Irish Math. Soc. Bulletin
Number 70, Winter 2012, 35–36
ISSN 0791-5578

CHARACTERIZING THE SPECTRA OF NONNEGATIVE MATRICES

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This is an abstract of the PhD thesis *Characterizing the Spectra of Nonnegative Matrices* written by Anthony Cronin under the supervision of Professor Thomas J Laffey at the School of Mathematical Sciences UCD and submitted in May 2012.

In this thesis we investigate the *nonnegative inverse eigenvalue problem* (NIEP). This is the problem of characterizing all possible spectra of entrywise nonnegative $n \times n$ matrices. We give a new inequality [1] relating Newton power sums. We then build on perturbation results of Guo [2] and Laffey [3]. Also presented are results for the class of *doubly companion matrices* and we address some questions of Monov [4]. We examine the classic spectrum $(3+t, 3, -2, -2, -2)$ and give some new results on the *diagonalizable and symmetric nonnegative inverse eigenvalue problems* (DNIEP and SNIEP respectively). The main results are:

Theorem 1. *Let $n > 1$ and A be a nonnegative $n \times n$ matrix. Then*

$$\Omega := n^2 s_3 - 3ns_1 s_2 + 2s_1^3 + \frac{n-2}{\sqrt{n-1}}(ns_2 - s_1^2)^{\frac{3}{2}} \geq 0,$$

where $s_k = trace(A^k), k = 1, 2, 3$.

Theorem 2. *Suppose that c is a real number such that, for all integers $n \geq 3$, and all lists $\sigma := (\rho, \lambda, \overline{\lambda}, \lambda_4, \ldots, \lambda_n)$ with Perron root ρ and $\lambda \notin \mathbb{R}$, the realizability of σ implies the realizability of $\sigma_c := (\rho + ct, \lambda + t, \overline{\lambda} + t, \lambda_4, \ldots, \lambda_n)$, then $c \geq 2$.*

Theorem 3. *Let $\sigma = (\rho, \lambda_2, \overline{\lambda_2}, \ldots, \lambda_n)$ be realizable by a nonnegative matrix A, where ρ is the Perron root and λ_2 and $\overline{\lambda_2}$ are non-real complex conjugates. Then, given any $\epsilon > 0$, the list $(\rho + (2+\epsilon)t, \lambda_2 + t, \overline{\lambda_2} + t, \lambda_4, \ldots, \lambda_n)$ is realizable for all sufficiently small $t > 0$.*

2010 *Mathematics Subject Classification.* 15A18,15A29.
Key words and phrases. Nonnegative matrices, spectrum, eigenvalues.
Received on 21-10-2012.
Support from SFIRFP is gratefully acknowledged.

Theorem 4. *If $\sigma = (\rho, \lambda_2, \dots, \lambda_j, \overline{\lambda_j}, \dots, \lambda_n)$, where ρ is the Perron root, is realizable by a nonnegative circulant matrix , then for all $t > 0$ the list $(\rho + ct, \lambda_2, \dots, \lambda_j + t, \overline{\lambda_j} + t, \dots, \lambda_n)$ is also realizable by a nonnegative circulant matrix for*

$$c \geq \begin{cases} 2, & \text{if } n \text{ is even} \\ 2\cos(\frac{\pi}{n}), & \text{if } n \text{ is odd} \end{cases}$$

Theorem 5. *Let*

$$\begin{aligned} f_1(x) &= x^n - a_1 x^{n-1} - \cdots - a_{n-1}x - a_n \\ f_2(x) &= x^n - b_1 x^{n-1} - \cdots - b_{n-1}x - b_n \\ f_3(x) &= u_0 + u_1 x + \cdots + u_{n-1}x^{n-1} \end{aligned}$$

and

$$A = \begin{pmatrix} C(f_1) & N \\ R(f_3) & C(f_2) \end{pmatrix}$$

where $C(f_i)$ is the companion matrix of $f_i(x)$ for $i = 1, 2$, N is the matrix of all zeros except for a 1 in position $(n, 1)$ and R is the $n \times n$ matrix with last row $(u_0, u_1, \cdots, u_{n-1})$ and all other rows zero. Then
(a) A has characteristic polynomial $f(x) := f_1(x)f_2(x) - f_3(x)$,
(b) if A is nonnegative and $f(x) = x^k F(x)$ where $F(x)$ is a polynomial of degree $2n - k$, where $0 \leq k \leq n$, then $F(x)$ is the characteristic polynomial of a $(2n - k) \times (2n - k)$ nonnegative matrix.

Theorem 6. *Let $f(x) = x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_{n-1}x + p_n$ be the characteristic polynomial of a nonnegative $n \times n$ matrix A. Then $g(x) = \frac{f'(x)}{n}$ is the characteristic polynomial of a nonnegative $(n-1) \times (n-1)$ matrix for $n \leq 4$ and for $n \in \{5, 6\}$ when $trA = 0$.*

Theorem 7. *SNIEP $\neq$ DNIEP*

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