in this case, x is negative and so we omit it. As a consequence, we get $(x, y) = (2^{\frac{k}{2}} \frac{v_{n+m}}{v_n}, 2^{\frac{k+4}{2}} \frac{B_m}{v_n})$. Now let k be odd. Then by Theorem 4.1, we have $v_n y = 2^{\frac{k-1}{2}} Q_{2m+1}$. Since $v_n = Q_{2n}$ and $v_n | v_n y$, it follows that $Q_{2n} | 2^{\frac{k-1}{2}} Q_{2m+1}$, implying that $Q_{2n} | 2^{\frac{k+1}{2}} \frac{Q_{2m+1}}{2}$.

It can be easily seen from (32) that $(Q_{2n},\ Q_{2m+1}/2) = 1$. Hence, $Q_{2n}|2^{\frac{k+1}{2}}$ and this is possible only when $n = 0$. Thus, the main equation $x^2 - 8B_nxy - 2y^2 = 2^k$ turns into the equation $x^2 - 2y^2 = 2^k$, whose solutions are $(x, y) = (2^{\frac{k+1}{2}}P_{2m+1},\ 2^{\frac{k-3}{2}}Q_{2m+1})$ by Theorem 4.1.

Conversely, if k is even and

$$(x, y) = (2^{\frac{k}{2}}\frac{v_{n+m}}{v_n}, 2^{\frac{k+4}{2}}\frac{B_m}{v_n})$$

with $m \geq 1$, $n|m$ and $m = 2rn$ for some $r > 0$, then by (21), it follows that $x^2 - 8B_nxy - 2y^2 = 2^k$. And if k is odd and $(x, y) = (2^{\frac{k+1}{2}}P_{2m+1}, 2^{\frac{k-3}{2}}Q_{2m+1})$ with $m \geq 0$, then $x^2 - 8B_nxy - 2y^2 = 2^k$ with $n = 0$. This completes the proof. □

Theorem 4.3. *If k is even, then the equation $x^2 - 8B_nxy - 2y^2 = -2^k$ has positive integer solutions only when $n = 0$ and the solutions are given by $(x, y) = (2^{\frac{k-2}{2}}Q_{2m+1},\ 2^{\frac{k}{2}}P_{2m+1})$ with $m \geq 0$. If k is odd, then all positive integer solutions of the equation $x^2 - 8B_nxy - 2y^2 = -2^k$ are given by $(x, y) = (2^{\frac{k+5}{2}}\frac{B_{n+m}}{v_n},\ 2^{\frac{k-1}{2}}\frac{v_m}{v_n})$ with $m \geq 1$, $n|m$ and $m = (2r + 1)n$ for some $r > 0$.*

Proof. When k is even, the method is similar to that used in Theorem 4.2 for the case when k is odd. So, we immediately have $v_ny = 2^{\frac{k+2}{2}}P_{2m+1}$. Since $4 \nmid v_n$ by (30) and $v_ny = 2^{\frac{k+2}{2}}P_{2m+1}$, it clearly follows that $v_n|2^{\frac{k+2}{2}}P_{2m+1}$. Since $v_n = Q_{2n}$, it can be easily seen from (31) that $(v_n, P_{2m+1}) = 1$. So, $v_n|2^{\frac{k+2}{2}}$ and this is possible only when $n = 0$. Hence, the main equation $x^2 - 8B_nxy - 2y^2 = -2^k$ turns into the equation $x^2 - 2y^2 = -2^k$, whose solutions are $(x, y) = (2^{\frac{k-2}{2}}Q_{2m+1},\ 2^{\frac{k}{2}}P_{2m+1})$ by Theorem 4.1. Now let k be odd. Multiplying both sides of the equation $x^2 - 8B_nxy - 2y^2 = -2^k$ by 4 and completing the square give $(2x - 8B_ny)^2 - (64B_n^2 + 8)y^2 = -2^{k+2}$. Using the fact that $64B_n^2 + 8 = 2v_n^2$ by (13), the previous equation becomes $(2x - 8B_ny)^2 - 2(v_ny) = -2^{k+2}$. Then by Theorem 4.1, we obtain $|2x - 8B_ny| = 2^{\frac{k+5}{2}}B_m$ and $v_ny = 2^{\frac{k-1}{2}}v_m$. Since $4 \nmid v_n$ and v_n is even, it follows that $(4, v_n) = 2$ and therefore $\frac{v_n}{2}y = 2^{\frac{k-1}{2}}\frac{v_m}{2}$. It can be easily seen that $(\frac{v_n}{2}, 2^{\frac{k-1}{2}}) = 1$. Thus, we get $\frac{v_n}{2}|\frac{v_m}{2}$, that is $v_n|v_m$. This implies from (25) that $n|m$ and $m = (2r + 1)n$ for some $r > 0$. Hence, we get $y = 2^{\frac{k-1}{2}}\frac{v_m}{v_n}$. Assume first that $2x - 8B_ny = 2^{\frac{k+5}{2}}B_m$. Substituting the value of y into the previous equation, we have

$x = 2^{\frac{k+5}{2}} \frac{B_n v_m + B_m v_n}{2v_n}$. Then by (19), we conclude that $x = 2^{\frac{k+5}{2}} \frac{B_{m+n}}{v_n}$. Now assume that $2x - 8B_n y = -2^{\frac{k+5}{2}} B_m$. In a similar manner, we get $x = 2^{\frac{k+3}{2}} \frac{B_n v_m - B_m v_n}{2v_n}$. By (20), we conclude that $x = 2^{\frac{k+3}{2}} \frac{B_{n-m}}{v_n}$. But in this case since $n - m < 0$, it follows from (9) that $B_{n-m} < 0$ and therefore we see that x is negative. So, we omit it. Conversely, if k is even and $(x, y) = (2^{\frac{k-2}{2}} Q_{2m+1}, 2^{\frac{k}{2}} P_{2m+1})$ with $m \geq 0$, then by (12), $x^2 - 8B_n xy - 2y^2 = -2^k$ with $n = 0$. And if k is odd and $(x, y) = (2^{\frac{k}{2}} \frac{B_{n+m}}{v_n}, 2^{\frac{k+4}{2}} \frac{v_m}{v_n})$ with $m \geq 1$, $n|m$ and $m = (2r + 1)n$ for some $r > 0$, then by (22), $x^2 - 8B_n xy - 2y^2 = -2^k$. This completes the proof. □

Now we consider the equations $x^2 - 8B_n xy^2 - 2y^4 = \pm 2^k$ and $x^4 - 8B_n x^2 y - 2y^2 = \pm 2^k$, respectively.

Theorem 4.4. *If* $k \equiv 0, 2, 3 \pmod 4$, *then the equation* $x^2 - 8B_n xy^2 - 2y^4 = 2^k$ *has no solutions* x *and* y. *If* $k \equiv 1 \pmod 4$, *then the equation* $x^2 - 8B_n xy^2 - 2y^4 = 2^k$ *has positive integer solutions only when* $n = 0$ *and the solution is given by* $(x, y) = (2^{\frac{k+1}{2}}, 2^{\frac{k-1}{4}})$.

Proof. Firstly, assume that k is even in $x^2 - 8B_n xy^2 - 2y^4 = 2^k$. Then by Theorem 4.2, it follows that $(x, y^2) = (2^{\frac{k}{2}} \frac{v_{n+m}}{v_n}, 2^{\frac{k+4}{2}} \frac{B_m}{v_n})$ with $m \geq 1$, $n|m$ and $m = 2rn$ for some $r > 0$. Hence, we obtain $y^2 = 2^{\frac{k+4}{2}} \frac{B_m}{v_n}$. Now we divide the proof into two cases.

Case 1 : Let $k \equiv 0 \pmod 4$. Then the equation $y^2 = 2^{\frac{k+4}{2}} \frac{B_m}{v_n}$ clearly follows that $B_m = v_n u^2$, which is impossible by Theorem 3.11.

Case 2 : Let $k \equiv 2 \pmod 4$. Then the equation $y^2 = 2^{\frac{k+4}{2}} \frac{B_m}{v_n}$ yields that $B_m = 2v_n u^2$, which is impossible by Theorem 3.12.

Secondly, assume that k is odd. Then by Theorem 4.2, it follows that $(x, y^2) = (2^{\frac{k+1}{2}} P_{2m+1}, 2^{\frac{k-3}{2}} Q_{2m+1})$ with $m \geq 0$. This shows that $y^2 = 2^{\frac{k-3}{2}} Q_{2m+1}$. Again dividing the remainder of the proof into two cases, we have

Case 1 : Let $k \equiv 3 \pmod 4$. Then we obtain $Q_{2m+1} = u^2$ for some $u > 0$. By (29), since Q_{2m+1} is even, it follows that u is even and therefore $4|Q_{2m+1}$, which is impossible by (30).

Case 2 : Let $k \equiv 1 \pmod 4$. Then we have $Q_{2m+1} = 2u^2$ for some $u > 0$. By Theorem 3.6, we get $m = 0$. Thus $y^2 = 2^{\frac{k-1}{2}}$, implying that $y = 2^{\frac{k-1}{4}}$ and $x = 2^{\frac{k+1}{2}}$. This completes the proof. □

Theorem 4.5. *If* $k \equiv 2, 3 \pmod 4$, *then the equation* $x^2 - 8B_n xy^2 - 2y^4 = -2^k$ *has no solutions* x *and* y. *If* $k \equiv 0 \pmod 4$, *then all* *positive integer solutions of the equation* $x^2-8B_nxy^2-2y^4=-2^k$ *are given by* $(x,y)=(2^{\frac{k}{2}},2^{\frac{k}{4}})$ *or* $(x,y)=(239\cdot 2^{\frac{k}{2}},13\cdot 2^{\frac{k}{4}})$. *If* $k\equiv 1(\text{mod } 4)$, *then there is only one positive integer solution of the equation* $x^2-8B_nxy^2-2y^4=-2^k$ *given by* $(x,y)=(2^{\frac{k+5}{2}}B_n,2^{\frac{k-1}{4}})$.

Proof. Assume that k is even in $x^2-8B_nxy^2-2y^4=-2^k$. Then by Theorem 4.3, it follows that $(x,y^2)=(2^{\frac{k-2}{2}}Q_{2m+1},2^{\frac{k}{2}}P_{2m+1})$ with $m\geq 0$. Hence, we obtain $y^2=2^{\frac{k}{2}}P_{2m+1}$. Dividing the proof into two cases, we have

Case 1 : Let $k\equiv 0(\text{mod } 4)$. Then from the equation $y^2=2^{\frac{k}{2}}P_{2m+1}$, we obtain $P_{2m+1}=u^2$ for some $u>0$. By Theorem 3.5, we get $2m+1=1$ or $2m+1=7$. This implies that $m=0$ or $m=3$. If $m=0$, then we immediately have $x=2^{\frac{k}{2}}$ and $y=2^{\frac{k}{4}}$. If $m=3$, then we obtain $x=239\cdot 2^{\frac{k}{2}}$ and $y=13\cdot 2^{\frac{k}{4}}$.

Case 2 : Let $k\equiv 2(\text{mod } 4)$. Then the equation $y^2=2^{\frac{k}{2}}P_{2m+1}$ becomes $P_{2m+1}=2u^2$, which is impossible since $2\nmid P_{2m+1}$ by (28).

Now assume that k is odd. Then by Theorem 4.3, we have

$$(x,y^2)=(2^{\frac{k+5}{2}}\frac{B_{n+m}}{v_n},2^{\frac{k-1}{2}}\frac{v_m}{v_n})$$

with $m\geq 1$, $n|m$ and $m=(2r+1)n$ for some $r>0$. Hence, we obtain $y^2=2^{\frac{k-1}{2}}\frac{v_m}{v_n}$. Now we divide the remainder of the proof into two cases.

Case 1 : Let $k\equiv 1(\text{mod } 4)$. Then the equation $y^2=2^{\frac{k-1}{2}}\frac{v_m}{v_n}$ implies that $v_m=v_nu^2$ for some $u>0$. By Theorem 3.13, this is possible only when $n=m$. Hence, we get $x=2^{\frac{k+5}{2}}\frac{B_{2n}}{v_n}$. Also using (14) for the value of x gives that $x=2^{\frac{k+5}{2}}B_n$. Thus, we conclude that $(x,y)=(2^{\frac{k+5}{2}}B_n,2^{\frac{k-1}{4}})$.

Case 2 : Let $k\equiv 3(\text{mod } 4)$. So, we immediately have $v_{n+m}=2v_nu^2$ for some $u>0$. Since v_n is even by (29), it is clear that $4|v_{n+m}$. But this is impossible by (30). This completes the proof. □

Theorem 4.6. *If* $k\equiv 0,1,2(\text{mod } 4)$, *then the equation* $x^4-8B_nx^2y-2y^2=2^k$ *has no positive integer solutions* x *and* y. *If* $k\equiv 3(\text{mod } 4)$, *then all positive integer solutions of the equation* $x^4-8B_nx^2y-2y^2=2^k$ *are given by* $(x,y)=(2^{\frac{k+1}{4}},2^{\frac{k-1}{2}})$ *or* $(x,y)=(13\cdot 2^{\frac{k+1}{4}},239\cdot 2^{\frac{k-1}{2}})$.

Proof. Assume that $x^4-8B_nx^2y-2y^2=2^k$ for some positive integers x and y. If k is even, then by Theorem 4.2, we have $(x^2,y)=(2^{\frac{k}{2}}\frac{v_{n+m}}{v_n},2^{\frac{k+4}{2}}\frac{B_m}{v_n})$ with $m\geq 1$, $n|m$ and m/n is even. Hence, we get $x^2=2^{\frac{k}{2}}\frac{v_{n+m}}{v_n}$.

Case 1 : Let $k \equiv 0 \pmod 4$. We readily obtain from $x^2 = 2^{\frac{k}{2}} \frac{v_{n+m}}{v_n}$ that $v_{n+m} = v_n u^2$ for some $u > 0$. By Theorem 3.13, this is possible only when $n + m = n$, implying that $m = 0$, which contradicts the fact that $m \geq 1$.

Case 2 : Let $k \equiv 2 \pmod 4$. So, we immediately have $v_{n+m} = 2v_n u^2$ for some $u > 0$. Since v_n is even by (29), we see that $4 | v_{n+m}$, which is impossible by (30).

If k is odd, then by Theorem 4.2, we have

$$(x^2, y) = (2^{\frac{k+1}{2}} P_{2m+1}, 2^{\frac{k-3}{2}} Q_{2m+1})$$

with $m \geq 0$. This implies that $x^2 = 2^{\frac{k+1}{2}} P_{2m+1}$.

Case 1 : Let $k \equiv 1 \pmod 4$. Then form $x^2 = 2^{\frac{k+1}{2}} P_{2m+1}$, we obtain $P_{2m+1} = 2u^2$, which is impossible since $2 \nmid P_{2m+1}$ by (28).

Case 2 : Let $k \equiv 3 \pmod 4$. Then the equation $x^2 = 2^{\frac{k+1}{2}} P_{2m+1}$ gives that $P_{2m+1} = u^2$ for some $u > 0$. By Theorem 3.5, we get $2m + 1 = 1$ or $2m + 1 = 7$, implying that $m = 0$ or $m = 3$. Substituting these values of m into $(x^2, y) = (2^{\frac{k+1}{2}} P_{2m+1}, 2^{\frac{k-3}{2}} Q_{2m+1})$, we conclude that $(x, y) = (2^{\frac{k+1}{4}}, 2^{\frac{k-1}{2}})$ or $(x, y) = (13 \cdot 2^{\frac{k+1}{4}}, 239 \cdot 2^{\frac{k-1}{2}})$. This completes the proof. □

Theorem 4.7. *If $k \equiv 1, 2, 3 \pmod 4$, then the equation $x^4 - 8B_n x^2 y - 2y^2 = -2^k$ has no positive integer solutions x and y. If $k \equiv 0 \pmod 4$, then there is only one positive integer solution of the equation $x^4 - 8B_n x^2 y - 2y^2 = -2^k$ given by $(x, y) = (2^{\frac{k}{4}}, 2^{\frac{k}{2}})$.*

Proof. Assume that $x^4 - 8B_n x^2 y - 2y^2 = -2^k$ for some positive integers x and y. If k is even, then by Theorem 4.3, it follows that $(x^2, y) = (2^{\frac{k-2}{2}} Q_{2m+1}, 2^{\frac{k}{2}} P_{2m+1})$ with $m \geq 0$. Hence, we get $x^2 = 2^{\frac{k-2}{2}} Q_{2m+1}$.

Case 1 : Let $k \equiv 0 \pmod 4$. Hence, we immediately have from $x^2 = 2^{\frac{k-2}{2}} Q_{2m+1}$ that $Q_{2m+1} = 2u^2$. By Theorem 3.6, we get $m = 0$. This yields that $(x, y) = (2^{\frac{k}{4}}, 2^{\frac{k}{2}})$.

Case 2 : Let $k \equiv 2 \pmod 4$. Hence, we readily obtain $Q_{2m+1} = u^2$ for some $u > 0$. Since Q_{2m+1} is even by (29), it is clear that u is even and therefore $4 | Q_{2m+1}$, which is impossible by (30).

If k is odd, then by Theorem 4.3, it follows that $(x^2, y) = (2^{\frac{k+5}{2}} \frac{B_{n+m}}{v_n}, 2^{\frac{k-1}{2}} \frac{v_m}{v_n})$ with $m \geq 1$, $n | m$ and $m = (2r + 1)n$ for some $r > 0$. This implies that $x^2 = 2^{\frac{k+5}{2}} \frac{B_{n+m}}{v_n}$.

Case 1 : Let $k \equiv 1 (\mathrm{mod}\ 4)$. Then from $2^{\frac{k+5}{2}} \frac{B_{n+m}}{v_n}$, we obtain $B_{n+m} = 2v_n u^2$, which is impossible by Theorem 3.12.

Case 2 : Let $k \equiv 3 (\mathrm{mod}\ 4)$. Then we have $B_{n+m} = v_n u^2$, which is also impossible by Theorem 3.11. This completes the proof. □

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Olcay Karaatlı is a research assistant and also a Ph.D student of Sakarya University, Turkey. He got his master from Sakarya University. His research interest is number theory, especially generalized Fibonacci and Lucas sequences and Diophantine equations.

Refik Keskin is a professor of Sakarya University, Turkey. He got his Ph. D from Karadeniz Technical University. His research interest is number theory, especially generalized Fibonacci and Lucas sequences and Diophantine equations.

Huilin Zhu is an assisant professor at Xiamen University, China. He got his Ph. D from Wuhan University and had postdoctoral experience at Shandong University. His research interests are number theory and cryptography, especially Diophantine equations such as families of elliptic curves, generalized

Lebesgue-Nagell equations, ternary pure exponential Diophantine equations and the generalized Fermat conjecture.

(Karaatlı, Keskin) Department of Mathematics, Sakarya University, Sakarya, Turkey

(Zhu) School of Mathematical Sciences, Xiamen University, Xiamen, China

E-mail address, Karaatlı: okaraatli@sakarya.edu.tr

E-mail address, Keskin: rkeskin@sakarya.edu.tr

E-mail address, Zhu: hlzhu@xmu.edu.cn

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THE NUMBER OF QUADRATIC SUBFIELDS

JEAN B NGANOU

Abstract. We use elementary field theory to compute the number of intermediate fields of degree 2 of any finite extension of fields of characteristic not 2. This leads to necessary and sufficient conditions for such extensions to have no intermediate fields of degree 2, or to have a unique intermediate field of degree 2. We obtain several applications including the number of quadratic subfields of cyclotomic extensions.

1. Introduction

The Fundamental Theorem of Galois Theory is by far the most important connection between field and group theory. It asserts that for finite Galois extensions, there is a one-to-one correspondence between the intermediate fields and the subgroups of the Galois group. Moreover, under this correspondence subgroups of index n correspond to intermediate fields of degree n. Therefore, this produces an indirect method for finding the number of intermediate fields of a given degree. One would use group-theoretic facts to find the number of subgroups of the correspoding index. Unfortunately, for non Galois extensions, there is no obvious relationship between the intermediate fields of the extensions and the subgroups of their Galois groups.

The aim of this note is to find a field-theoretic formula for the number of intermediate fields of degree 2 for an arbitrary finite extension K/F with F of characteristic not 2. Such a formula would allow us to answer questions about the existence of subfields of degree 2 without having to compute the Galois group, even for non Galois extensions. Our strategy consists in the following steps: starting

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with a finite extension K/F, we create a new extension Q/F that is finite Galois such that K/F and Q/F have the same intermediate fields of degree 2. We can now apply the Fundamental Theorem of Galois Theory to compute the number of subfields of degree 2 of the extension Q/F, and therefore those of K/F. We also obtain that the extension Q is the fixed field of the subgroup of squares of the Galois group of K/F. In all that follows, F will denote a field of characteristic different from 2 and K/F a finite extension(K is a finite dimensional vector space over F). In addition, for every integer $k \geq 2$, C_k shall denote the cyclic group of order k.

Given a field extension K/F, the Galois group of K/F is denoted by $\mathcal{G}(K/F)$, and is the set of autormorphisms of K that fix every element in F. Given a subgroup H of $\mathcal{G}(K/F)$, the fixed field of H is the intermediate field of K/F defined by $\mathcal{F}(H) := \{a \in K : \sigma(a) = a, \text{ for all } \sigma \in H\}$. An extension K/F is called algebraic if every element of K is a solution to a polynomial equation with coefficients in F. A Galois extension is any algebraic exetnsion K/F satisfying $\mathcal{F}(\mathcal{G}(K/F)) = F$. For the convenience of the reader, we recall the Fundamental Theorem of Galois Theory.

Let K/F be a finite Galois extension with Galois group G. If E is an intermediate field of K/F, let $\mathcal{G}(K)$ denote $\mathcal{G}(K/E)$. Then $\mathcal{F}$ is a bijection from the the subgroups of G to the intermediate fields, with inverse $\mathcal{G}$ such that for every E, $[E : F] = [G : \mathcal{G}(E)]$ and $[K : E] = |\mathcal{G}(E)|$.

We also recall the following result about the number of subgroups of index 2 in any group. The subgroup of squares of a group G is denoted by G^2, and is the subgroup of G generated by $\{g^2 : g \in G\}$. It is easy to see that $G^2 = \{g_1^2 g_2^2 \cdots g_n^2 : g_i \in G, n \geq 1\}$. It is known that, every arbitrary finite group G has exactly $[G : G^2] - 1$ subgroups of index 2 [4, Corollary 1].

We hope the article is accessible to readers with limited background.

2. Number of intermediate fields of degree 2

Let $Q(K/F) := F(\{\alpha \in K : \alpha^2 \in F\})$, then $F \subseteq Q(K/F) \subseteq K$. Since K/F is finite, there exists a subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq K$ with $\alpha_i^2 \in F$ such that $\alpha_{i+1} \notin F(\alpha_1, \alpha_2, \ldots, \alpha_i)$ for all $i \leq n - 1$ and $Q(K/F) = F(\alpha_1, \alpha_2, \ldots, \alpha_n)$. When it is clear what extension K/F is being considered, we will simply write Q for $Q(K/F)$. We will call

$Q(K/F)$ the *quadratic closure* of F in K, or simply the *quadratic closure* of the extension K/F.
Let E be an intermediate field of K/F such that $[E : F] = 2$, then since $Char(F) \neq 2$, there exists $\beta \in E$ with $\beta^2 \in F$ such $E = F(\beta)$. Hence, $E \subseteq Q$ and therefore, every intermediate field of K/F of degree 2 is an intermediate field of Q/F of degree 2. The converse is clearly true. Therefore, intermediate fields of K/F of degree 2 and intermediate fields of Q/F of degree 2 are the same. On the other hand, note that Q is the splitting field of $(x^2 - \alpha_1^2)(x^2 - \alpha_2^2) \cdots (x^2 - \alpha_n^2)$ over F. We have obtained the following key fact about the extension Q/F, which shall play a central role in the entire article.

Theorem 2.1. *For every finite extension K/F, the extension Q/F is a finite Galois extension. Furthermore, K/F and Q/F have the same intermediate subfields of degree 2 over F.*

The Galois group of Q/F is elementary Abelian of exponent 2 as we prove next.

Theorem 2.2. *Let $G = \mathcal{G}(Q/F)$. Then $G \cong C_2 \times C_2 \times \cdots \times C_2$.*

Proof. To see this, first as observed above, there exists a subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq K$ with $\alpha_i^2 \in F$ such that $\alpha_{i+1} \notin F(\alpha_1, \alpha_2, \ldots, \alpha_i)$ for all $i \leq n-1$ and $Q = F(\alpha_1, \alpha_2, \ldots, \alpha_n)$. It follows from the tower formula that $[Q : F] = 2^n$, and the Fundamental Theorem of Galois Theory that $|G| = 2^n$. In addition, for every $\sigma \in G$, $\sigma(\alpha_i) = \pm\alpha_i$ for all i and consequently $\sigma^2 = id$. It follows from a well-known exercise that G is Abelian, and since G has exponent 2, by the Fundamental Theorem of Finite Abelian groups, $G \cong C_2 \times C_2 \times \cdots \times C_2$. □

Appealing to the Fundamental Theorem of Galois Theory, we can count the number of intermediate fields of Q/F of degree 2 by counting the number of subgroups of index 2 in $\mathcal{G}(Q/F)$.

Corollary 2.3. *Let K/F be finite extension with quadratic closure Q. Then, Q/F has exactly $[Q : F] - 1$ intermediate fields of degree 2.*

Proof. From Theorem 2.2, we have $\mathcal{G}(Q/F) \cong C_2 \times C_2 \times \cdots \times C_2$, where C_2 is the cyclic group of order 2. Thus, $\mathcal{G}(Q/F)^2$ is the trivial group and by [4, Corollary 1], $\mathcal{G}(Q/F)^2$ has $|\mathcal{G}(Q/F)| - 1$ subgroups of index 2. Now, it follows from Fundamental Theorem of Galois Theory, that Q/F has $[Q : F] - 1$ subfields of degree 2. □

Since K/F and Q/F have the same intermediate fields of degree 2 (Theorem 2.1), we deduce;

Corollary 2.4. *Every finite extension* K/F *has exactly* $[Q : F] - 1$ *intermediate fields of degree* 2 *where* Q *is the quadratic closure of* K/F.

Corollary 2.5. *A finite extension* K/F *has a unique intermediate field of degree* 2 *if and only if* $[Q : F] = 2$.

One can apply the Fundamental Theorem of Galois Theory and the results in [4] about subgroups of index 2 to obtain the following.

Theorem 2.6. *For every finite Galois extension* K/F *with Galois group* G *and quadratic closure* Q,

$$Q = \mathcal{F}(G^2) \text{ and } \mathcal{G}(K/Q) = G^2$$

3. Applications

We can apply the results above to compute the number of quadratic subfields in some typical extensions. The quadratic closure Q/F of an extension as introduced here could be difficult to compute, but in many cases as in the following Example, it is quite simple and illustrative.

Example 3.1. Consider the extension $\mathbb{Q}(\sqrt{2}, \sqrt[4]{3})/\mathbb{Q}$, which is not Galois. It is easy to see that $[\mathbb{Q}(\sqrt{2}, \sqrt[4]{3}) : \mathbb{Q}] = 8$. It is also clear that $\sqrt{2}, \sqrt{3} \in Q(\mathbb{Q}(\sqrt{2}, \sqrt[4]{3})/\mathbb{Q})$ and $\sqrt[4]{3} \notin Q(\mathbb{Q}(\sqrt{2}, \sqrt[4]{3})/\mathbb{Q})$. Hence $Q(\mathbb{Q}(\sqrt{2}, \sqrt[4]{3})/\mathbb{Q}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ and $[Q(\mathbb{Q}(\sqrt{2}, \sqrt[4]{3})/\mathbb{Q}) : \mathbb{Q}] = 4$. Therefore, $\mathbb{Q}(\sqrt{2}, \sqrt[4]{3})/\mathbb{Q}$ has three intermediate fields of degree 2.

The next example justifies why the characteristic assumption on the base field cannot be dropped.

Example 3.2. Let $k = \mathbb{F}_2(t)$ be the field of rational functions in t over the Galois field of two elements, $K = k(x, y)$ be the field of rational functions in two variables over k and $F = k(x^2, y^2)$. Then for each $a \in k$, there is a degree 2 field extension $L_a = F(x + ay)$ of F. It is elementary to see $L_a = L_b$ if and only if $a = b$. Thus, since $|k|$ is infinite, there are infinitely many degree 2 subextensions of K/F even though $[K : F] = 4$ is finite.

As another application, we investigate degree 2 subfields of the cyclotomic extensions. A simple and complete treatment of cyclotomic extensions can be found in [3, §7]. Recall that for $n \geq 1$, the nth cyclotomic extension of $\mathbb{Q}$ is $\mathbb{Q}_n := \mathbb{Q}(\omega)$ where ω is a primitive nth root of unity. It is well known that $\mathbb{Q}_n/\mathbb{Q}$ is Galois and $\mathcal{G}(\mathbb{Q}_n/\mathbb{Q}) \cong U(n)$, the group of units of the ring $\mathbb{Z}_n$ of integers modulo n. The order of $U(n)$ is denoted by $\phi(n)$ and is equal to the number of positive integers less than or equal to n that are relatively prime to n.

Example 3.3. Let $N_2(\mathbb{Q}_n/\mathbb{Q})$ denote the exact number of degree 2 intermediate fields of $\mathbb{Q}_n/\mathbb{Q}$. Then we have the following formulae:

1. For every nonnegative integer r

$$N_2(\mathbb{Q}_{2^r}/\mathbb{Q}) = \begin{cases} 0 & \text{if } r = 0, 1 \\ 1 & \text{if } r = 2 \\ 3 & \text{if } r \geq 3 \end{cases}$$

2. If $n = 2^r p_1^{t_1} p_2^{t_2} \cdots p_s^{t_s}$ where $r \geq 0$, $s \geq 1$ and $p_1, p_2, \cdots, p_s$ are distinct odd primes.

$$N_2(\mathbb{Q}_n/\mathbb{Q}) = \begin{cases} 2^s - 1 & \text{if } r = 0, 1 \\ 2^{s+1} - 1 & \text{if } r = 2 \\ 2^{s+2} - 1 & \text{if } r \geq 3 \end{cases}$$

Recall that $\mathcal{G}(\mathbb{Q}_n/\mathbb{Q}) \cong U(n)$ (see for instance [3, Corollary 7.8]. So, by Corollary 2.3, $N_2(\mathbb{Q}_n/\mathbb{Q}) = [Q(\mathbb{Q}_n/\mathbb{Q}) : \mathbb{Q}] - 1 = [U(n) : U(n)^2] - 1$. Therefore, we need to compute $[U(n) : U(n)^2]$ in each case. We will use the decomposition of the U-groups into cyclic groups as found in [1, pp159-160] or [2]. We will also use the facts that if C is a cyclic group of order m, it follows from [1, Theorem. 4.2] that C^2 is cyclic of order $m/gcd(2, m)$; and that $(G_1 \oplus G_2)^2 = G_1^2 \oplus G_2^2$ for every groups G_1, G_2 [4, Theorem 4].
1. First note that the case $r = 0$ is obvious as $\mathbb{Q}_1 = \mathbb{Q}$. On the other hand, we have $U(2) \cong \{0\}$, $U(4) \cong C_2$, $U(2^r) \cong C_2 \oplus C_{2^{r-2}}$ for $r \geq 3$. So $U(2)^2 \cong U(4)^2 \cong \{0\}$, and $U(2^r)^2 \cong C_{2^{r-3}}$ for $r \geq 3$. Hence $[U(2) : U(2)^2] = 1$ and $[U(4) : U(4)^2] = 2$ and for $r \geq 3$, $[U(2^r) : U(2^r)^2] = 2^{r-1}/2^{r-3} = 4$. Therefore the formula is justified.
2. First assume $r \geq 3$, then $U(n) \cong U(2^r) \oplus U(p_1^{t_1}) \oplus \cdots \oplus U(p_s^{t_s}) \cong C_2 \oplus C_{2^{r-2}} \oplus C_{p_1^{t_1} - p_1^{t_1 - 1}} \oplus \cdots \oplus C_{p_s^{t_s} - p_s^{t_s - 1}}$. Hence, $U(n)^2 \cong C_{2^{r-3}} \oplus C_{(p_1^{t_1} - p_1^{t_1-1})/2} \oplus \cdots \oplus C_{(p_s^{t_s} - p_s^{t_s-1})/2}$. Thus, $|U(n)^2| = \phi(n)/2^{s+2}$ and $[U(n) : U(n)^2] = 2^{s+2}$. For $r = 2$, as above $U(n) \cong C_2 \oplus C_{p_1^{t_1} - p_1^{t_1-1}} \oplus \cdots \oplus C_{p_s^{t_s} - p_s^{t_s-1}}$, so $U(n)^2 \cong C_{(p_1^{t_1} - p_1^{t_1-1})/2} \oplus \cdots \oplus C_{(p_s^{t_s} - p_s^{t_s-1})/2}$. So,

$|U(n)^2| = \phi(n)/2^{s+1}$ and $[U(n) : U(n)^2] = 2^{s+1}$.
We leave the cases $r = 0, 1$ as a simple verification exercise.

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Jean B Nganou earned his PhD in Mathematics from New Mexico State University, and is currently an Instructor in Mathematics at the University of Oregon. Dr. Nganou's interests include algebraic logic (MV-algebras, BL-algebras, residuated lattices), hyperstructures, non-commutative ring theory, elementary group theory, and mathematics for elementary school teachers.

Department of Mathematics, University of Oregon, Eugene, OR 97403, OR 48710-0001

E-mail address: nganou@uoregon.edu

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Siddartha Sen: The Joy of Understanding and Solving Problems: A Guide to School Mathematics, Lulu.com (USA), 2012. ISBN:978-1-291-42495-9, EUR 36.75, 607 pp.

REVIEWED BY MICHAEL BRENNAN

The Joy of Understanding and Solving Problems — A Guide to School Mathematics makes some new inroads into school mathematics but few inroads into joy. Siddartha Sen has a gift for bringing innovative examples and fresh motivations to old and well-worn topics, but in his stout-hearted ambition to present the whole of school mathematics, and more besides, in an expository fashion in a single volume, the novelty that should be at the heart of joy soon drains away. An Irish reader will notice that some of the material, as much as 100 pages of the 600 in the book, including vectors, conics and group theory, is no longer on any second-level maths syllabus.

The book aims high. In a long preface on the nature of maths and maths learning, the author declares his intention that the book should implement, embody perhaps, the idea that (quoting mythologist Joseph Campbell) a civilisation is on the rise when there is "integrity and cogency in [that civilisation's] supportive canons of myths, for not authority but aspiration is the motivator, builder and transformer of civilisation".

Maintaining such an ideal among the nuts and bolts of a didactic course in school mathematics was always going to be challenge, and despite the sprinkling of novel motivational examples, the book overall does not escape its own gunfire: "mathematics is taught in schools as a collection of rules for solving certain examination problems". Most of the methods, apart from the most elementary, where the derivation can be explained, are introduced conventionally by rules.

While Sen's aspiration is that the senior school student, at whom he

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says the book is aimed, will hear that "mathematics is an enjoyable human activity created by people with great imagination who have a passion to solve problems and find patterns", normal 16-year olds will worry about whether they have enough imagination to create mathematics, and if not, how they fit in.

Long chapters of worded exposition mean that this is not a textbook. In places it justifies the choice of topics on a school syllabus, in other places it merely rehearses the material. How a second-level student would use it is not clear. Good maths students could refer to it for reassurance or further study, but this is hardly its purpose: the students that inspirational books want to capture are the weaker ones, those who would love to be inspired.
Sen works through Leaving Certificate Euclidean Geometry in a verbalising manner, where a school text would be terse. Verbalising does not always mean clarifying. In the theorem about corresponding sides in similar triangles being proportional, the proof moves from "$\lambda^2(BX^2 + AX^2) = FY^2 + AY^2$" to "This equation has to hold as an identity. Thus $AY = \lambda AX, FY = \lambda BX$". The normal 16-year old would be tempted to learn this passage by heart. When dealing with the Geometry it might have been better to have focussed on how the student will meet the new demand by Project Maths: that he/she be able to "use the following terms related to logic and deductive reasoning: is equivalent to, if and only if, proof by contradiction".

Siddartha Sen's book should be reduced and rewritten. It does not have to start with addition of integers — some threshold can be assumed. And rather than attempting to compete exhaustively with school texts, it should focus on a few interesting ideas about pattern and structure and mathematical inquiry. That would give it a chance to become the inspirational, joyful handbook that Sen intended. But in the rewriting, a lot needs to change. The book is littered with editorial typos, and these intrude on the mathematics. The main page on quaternions is full of errors ($jk = i$, then $jk = -i$, ..., $ik = j$ ...). There is repetition of topics, and when this happens the book does not reference itself.

One of the potentially best chapters in the book, “Using Mathematics”, starts on page 534. Apart from explaining, again, how to calculate $5-3+1$, and the rules of fractions, both covered 500 pages earlier, it contains a section on Maths Muscle Building, where Sen seeks to use the algebra that has been learned, though principally to solve algebraic problems. This idea could be expanded to include mathematical modelling for second-level pupils, and the black art of converting a worded problem to a symbolic one teased out and revealed to all, student and teacher alike. Acquiring this art is the biggest challenge facing our second-level maths teachers today.

In hurling counties in Ireland there is the concept of the “makings” of a hurley, a J-shaped plank cut from the lower half and root of a young ash-tree. It looks like a hurley but one could not play with it. Before hand-drills/sanders became household items, a person would reduce the makings to a proper hurley using a spoke-shave. The Joy of Understanding and Solving Problem has the makings of a new type of maths book. The author must take a spoke-shave to it.

Michael Brennan is a Research Associate with the Trinity Irish Art Research Centre (TRIARC) at TCD, examining the structure of motifs in early medieval art. He is a Council member of the Irish Mathematics Teachers Association.

School of Humanitities, Trinity College Dublin
E-mail address: brennamn@tcd.ie

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Paul J. Nahin: Will You Be Alive 10 Years from Now?, Princeton University Press, 2014. ISBN:978-0-691-15680-4, USD 27.95, 220+xxvi pp.

REVIEWED BY GABRIELLE KELLY

This book outlines some of the more interesting results that arise when studying probability or statistics for the first time. The book is laid out in an Introduction, followed by 25 chapters and a final chapter with challenge problem solutions. The title comes from one particular chapter in the book (chapter 23) that deals with life expectancy and is not indicative of the type of problem dealt with in general. The most common topics the book deals with are combinatorial problems. Although the author claims there is little in the book by way of statistical problems, I found this to be not so. Topics covered range from distribution of maxima and minima of uniform random variables to the memoryless property of the exponential distribution and these arise naturally when studying statistics. Each chapter has some topic that is formulated by way of a question with a theoretical answer and a simulation solution using MATLAB is also given. The introduction is interesting from a historical point of view and also gives a flavour of the type of problems and results that follow. It is interesting to classify the chapters by type of problem. Chapter 2 is a probability counting problem. Chapter 3 presents an occupancy problem. Chapter 4 deals with the geometric distribution. Chapter 5 deals with the uniform distribution. Chapter 6 is estimating population size. Chapter 7 presents a quite tricky problem on the probability a chain letter will end. Chapter 8 and 9 are probability problems involving enumerating all possibilities. Chapter 10 concerns statistical hypothesis testing. Chapter 11 is a runs problem. Chapter 12 is a probability problem with an infinite sum involving Eulers constant and one of the more interesting chapters. Chapter 13 is a combinatorial problem. Chapter 14 reads more like a lesson on integration. Chapter 15 is one of the few

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where the solution to a probability problem is motivated by an important application. Chapter 16 deals with the uniform distribution again. Chapter 17 deals with false positive and negative rates from test results. Chapter 18 mainly involves doing an intricate integral. Chapter 19 dealing with the memoryless property of the exponential distribution gives one of the more interesting results. Chapter 20 is the ballot problem (but not motivated by a random walk as most texts). Chapter 21 deals with maxima of uniform random variables. Chapter 22 involves setting up a double difference equation to solve a probability problem winning at Ping-Pong or squash. Chapter 23 is a little disappointing as it assumes a hypothetical life-expectancy distribution is available. Chapter 24 presents further counting problems. Chapter 25 presents Newcombs paradox - another interesting chapter - this one without a solution and providing food for thought. The solutions to challenge problems are in Chapter 26 many of which involve geometry and integration. Thus many topics arise naturally when studying statistics — most students of statistics are familiar with the distribution of maximum and minimum of random variables, conditional probability, the Gamblers ruin problem, false positive and negative rates among others. A drawback of the book is that there is no intrinsic order to the Chapters. Thus for a student of probability the solutions to many of the problems are just one-off and the lack of coherent structure mitigates against learning how to do them. Also many of the problems posed are artificial and are not motivated by important everyday problems e.g. Steve's elevator problem in Chapter 3 or the chicken in boxes problem Chapter 24. However, what is considered an interesting problem can be subjective as the author indicates by the Note to what he terms the Plum Pudding problem in Chapter 21. On the plus side, the book provides useful problems for an instructor wishing to improve their student's ability at combinatorics, statistical distribution theory and calculus (specifically integration). It also provides useful MATHLAB problems. In addition the book showcases some of the famous problems in probability and statistics like the Gambler's ruin problem, Simpson's paradox (both in the Introduction) and the ballot problem. Thus the book also provides motivation for an interested student or reader to pursue the study of probability and statistics to a deeper level.

Gabrielle Kelly Dr. Gabrielle E. Kelly is a senior lecturer in Statistics in the School of Mathematical Sciences in University College Dublin (UCD). She obtained a Ph.D. in Statistics from Stanford University in 1981. She then worked in UCC, Columbia University in New York and University College London before joining UCD in 1990. She has published extensively in both applied and theoretical statistics. Her present research interests include spatial statistics, the change-point problem and population genetics.

School of Mathematical Sciences, University College Dublin
E-mail address: gabrielle.kelly@ucd.ie

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David Hand: The Improbability Principle, Scientific American and Farrar, Strauss and Giroux in USA; Bantam in UK, 2014. ISBN:978-0593072813, GBP 20, 269 pp.

REVIEWED BY GABRIELLE KELLY

The longer title of this book is The Improbability Principle : Why incredibly unlikely things keep happening. Hand's thesis is that extremely improbable events are commonplace.

The introductory chapters discuss phenomena such as superstitions, prophecies, gods and miracles, parapsychology and the paranormal, psychic powers, synchronicity and morphic resonance. Hand explains these ideas are all inventions designed to explain surprising phenomena and all that is really needed to explain them are the basic laws of probability.

Hand defines five strands (or "Laws") contributing to the Improbability Principle: the law of inevitability, the law of truly large numbers, the law of selection, the law of the probability lever and the law of near enough. Putting the laws together leads to 'extraordinary' events: financial crashes, winning the lottery twice, being struck by lightning seven times etc.

A chapter is devoted to each of the above laws, each of which is derived from and motivated by examples from everyday life. My favourite is in regard to the distribution of market fluctuations. If the distribution is normal a 5-sigma event has probability 1 in 3.5 million. The same probability assuming a Cauchy distribution is 1 in 16. This might explain why financial crashes happen all the time and comes under Hand's law of the probability lever. Most examples are nontrivial and require a subtlety of thought, some touching on principles in physics and finance.

While the above chapters are very entertaining, the penultimate two chapters explore frailties in our ways of thinking about the world and discuss the improbability of our universe. The exposure of our frailty is what makes this book enriching. You will undoubtedly encounter an example that 'corrects' your way of thinking about

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some phenomenon. It will change how you think about coincidences in particular. The final chapter explores how the idea of significance in statistics is based on probability concepts.

There are two appendices, one explaining how large numbers are written in mathematics and the other explaining the axioms of probability.

The book is written in an easy style and all arguments are easy to follow. It is very well written and is both entertaining and informative. It is accessible to a lay person and yet should appeal to anyone working in a numerate discipline. Even statisticians or probabilists accustomed to thinking about probability every day will find something of interest in this book. Several months after reading it I find several of the 'laws' have stayed with me. I strongly recommend it.

School of Mathematical Science, UCD Dublin
E-mail address: `gabrielle.kelly@ucd.ie`

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PROBLEMS

IAN SHORT

Problems

The first problem was contributed by Finbarr Holland of University College Cork.

Problem 73.1. Let U_n denote the Chebyshev polynomial of the second kind of degree n, which is the unique polynomial that satisfies the equation $U_n(\cos\theta) = \sin((n+1)\theta)/\sin\theta$. The polynomial U_{2n} satisfies $U_{2n}(t) = p_n(4t^2)$, where

$$p_n(z) = \sum_{k=0}^{n} (-1)^k \binom{2n-k}{k} z^{n-k}.$$

Prove that p_n is irreducible over the integers when $2n+1$ is a prime number.

The second and third problems were passed on to me by Tony Barnard of King's College London.

Problem 73.2. Find all positive integers a, b, and c such that

$$\begin{aligned} bc &\equiv 1 \pmod{a} \\ ca &\equiv 1 \pmod{b} \\ ab &\equiv 1 \pmod{c}. \end{aligned}$$

Problem 73.3. Prove that

$$\frac{1}{10\sqrt{2}} < \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \cdots \times \frac{99}{100} < \frac{1}{10}.$$

Solutions

Here are solutions to the problems from *Bulletin* Number 71. The first problem was solved by J.P. McCarthy of University College Cork and also by the North Kildare Mathematics Problem Club.

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We give McCarthy's solution to (a) and the NKMPC's solution to (b). The problem was also solved by the proposer.

Problem 71.1. For $n = 0, 1, 2, \ldots,$ the triangular numbers T_n and Jacobsthal numbers J_n are given by the formulas

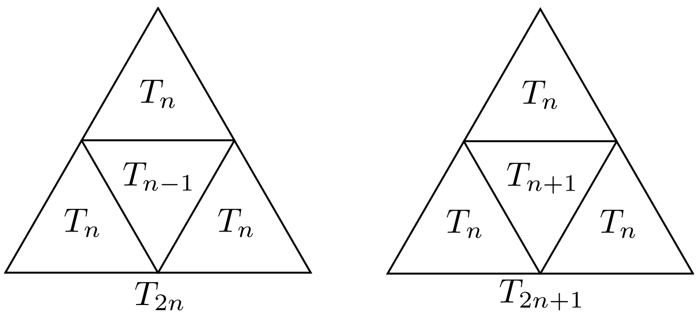
$$T_n = \frac{n(n+1)}{2} \quad \text{and} \quad J_n = \frac{2^n - (-1)^n}{3}.$$

(a) Prove that for each integer $n \geqslant 3$ there exist positive integers a, b, and c such that $T_n = T_a + T_b T_c$.
(b) Prove that infinitely many square numbers can be expressed in the form $J_a J_b + J_c J_d$ for positive integers a, b, c, and d.

Solution 71.1. (a) It is straightforward to check that

$$T_{2n} = T_{n-1} + 3T_n$$
$$T_{2n+1} = T_{n+1} + 3T_n.$$

These equations are illustrated in the figure below.



Since $T_2 = 3$, the result follows immediately.
(b) We have

$$J_{2n} + J_{2n+1} = \frac{2^{2n} - 1}{3} + \frac{2^{2n+1} + 1}{3} = 2^{2n}.$$

Therefore the square of each positive integer power of 2 can be written as a sum of two Jacobsthal numbers, and since $J_1 = 1$ the result follows immediately. □

The second problem was solved separately by Niall Ryan of the University of Limerick, the North Kildare Mathematics Problem Club, and the proposer. All solutions were in the same spirit, and we present a solution based on that of the proposer.

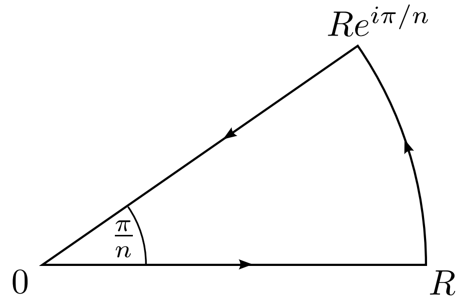
Problem 71.2. Prove that for each integer $n \geqslant 3$,

$$\int_0^\infty \frac{x-1}{x^n - 1}\, dx = \frac{\pi}{n \sin(2\pi/n)}.$$

Solution 71.2. Let

$$f(z) = \frac{1}{z^{n-1} + \cdots + z + 1}.$$

This function is analytic on a region containing the closed contour shown below.



Applying Cauchy's theorem to f, and letting $R \to \infty$, we obtain

$$\int_0^\infty \frac{x-1}{x^n-1}\,dx + e^{i\pi/n}\int_0^\infty \frac{e^{i\pi/n}x - 1}{x^n+1}\,dx = 0.$$

Therefore

$$\begin{aligned}\int_0^\infty \frac{x-1}{x^n-1}\,dx &= -\mathrm{Re}\left[e^{i\pi/n}\int_0^\infty \frac{e^{i\pi/n}x-1}{x^n+1}\,dx\right]\\ &= \cos\left(\frac{\pi}{n}\right)K(n,0) - \cos\left(\frac{2\pi}{n}\right)K(n,1), \qquad (1)\end{aligned}$$

where, for non-negative integers m and n,

$$K(n,m) = \int_0^\infty \frac{x^m}{x^n+1}\,dx.$$

Using a similar contour to above but with angle $2\pi/n$ rather than π/n, one can obtain the well-known formula

$$K(n,m) = \frac{\pi}{n\sin((m+1)\pi/n)}, \quad n > m+1.$$

Substituting the expressions for $K(n,0)$ and $K(n,1)$ into (1) gives the required result. □

Problem 71.3 remains unsolved!

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com`.

Department of Mathematics and Statistics, The Open University, Milton Keynes MK7 6AA, United Kingdom

Instructions to Authors

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