**Irish Math. Soc. Bulletin**
Number 71, Summer 2013, 33–39
ISSN 0791-5578

# THE TENSOR PRODUCT OF A CSL AND AN ABSL

SAVVAS PAPAPANAYIDES

Abstract. We study the question that asks whether the tensor product of two reflexive subspace lattices is reflexive. In particular, we study the tensor product of a commutative subspace lattice $\mathcal{L}$ and an atomic boolean subspace lattice $\mathcal{M}$ and we prove that it is equal to the extended tensor product of the two subspace lattices. Furthermore, we give a description of the subspace lattice $\mathcal{L} \otimes \mathcal{M}$ and with the help of a result of Harrison in [3] we prove that it is reflexive. We also show that the lattice tensor product formula holds for any Arveson algebra of $\mathcal{L}$ and $\operatorname{alg} \mathcal{M}$.

## 1. Introduction

In this paper we consider every Hilbert space to be separable. If $H$ is a Hilbert space, then we set $\mathcal{B}(H)$ to be the set of all bounded operators acting on $H$ and $\mathcal{P}(H)$ to be the set of all orthogonal projections acting on $H$. If $P, Q \in \mathcal{P}(H)$ then we define $P \vee Q$ to be the projection with range $PH \vee QH$ and $P \wedge Q$ the projection with range $PH \wedge QH$. It is clear that $\mathcal{P}(H)$ is a lattice with respect to the binary operations of the intersection $\wedge$ and the closed linear span $\vee$. A strongly closed sublattice of $\mathcal{P}(H)$ (with respect to the binary operations of the intersection and the linear span) that contains 0 and the identity operator $I$ is called a subspace lattice.

If $\mathcal{A}$ is an operator algebra acting on a Hilbert space $H$ then we define

$$\operatorname{lat} \mathcal{A} = \{P \in \mathcal{P}(H) : PTP = TP \text{ for each } T \in \mathcal{A}\}.$$

Obviously $\operatorname{lat} \mathcal{A}$ is a subspace lattice. Subspace lattices of this form are called reflexive. Similarly, if $\mathcal{L}$ is a subspace lattice then we

2010 *Mathematics Subject Classification.* 47L35; 47L75.

*Key words and phrases.* Operator algebra, tensor product, extended tensor product, commutative subspace lattice, atomic boolean subspace lattice.

Received on 8-4-2013; revised 22-4-2013.

Support from my supervisor, Dr I.G. Todorov, is gratefully acknowledged.

define

$$\operatorname{alg}\mathcal{L} = \{T \in \mathcal{B}(H) : L^{\perp}TL = 0, \text{ for each } L \in \mathcal{L}\}.$$

Clearly $\operatorname{alg}\mathcal{L}$ is a weakly closed unital subalgebra of $\mathcal{B}(H)$.

Let $T_i$ be a bounded operator acting on a Hilbert space $H_i$ for $i = 1, 2$. We define $T_1 \otimes T_2$ be the bounded operator acting on the Hilbert space $H_1 \otimes H_2$ such that, if $x_1 \in H_1$ and $x_2 \in H_2$, then $T_1 \otimes T_2(x_1 \otimes x_2) = T_1x_1 \otimes T_2x_2$.

If $\mathcal{A}_1$ and $\mathcal{A}_2$ are ultraweakly closed algebras, then we denote by $\mathcal{A}_1 \otimes \mathcal{A}_2$ the ultraweakly closed algebra generated by the elementary tensors $A_1 \otimes A_2$ where $A_i \in \mathcal{A}_i$, $i = 1, 2$. Similarly, if $\mathcal{L}_1$ and $\mathcal{L}_2$ are subspace lattices we denote by $\mathcal{L}_1 \otimes \mathcal{L}_2$ the smallest subspace lattice containing all elementary tensors $L_1 \otimes L_2$ where $L_i \in \mathcal{L}_i$, $i = 1, 2$.

Given two subspace lattices $\mathcal{L}$ and $\mathcal{M}$, the algebra tensor product formula (ATPF) holds for $\mathcal{L}$ and $\mathcal{M}$ if $\operatorname{alg}(\mathcal{L} \otimes \mathcal{M}) = \operatorname{alg}\mathcal{L} \otimes \operatorname{alg}\mathcal{M}$. Analogously, the lattice tensor product formula (LTPF) holds for two operator algebras $\mathcal{A}$ and $\mathcal{B}$ if $\operatorname{lat}(\mathcal{A} \otimes \mathcal{B}) = \operatorname{lat}\mathcal{A} \otimes \operatorname{lat}\mathcal{B}$. The LTPF was first introduced by Hopenwasser in [4].

Let $Y$ be a compact metric space and $\nu$ a finite regular Borel measure on $Y$. If $A \subseteq Y$ is measurable, then we denote by $M_A$ the map that sends $\psi \in L^2(Y, \nu)$ to $\psi\chi_A$ where $\chi_A$ is the characteristic function on $A$.

Let $\mathcal{L}$ be a commutative subspace lattice (CSL). It follows from Arveson [1] that there exists a compact metric space $X$, a standard preorder $\leq$ on $X$ and a finite regular Borel measure $\mu$ on $X$ such that $\mathcal{L}$ is unitarily equivalent to

$$\mathcal{L}(X, \mu, \leq) = \{M_B : B \subseteq X \text{ measurable and almost increasing}\}.$$

A subset $B \subseteq X$ is almost increasing if there exists a null subset $\Gamma \subseteq X$ such that $B\backslash\Gamma$ is increasing. If $\mathcal{M}$ be a subspace lattice acting on a Hilbert space $K$, then a function $\phi : X \to \mathcal{M}$ is almost increasing if there exists a null subset $\gamma \subseteq X$ such that $\phi$ is increasing on $X\backslash\gamma$. Also, the function $\phi$ is measurable if the map $x \to (\phi(x)\xi, \eta)$ is measurable for all $\xi, \eta \in K$. We define $L^\infty(X, \mu, \leq, \mathcal{M})$ to be the space of all essentially bounded, $\mathcal{M}$-valued, almost increasing and measurable functions on $X$. If $\phi \in L^\infty(X, \mu, \leq, \mathcal{M})$, then we denote by $M_\phi$ the map from $L^2(X, \mu, K)$ to $L^2(X, \mu, K)$ such that $(M_\phi f)(x) = \phi(x)f(x)$ for all $f \in L^2(X, \mu, K)$ and for all $x \in X$. The **extended tensor product** of $\mathcal{L}$ and $\mathcal{M}$ is defined to be the space

$$\{M_\phi : \phi \in L^\infty(X, \mu, \leq, \mathcal{M})\}$$

and it is denoted by $\mathcal{L}\otimes_{ext}\mathcal{M}$. In many occasions it is easier to identify $\mathcal{L}\otimes_{ext}\mathcal{M}$ with $L^\infty(X, \mu, \leq, \mathcal{M})$ through the map that sends $\phi$ to $M_\phi$ for all $\phi \in L^\infty(X, \mu, \leq, \mathcal{M})$. Also, if $B \subseteq X$ is almost increasing and measurable and $L \in \mathcal{M}$, then through the map that sends $M_B \otimes L$ to the function $x \to \chi_B(x)L$, where $x \in X$, we identify $\mathcal{L}\otimes\mathcal{M}$ with a subset of $\mathcal{L}\otimes_{ext}\mathcal{M}$ and we consider $\mathcal{L}\otimes\mathcal{M} \subseteq \mathcal{L}\otimes_{ext}\mathcal{M}$.

The extended tensor product was firstly introduced by Harrison in [3]. One of the main results obtained in that paper is that the extended tensor product of a completely distributive CSL $\mathcal{L}$ and any subspace lattice $\mathcal{M}$ is equal to their tensor product. An interesting question emerging from this result is whether the equality between the tensor product and the extended tensor product still holds, if we remove the property of complete distributivity from the subspace lattice $\mathcal{L}$ to the subspace lattice $\mathcal{M}$. The main result of this paper answers the previous question positively in the case where the subspace lattice $\mathcal{M}$ is an atomic Boolean subspace lattice (ABSL). Also, if $\mathcal{M}$ is an ABSL, it follows that the tensor product of $\mathcal{L}$ and $\mathcal{M}$ is reflexive and that the LTPF holds for every Arveson algebra of $\mathcal{L}$ and $\operatorname{alg}\mathcal{M}$. Furthermore, we have a description of $\mathcal{L}\otimes\mathcal{M}$ in terms of the elements of $\mathcal{L}$ and the atoms of $\mathcal{M}$.

## 2. The main results

Recall at this point of the paper that a subspace lattice $\mathcal{M}$ is an ABSL if it is distributive, complemented (i.e. for every $P \in \mathcal{M}$ there exists an element $P' \in \mathcal{M}$ such that $P \wedge P' = 0$ and $P \vee P' = I$) and there exists a subset $\mathcal{K} \subset \mathcal{M}$ of non-zero elements (called the atoms) such that (i) if $M \in \mathcal{M}$, $K \in \mathcal{K}$ and $0 \subseteq M \subseteq K$, then either $M = 0$ or $M = K$ and (ii) if $M \in \mathcal{M}$ then $M$ is equal to the closed span of the atoms that it majorises.

**Lemma 2.1.** *If $\mathcal{M}$ is an ABSL acting on a separable Hilbert space $H$, then $\mathcal{M}$ has at most a countable number of atoms.*

*Proof.* Let $(E_j)_{j\in J}$ be the set of atoms of $\mathcal{M}$ and $(e_k^{(j)})_{k\in K_j}$ be an orthonormal basis of $E_jH$, $j \in J$. Since $H$ is separable, it follows that $K_j$ is at most countable for all $j \in J$. It is also clear that

$\bigvee_{j\in J}(\bigvee_{k\in K_j} e_k^{(j)}) = H$. The class of subsets of $\mathcal{T} = \bigcup_{j\in J}\{e_k^{(j)} : k \in K_j\}$ whose closed span equals $H$ is not empty as $\mathcal{T}$ is such a set itself. Also, if there exists $\mathcal{S} \subseteq \mathcal{T}$ and $i \in J$ such that $\mathcal{S} \cap E_i = \emptyset$, then $\vee\{s : s \in \mathcal{S}\} \subseteq \bigvee_{j\neq i} E_j \neq H$. Hence, for every subset $\mathcal{S}$ of $\mathcal{T}$ whose closed span is equal to $H$ and for every $i \in J$ we have that $\mathcal{S}\cap E_i \neq \emptyset$ and thus the cardinality of each of those sets is bigger or equal to the cardinality of $J$.

Let $(f_n)_{n\in\mathbb{N}}$ be an orthonormal basis of $H$. Obviously, for each $n \in \mathbb{N}$, there exists a sequence $(a_l^{(n)})_{l\in\mathbb{N}}$ such that

$$a_l^{(n)} = \sum_{i=1}^{k_l^{(n)}} \mu_i^{(l,n)} e_i^{(l,n)}, \mu_i^{(l,n)} \in \mathbb{C},$$

where $k_l^{(n)} \in \mathbb{N}$, $e_i^{(l,n)} \in \bigcup_{j\in J}\{e_k^{(j)} : k \in K_j\}$ for all $1 \leq i \leq k_l^{(n)}$ and all $l \in \mathbb{N}$, and $f_n = \lim_{l\to\infty} a_l^{(n)}$. Let $\mathcal{V} = \{e_i^{(l,n)} : 1 \leq i \leq k_l^{(n)}$ and $l, n \in \mathbb{N}\}$. Then $\mathcal{V}$ is at most countable and the closed span of its elements is equal to $H$. Since $\mathcal{V} \subseteq \mathcal{T}$, it follows from our previous observation that the cardinality of $J$ is smaller than the cardinality of $\mathcal{V}$ and thus it is at most countable proving our lemma. □

**Theorem 2.2.** *Let $X$ be a compact metric space, $\mu$ be a finite Borel measure, $\leq$ be a standard preorder, $\mathcal{L} = \mathcal{L}(X, \mu, \leq)$ be a CSL acting on the Hilbert space $K = L^2(X, \mu)$ and $\mathcal{M}$ be an ABSL acting on a Hilbert space $H$ with atoms $\{E_j : j \in J\}$. Then every element of $\mathcal{L}\otimes_{ext}\mathcal{M}$ can be written in the form $\bigvee_{j\in\mathbb{N}}(M_{\alpha_j} \otimes E_j)$ where $\alpha_j \subseteq X$ is almost increasing and measurable and thus $\mathcal{L}\otimes_{ext}\mathcal{M} = \mathcal{L} \otimes \mathcal{M}$.*

*Proof.* In the proof of this theorem we identify $L^\infty(X, \mu, \leq, \mathcal{M})$ with $\mathcal{L}\otimes_{ext}\mathcal{M}$. By Lemma 2.1, $\mathcal{M}$ has at most a countable number of atoms. Since $\mathcal{L}\otimes\mathcal{M} \subseteq \mathcal{L}\otimes_{ext}\mathcal{M}$, we only need to prove the opposite inclusion. Fix $P \in \mathcal{L}\otimes_{ext}\mathcal{M}$ and let $\beta_j = \{x \in X : P(x) \geq E_j\}$ for all $j \in J$. Let $j \in J$ and $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $E_j$. It follows that

$$\begin{aligned} \beta_j &= \{x \in X : P(x) \geq E_j\} \\ &= \{x \in X : (P(x)e_k, e_k) = (E_j e_k, e_k),\ k \in \mathbb{N}\} \\ &= \bigcap_{k \in \mathbb{N}} \{x \in X : (P(x)e_k, e_k) = 1\}. \qquad (1) \end{aligned}$$

The function $x \to (P(x)\xi, \eta)$ is measurable by definition for all $\xi, \eta \in H$ and thus the set $\{x \in X : (P(x)e_k, e_k) = 1\}$ is measurable for all $k \in \mathbb{N}$. It follows from (1) that $\beta_j$ is measurable for all $j \in J$. Also, $P$ is almost increasing and thus there exists a null set $A$ such that $P$ is increasing on $X \backslash A$. If for some $j \in J$, $x \in \beta_j \backslash A$, $y \in X \backslash A$ and $x \leq y$, then by the definition of $\beta_j$, we have that $P(x) \geq E_j$. Since $x \leq y$ and $P$ is increasing on $X \backslash A$, it follows that $P(y) \geq P(x) \geq E_j$. By the definition of $\beta_j$, we have that $y \in \beta_j$. Hence, $\beta_j$ is increasing in $X \backslash A$ and thus it is almost increasing. Since $j \in J$ is arbitrary, $\beta_j$ is almost increasing for all $j \in J$.

It now suffices to show that $P = \bigvee_{j \in J} (M_{\beta_j} \otimes E_j)$. It is well known that there exists a null subset $N \subseteq X$ such that $(\bigvee_{j \in J} (M_{\beta_j} \otimes E_j))(x) = \bigvee_{j \in J} ((M_{\beta_j} \otimes E_j)(x))$ for all $x \in X \backslash N$ (for a detailed proof of this statement see Proposition 1.5.2 in [5]). It is easy to see that $(M_{\beta_j} \otimes E_j)(x) \leq P(x)$ for all $x \in X \backslash N$ and for all $j \in J$, and thus $(\bigvee_{j \in J} (M_{\beta_j} \otimes E_j))(x) \leq P(x)$ for all $x \in X \backslash N$. Since $\mathcal{M}$ is an ABSL, $P(x)$ is equal to the span of the atoms that it contains for all $x \in X$. Hence, in order to prove the opposite inclusion it is enough to show that all atoms contained in $P(x)$ are also contained in $(\bigvee_{j \in J} (M_{\beta_j} \otimes E_j))(x)$ for all $x \in X \backslash N$. Fix $x \in X \backslash N$ and $i \in J$ such that $E_i \leq P(x)$. It follows from the definition of $\beta_i$ that $x \in \beta_i$ and thus

$$E_i \leq \bigvee_{j \in J} ((M_{\beta_j} \otimes E_j)(x)) = (\bigvee_{j \in J} (M_{\beta_j} \otimes E_j))(x).$$

Since $E_i$ is an arbitrary atom contained in $P(x)$, it follows that

$$P(x) = (\bigvee_{j \in J} (M_{\beta_j} \otimes E_j))(x).$$

Since $N$ is null and $x \in X \backslash N$ is arbitrary, $P = \bigvee_{j \in J} (M_{\beta_j} \otimes E_j)$ and the theorem is proved. $\square$

Recall at this point from Arveson [1] that for every CSL $\mathcal{L}$ there exists a minimal weak* closed algebra $\mathcal{A}_{\min}$ such that i) $\mathcal{A}_{\min}$ contains a maximal abelian selfadjoint algebra (masa), and ii) $\operatorname{lat}\mathcal{A}_{\min} = \mathcal{L}$. Every weak* closed algebra satisfying those two conditions is called an Arveson algebra.

**Corollary 2.3.** *Let $\mathcal{L}$ be a CSL acting on a separable Hilbert space $K$, $\mathcal{B}$ be an Arveson algebra of $\mathcal{L}$, $\mathcal{M}$ be an ABSL and $\mathcal{A} = \operatorname{alg}\mathcal{M}$. Then the LTPF holds for $\mathcal{A}$ and $\mathcal{B}$ and $\mathcal{L} \otimes \mathcal{M}$ is reflexive.*

*Proof.* Recall from Section 1 that there exists a compact metric space $X$, a finite Borel measure $\mu$ and is a standard preorder $\leq$ acting on $X$ such that $\mathcal{L}$ is unitarily equivalent to $\mathcal{N} = \mathcal{L}(X, \mu, \leq)$. By [3, Theorem 1], we have that

$$\mathcal{N} \otimes_{ext} \operatorname{lat}\mathcal{A} = \operatorname{lat}(\mathcal{A}_{\min} \otimes \mathcal{A})$$

where $\mathcal{A}_{\min}$ is the smallest ultraweakly closed algebra containing a masa for which $\mathcal{N} = \operatorname{lat}\mathcal{A}_{\min}$. Since every ABSL is reflexive [2], we have that

$$\mathcal{N} \otimes_{ext} \mathcal{M} = \operatorname{lat}(\mathcal{A}_{\min} \otimes \mathcal{A}).$$

It follows by Theorem 2.2 that $\mathcal{N} \otimes \mathcal{M} = \operatorname{lat}(\mathcal{A}_{\min} \otimes \mathcal{A})$ and thus $\mathcal{N} \otimes \mathcal{M}$ is reflexive. Hence $\mathcal{L} \otimes \mathcal{M}$ is reflexive and if $\mathcal{B}_{\min}$ is the smallest ultraweakly closed algebra containing a masa for which $\mathcal{L} = \operatorname{lat}\mathcal{B}_{\min}$, then

$$\mathcal{L} \otimes \mathcal{M} = \operatorname{lat}(\mathcal{B}_{\min} \otimes \mathcal{A}) \supseteq \operatorname{lat}(\mathcal{B} \otimes \mathcal{A}) \supseteq \operatorname{lat}\mathcal{B} \otimes \operatorname{lat}\mathcal{A} = \mathcal{L} \otimes \mathcal{M}$$

and the LTPF holds for $\mathcal{A}$ and $\mathcal{B}$. □

## References


[1] W. Arveson, *Operator Algebras and Invariant Subspaces*, Ann. of Math. (2) 100 (1974), 433 - 532

[2] P. R. Halmos, *Reflexive lattices of subspaces*, J. London Math. Soc. (2) 4 (1971), 257 - 263

[3] K. J. Harrison, *The tensor product formula for reflexive subspace lattices*, Canad. Math. Bull. 38 (1995), no. 3, 308 - 316

[4] A. Hopenwasser, *Tensor products of reflexive subspace lattices*, Michigan Math. J. 31 (1984), no. 3, 359 - 370

[5] S. Papapanayides (2011). *Properties of subspace lattices related to reflexivity*. PhD Thesis. Queen's University Belfast: U.K.

**Savvas Papapanayides** studied at the Universities of Cyprus and Bath before obtaining his Ph.D. in 2011 from Queen's University, Belfast. His research interests relate to Operator Algebras and Operator Theory.

(April 2013) School of Mathematics and Physics, Queen's University Belfast

*E-mail address*, S. Papapanayides: spapapanayides01@qub.ac.uk