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TWO TRIGONOMETRIC IDENTITIES

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Abstract. We show that the trigonometric identities

$$\prod_{k=1}^{n-1}\Big\{1-\cos\frac{2k\pi}{n}\Big\}^{(n-k)\ell+m}=2^{(1-n)(\ell n/2+m)}\,n^{\ell n+2m}$$

and

$$\prod_{k=1}^{n-1}\Big\{\cos(2\theta)-\cos\frac{2k\pi}{n}\Big\}^{(n-k)\ell+m}=2^{(1-n)(\ell n/2+m)}\Big\{\frac{\sin(n\theta)}{\sin\theta}\Big\}^{\ell n+2m}$$

are valid for all $\ell,m\in\mathbf{Z}$ and $2\le n\in\mathbf{N}$. They extend the results due to Baica and Gregorac, who proved the identities for the special case $\ell=1$, $m=-1$. Moreover, we determine all ℓ,m,n such that the first trigonometric product just displayed is an integer.

In 1986, Baica [1] applied methods from cyclotomic fields to provide a rather long and complicated proof for the following interesting trigonometric identity:

$$\prod_{k=1}^{n-1}\Big\{1-\cos\frac{2k\pi}{n}\Big\}^{n-k-1}=2^{(1-n)(n/2-1)}\,n^{n-2}\tag{1}$$

where $n=2,3,4,\cdots$. Baica also remarked that "any proof avoiding cyclotomic fields could be very difficult, if not insoluble" [1, P. 705].

In 1989, Gregorac [3] used properties of Chebyshev polynomials to present a new proof of (1). Actually, he proved the identity

$$\prod_{k=1}^{n-1}\Big\{\cos(2\theta)-\cos\frac{2k\pi}{n}\Big\}^{n-k-1}=2^{(1-n)(n/2-1)}\Big\{\frac{\sin(n\theta)}{\sin\theta}\Big\}^{n-2}\tag{2}$$

for $n=2,3,4,\cdots$, which, letting θ tend to 0, leads to (1).

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Here, we extend (1) and (2). First, we offer an elementary short and simple proof of a generalization of Baica's identity. In order to verify our result we only make use of three well-known properties of sine and cosine,

$$1-\cos(2\theta)=2\,\sin^2\theta, \tag{3}$$
$$\sin(\pi-\theta)=\sin\theta, \tag{4}$$
$$\prod_{k=1}^{n-1}\sin\frac{k\pi}{n}=2^{1-n}\,n. \tag{5}$$

Formula (5) as well as many related formulas involving trigonometric functions can be found in [2, Eq. 4.14].

We have the following extension of identity (1).

Theorem 1. *Let ℓ, m be integers and let $n\geq 2$ be a natural number. Then,*

$$\prod_{k=1}^{n-1}\Big\{1-\cos\frac{2k\pi}{n}\Big\}^{(n-k)\ell+m}=2^{(1-n)(\ell n/2+m)}\,n^{\ell n+2m}. \tag{6}$$

Proof. Applying (3) yields

$$\prod_{k=1}^{n-1}\Big\{1-\cos\frac{2k\pi}{n}\Big\}^{(n-k)\ell+m}=2^{(n-1)(\ell n/2+m)}\prod_{k=1}^{n-1}\Big\{\sin\frac{k\pi}{n}\Big\}^{2[(n-k)\ell+m]}. \tag{7}$$

From (4) we conclude that

$$\prod_{k=1}^{n-1}\Big\{\sin\frac{k\pi}{n}\Big\}^{(n-k)\ell+m}=\prod_{k=1}^{n-1}\Big\{\sin\frac{(n-k)\pi}{n}\Big\}^{k\ell+m}=\prod_{k=1}^{n-1}\Big\{\sin\frac{k\pi}{n}\Big\}^{k\ell+m}. \tag{8}$$

Using (8) and (5) gives

$$\begin{aligned}\prod_{k=1}^{n-1}\Big\{\sin\frac{k\pi}{n}\Big\}^{2[(n-k)\ell+m]} &= \prod_{k=1}^{n-1}\Big\{\sin\frac{k\pi}{n}\Big\}^{(n-k)\ell+m}\prod_{k=1}^{n-1}\Big\{\sin\frac{k\pi}{n}\Big\}^{(n-k)\ell+m}\\ &= \prod_{k=1}^{n-1}\Big\{\sin\frac{k\pi}{n}\Big\}^{(n-k)\ell+m}\prod_{k=1}^{n-1}\Big\{\sin\frac{k\pi}{n}\Big\}^{k\ell+m}\\ &= \prod_{k=1}^{n-1}\Big\{\sin\frac{k\pi}{n}\Big\}^{\ell n+2m} = \big(2^{1-n}\,n\big)^{\ell n+2m}. \qquad (9)\end{aligned}$$

Combining (7) and (9) leads to (6). □

Next, we extend Gregorac's identity (2). We need the following formulas:

$$\sin\Big(\frac{\pi}{2}-\theta\Big) = \cos\theta, \tag{10}$$

$$\sin(2\theta) = 2\sin\theta\cos\theta, \tag{11}$$

$$\cos y - \cos x = 2\sin\frac{x+y}{2}\sin\frac{x-y}{2}, \tag{12}$$

$$\frac{\sin(n\theta)}{\sin\theta} = 2^{n-1}\prod_{k=1}^{n-1}\Big\{\cos\theta - \cos\frac{k\pi}{n}\Big\}. \tag{13}$$

Identity (13) is the well-known product representation for the Chebyshev polynomials of the second kind.

Theorem 2. *Let ℓ, m be integers and let $n \geq 2$ be a natural number. Then, for $\theta \in \mathbf{R}$,*

$$\prod_{k=1}^{n-1}\Big\{\cos(2\theta) - \cos\frac{2k\pi}{n}\Big\}^{(n-k)\ell+m} = 2^{(1-n)(\ell n/2+m)}\Big\{\frac{\sin(n\theta)}{\sin\theta}\Big\}^{\ell n+2m}. \tag{14}$$

Proof. Using (10) gives

$$\prod_{k=1}^{n-1}\sin\Big(\frac{k\pi}{2n}-\frac{\theta}{2}\Big) = \prod_{k=1}^{n-1}\sin\Big(\frac{(n-k)\pi}{2n}-\frac{\theta}{2}\Big) = \prod_{k=1}^{n-1}\cos\Big(\frac{k\pi}{2n}+\frac{\theta}{2}\Big). \tag{15}$$

Now, we apply (12), (15) and (11). Then we have

$$\begin{aligned}\prod_{k=1}^{n-1}\Big\{\cos\theta-\cos\frac{k\pi}{n}\Big\}=&\prod_{k=1}^{n-1}\Big\{2\sin\Big(\frac{k\pi}{2n}+\frac{\theta}{2}\Big)\sin\Big(\frac{k\pi}{2n}-\frac{\theta}{2}\Big)\Big\}\\=&\prod_{k=1}^{n-1}\Big\{2\sin\Big(\frac{k\pi}{2n}+\frac{\theta}{2}\Big)\cos\Big(\frac{k\pi}{2n}+\frac{\theta}{2}\Big)\Big\}\\=&\prod_{k=1}^{n-1}\sin\Big(\frac{k\pi}{n}+\theta\Big).\end{aligned}\tag{16}$$

From (4) and (12) we obtain

$$\begin{aligned}\prod_{k=1}^{n-1}\sin^2\Big(\frac{k\pi}{n}+\theta\Big)=&\prod_{k=1}^{n-1}\Big\{\sin\Big(\frac{k\pi}{n}+\theta\Big)\sin\Big(\frac{(n-k)\pi}{n}-\theta\Big)\Big\}\\=&\prod_{k=1}^{n-1}\Big\{\sin\Big(\frac{k\pi}{n}+\theta\Big)\sin\Big(\frac{k\pi}{n}-\theta\Big)\Big\}\\=&2^{1-n}\prod_{k=1}^{n-1}\Big\{\cos(2\theta)-\cos\frac{2k\pi}{n}\Big\}.\end{aligned}\tag{17}$$

Applying (13), (16) and (17) yields

$$\begin{aligned}\Big\{\frac{\sin(n\theta)}{\sin\theta}\Big\}^{2m}=&2^{2m(n-1)}\prod_{k=1}^{n-1}\Big\{\cos\theta-\cos\frac{k\pi}{n}\Big\}^{2m}\\=&2^{2m(n-1)}\prod_{k=1}^{n-1}\sin^{2m}\Big(\frac{k\pi}{n}+\theta\Big)\\=&2^{m(n-1)}\prod_{k=1}^{n-1}\Big\{\cos(2\theta)-\cos\frac{2k\pi}{n}\Big\}^{m}.\end{aligned}\tag{18}$$

From (4) and (12) we get

$$
\begin{aligned}
\prod_{k=1}^{n-1} \sin^n\Big(\frac{k\pi}{n}+\theta\Big) =& \prod_{k=1}^{n-1}\Big\{\sin^{n-k}\Big(\frac{k\pi}{n}+\theta\Big)\sin^k\Big(\frac{k\pi}{n}+\theta\Big)\Big\}\\
=& \prod_{k=1}^{n-1}\Big\{\sin^{k}\Big(\frac{(n-k)\pi}{n}+\theta\Big)\sin^k\Big(\frac{k\pi}{n}+\theta\Big)\Big\}\\
=& \prod_{k=1}^{n-1}\Big\{\sin^{k}\Big(\frac{k\pi}{n}-\theta\Big)\sin^k\Big(\frac{k\pi}{n}+\theta\Big)\Big\}\\
=& 2^{(1-n)n/2}\prod_{k=1}^{n-1}\Big\{\cos(2\theta)-\cos\frac{2k\pi}{n}\Big\}^k. \qquad (19)
\end{aligned}
$$

Combining (13), (16) and (19) gives

$$
\Big\{\frac{\sin(n\theta)}{\sin\theta}\Big\}^n = 2^{(n-1)n/2}\prod_{k=1}^{n-1}\Big\{\cos(2\theta)-\cos\frac{2k\pi}{n}\Big\}^k. \qquad (20)
$$

Finally, (18) and (20) lead to

$$
\begin{aligned}
\Big\{\frac{\sin(n\theta)}{\sin\theta}\Big\}^{2m+\ell n} &= 2^{(n-1)(\ell n/2+m)}\prod_{k=1}^{n-1}\Big\{\cos(2\theta)-\cos\frac{2k\pi}{n}\Big\}^{m+k\ell}\\
&= 2^{(n-1)(\ell n/2+m)}\prod_{k=1}^{n-1}\Big\{\cos(2\theta)-\cos\frac{2k\pi}{n}\Big\}^{(n-k)\ell+m}.
\end{aligned}
$$

This is equivalent to (14). □

Remark 1. Setting $\ell=1$ and $m=-1$ in (6) and (14), respectively, gives (1) and (2).

Remark 2. Let $\ell, m \in \mathbf{Z}$ and $2 \le n \in \mathbf{N}$ with $\ell n + 2m > 0$. Applying (14) and the well-known inequality

$$
\Big|\frac{\sin(n\theta)}{n\sin\theta}\Big| \le 1 \quad (n = 1, 2, 3, ...)
$$

we obtain for all $\theta \in \mathbf{R}$:

$$
\prod_{k=1}^{n-1}\Big\{\cos(2\theta)-\cos\frac{2k\pi}{n}\Big\}^{(n-k)\ell+m} \le 2^{(1-n)(\ell n/2+m)} n^{\ell n+2m}. \qquad (21)
$$

Setting $\theta = 0$ we conclude from (6) that the given upper bound is sharp. If $\ell n + 2m < 0$, then (21) holds with "$\geq$" instead of "$\leq$".

The representation (6) reveals that if $\ell, m \in \mathbf{Z}$, $2 \leq n \in \mathbf{N}$, then the product

$$P_n(\ell, m) = \prod_{k=1}^{n-1} \left\{1 - \cos\frac{2k\pi}{n}\right\}^{(n-k)\ell+m}$$

is a rational number. In view of this result it is natural to ask whether there exist numbers ℓ, m, n such that $P_n(\ell, m)$ is an integer. The next theorem answers this question.

Theorem 3. *Let ℓ, m be integers and $n \geq 2$ a natural number. The product $P_n(\ell, m)$ is an integer if and only if* $\ell n + 2m = 0$

$$\text{or} \quad \ell n + 2m > 0 \quad \text{with} \quad n = 2^r \quad (r = 1, 2); \tag{22}$$

$$\text{or} \quad \ell n + 2m < 0 \quad \text{with} \quad n = 2^r \quad (3 \leq r \in \mathbf{N}). \tag{23}$$

Proof. Using (6) we obtain:
if $\ell n + 2m = 0$, then $P_n(\ell, m) = 1$;
if $n = 2^r$ $(r = 1, 2)$ and $\ell n + 2m > 0$, then

$$P_n(\ell, m) = 2^{(\ell n + 2m)/2} \in \mathbf{Z};$$

if $n = 2^r$ $(r \geq 3)$ and $\ell n + 2m < 0$, then

$$P_n(\ell, m) = 2^{-(2^r - 2r - 1)(\ell n + 2m)/2} \in \mathbf{Z}.$$

Now, let $P_n(\ell, m) \in \mathbf{Z}$. We asssume (for a contradiction) that none of (22), (23) and $\ell n + 2m = 0$ is satisfied. We have

$$\begin{aligned} P_2(\ell, m) &= 2^{\ell+m}, \\ P_3(\ell, m) &= \left\{\frac{3}{2}\right\}^{3\ell+2m}, \\ P_4(\ell, m) &= 2^{2\ell+m}, \\ P_5(\ell, m) &= \left\{\frac{5}{4}\right\}^{5\ell+2m}. \end{aligned}$$

Case 1. $\ell n + 2m > 0$.
Then, $P_3(\ell, m) \notin \mathbf{Z}$ and $P_5(\ell, m) \notin \mathbf{Z}$. Let $n \geq 6$. From

$$2^{(n-1)(\ell n + 2m)/2} \cdot K = n^{\ell n + 2m} \quad (K \in \mathbf{N}) \tag{24}$$

we conclude that 2 divides $n^{\ell n+2m}$. This implies that n is even. Let $n = 2^r q$, where $r \geq 1$ and q is odd. Then, (24) leads to

$$2^{((n-1)/2-r)(\ell n+2m)} \cdot K = q^{\ell n+2m}.$$

Since q is odd, we obtain

$$\frac{n-1}{2} - r \leq 0. \tag{25}$$

Hence,

$$2^r \leq 2^r q = n \leq 2r + 1.$$

If follows that $r = 1$ or $r = 2$. However, this contradicts (25), since $n \geq 6$.

Case 2. $\ell n + 2m < 0$.
Then, $P_n(\ell, m) \notin \mathbf{Z}$ for $n = 2, 3, 4, 5$. Let $n \geq 6$. From (24) we obtain

$$n^{-(\ell n+2m)} \cdot K = 2^{-(n-1)(\ell n+2m)/2}.$$

This yields $n = 2^r$ with $r \geq 3$. A contradiction. The proof is complete. □

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