

**Avner Ash and Robert Gross: *Elliptic Tales*, Princeton
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An elliptic curve E over the field of rationals \mathbb{Q} is a non-singular cubic curve in \mathbb{P}^2 on which there is a specified point with coordinates in \mathbb{Q} . The points on E with coefficients in \mathbb{Q} form an abelian group $E(\mathbb{Q})$. The Birch-Swinnerton-Dyer conjecture relates the rank of this group to the order to which the L -function $L(E, s)$ of E vanishes at $s = 1$. More specifically, the conjecture states that the rank of $E(\mathbb{Q})$, which by Mordell's Theorem is always finite, equals this order of vanishing. The generalization to number fields other than \mathbb{Q} is not discussed in this book.

Roots of the conjecture reach far back into the past. The version given above was formulated in 1965. Work by B. Gross and D. Zagier, and by V. Kolyvagin in the 1980s and extensions of their results established the conjectured equality for vanishing order zero or one in 2001. Despite much computational evidence, a clue for dealing with the conjecture in greater generality is unknown, and the Clay Mathematical Institute offers $\$10^6$ for its first complete proof. This suggests to any non-expert in the field, to which the book at hand is largely addressed, that the problem may be worth thinking about.

Elliptic Tales is written for an audience with little mathematical knowledge and experience. In Part I, after having identified the problem of sorting out a well-defined notion of the degree of a curve, the book familiarizes the reader with complex numbers, rings and fields, congruences between integers, algebraic closures, and eventually with the projective plane. Part II treats abelian groups and takes up the discussion of elliptic and of singular cubic curves. Part III relates elliptic curves to analysis. Generating functions, Taylor series and analytic continuation are introduced, and the book culminates in discussing the L -function of a curve and eventually the

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BSD conjecture. Overall, the authors try to give a more comprehensive picture of the field than is needed to just make sense out of the conjectured equality. For instance, though the notion of a local intersection number cannot be provided for general curves, Bézout's theorem together with a sketched proof is presented to show the heuristically established equality between the algebraic degree of a curve in \mathbb{P}^2 and its geometric degree.

The text covers roughly 250 pages and is written in a largely conversational style. Clear references to specific formulas are scarce, the authors prefer to repeat equations to accommodate the reader, and the cohesion between the individual parts of the book relies essentially on frequent reminders. The latter extend to repeatedly mentioning that part of the material has been treated in the authors' previous book already but on p. 214 the promotion peaks in the intimidating phrase "This paragraph will only make sense to you if you read *Fearless Symmetry* or learned these concepts from somewhere else". Writing a popular book about mathematical developments at the edge of current research without either scaring or boring its inexperienced audience is a commendable and highly non-trivial task. The exposition requires tentative "definitions" which are included in the normal text. This treatment risks sloppiness. On p. 23 the authors prematurely suggest that the geometric degree of a line is 1 though according to their provisional definition of the geometric degree as "the maximum number of points of intersection of any line with the object" (p. 20), it is actually infinite. Also, there is some confusion about the notions of "local intersection multiplicity" (p. 72) and "local intersection number" (p. 76), which are referred to by the same symbol $I(C, L, P)$, where C, L are two curves in the ordinary plane with common point P . In the context of Bézout's Theorem, on p. 85, the notation $I(\bar{C}, \bar{L}, P)$ is used for two projective curves \bar{C}, \bar{L} . This expression is not properly defined though a definition in terms of $I(C, L, P)$ is aimed at.

On the other hand, the notion of an integer "killing" an element of an abelian group gets promoted to a proper "definition" (p. 107) for no obvious reason. The term is used in the subsequent paragraph and never thereafter, and it does not appear in the index. A more structured approach would have added to clarity.

The discussion is slowed down by inclusion of many elementary calculations whose intermediate steps are worked out in full detail.

When they are integrated in the running text, the computations hinder the smooth flow of arguments. When they are relegated to an appendix, the result is sometimes worse: In Part II (p. 137 et seq. and p. 147 et seq.), the group law on the projective completion of $C : y^2 = x^3$ is discussed. This should have been done by noting the equivalence to the projective completion of $z = x^3$. (When intersected with a line, the sum of the roots x_1, x_2, x_3 is zero, providing the additive group law.) Similarly, the split-multiplicative reduction should be discussed for the curve $x^3 - 1 = xy$. (Here $x_1x_2x_3 = 1$.) Instead, the authors take a brute force approach requiring almost five pages of uninteresting computer-aided calculations.

While the exposition tends to be talkative (e.g., “We won’t attempt to explain the proof, but we can explain what the theorem says” on p. 111), some parts would have gained from further explanations. The appendix to Chapter 7 is devoted to Dirichlet’s Unit Theorem. For the ring $\mathbb{Z}[\zeta]$, with ζ a primitive p -th root of unity, the rank of the group of units is introduced, but no examples of units of infinite order are provided (p. 112). The next example is $\mathbb{Z}[\sqrt{N}]$, which is treated more extensively. No connection between the two examples is made, however.

The historical outline suffers from serious omissions. While important contributions of C. Breuil et al., D. Goldfeld, R. Taylor, and A. Wiles are mentioned, an acknowledgement of in particular the work by B. H. Gross and D. Zagier, and by V. A. Kolyvagin is missing. Their achievements are simply attributed to “the work of many people” (p. 238). The combination of arguments proving the conjecture for vanishing order zero and one is somewhat involved and it is a flaw of the book that no attempt is made to give a rough outline of the historical development and the names involved. Instead, the authors mention the so-called refined form of the BSD conjecture for which they provide an essentially literal copy from Wiles’ review article. The formulation involves many unexplained terms and is actually dispensable for the exposition. It would have been worthwhile, in contrast, to feature some of the mentioned plots (p. 231) that led Birch and Swinnerton-Dyer to their conjecture, even though a source from the AMS is available online. A major part of the last chapter on the congruent number problem is an excerpt from the authors’ previous book, as they tell us, and would indeed

have been more suitable for a warm-up opening story about the history of diophantine equations, which gets a raw deal here. The book culminates in the exercise to show that the specific number 1234567 is the area of a rational right triangle. This result is memorable but seems a bit anticlimactic. Overall it feels like the authors reached their summit somewhat short of breath.

Among the famous open problems in mathematics, the BSD conjecture is considered as one of the particularly beautiful ones. A professional mathematician who is not an expert on the field would appreciate a better-organised, more elliptical introduction to the subject than the one at hand, and there do exist excellent non-technical expositions. It is true, however, that none of these makes so great an effort to explain the basic concepts underlying the BSD conjecture to an untrained reader. Rather than lamenting the low level skills of the average first-year mathematics student this book responds by trying to actually provide relief. This should be recognized. As outlined above, however, the specific work-out bares some methodical and pedagogical deficiencies which make the book unsuitable as a mathematical guide for an inexperienced yet interested and gifted student.

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