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QUADRATIC MONOMIAL ALGEBRAS AND THEIR COHOMOLOGY

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Abstract. The aim of this note is to discuss and highlight the use of projective modules and projective resolutions in homological algebra. Using a minimal projective resolution by Sköldberg in [8] , we describe the calculation of the cohomology groups for the class of quadratic monomial algebras. The cohomology groups of an associative algebra are invariants of an algebra and provide a fundamental description of the structure of the algebra.

1. Introduction

In a forthcoming paper with Emil Sköldberg, the cohomology groups and cohomology ring structure are explicitly described for a particular class of associative algebras: the class of *quadratic monomial algebras.* Until recently, little was known about the multiplicative structure of the Hochschild cohomology ring for most classes of associative algebras. During the last decade or so, more light has been shed on this topic, with several papers published regarding the structure of the Hochschild cohomology rings for various classes of associative algebras, see for instance [1] and [2] . In [2], Claude Cibils computes the cohomology groups and ring structure for the class of radical square zero algebras. The aforementioned algebras are a subclass of algebras considered in this paper.

2. QUIVERS AND QUADRATIC MONOMIAL ALGEBRAS

In this section we begin by recapitulating the following definition of a quiver and from there we will present the main objects of interest in this article, namely the class of quadratic monomial algebras.

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Definition 1. A *quiver* $\Delta = (\Delta_0, \Delta_1)$, is an oriented graph, where Δ_0 denotes the set of vertices, and Δ_1 the set of arrows between the vertices. The origin and terminus of an arrow $a \in \Delta_1$, is denoted by $o(a)$ and $t(a)$ respectively.

We shall deal with finite connected quivers, that is the sets Δ_0 and Δ_1 are finite and the undirected graph will be connected. A *path* α in Δ, is an ordered sequence of arrows, $\alpha = a_1 \cdots a_n$, $a_i \in \Delta_1$ with $t(a_i) = o(a_{i+1})$ for $i = 1, \cdots, n-1$. We shall write $o(\alpha) = o(a_1)$ and $t(\alpha) = t(a_n)$ for the initial and terminal vertices of α respectively. An *oriented* cycle in Δ, is a path α, where $o(\alpha) = t(\alpha)$. The *length* or *degree* of α, denoted $|\alpha|$ is equal to the number of arrows in α and the set of all paths of length n, is denoted Δ_n. A vertex $e \in \Delta_0$ is considered to be a path of length zero with $o(e) = t(e) = e$. We shall allow Δ to have oriented cycles and multiple arrows between vertices.

Now we would like to make a semigroup out of the paths, and we will do this by first defining the set $\hat{\Delta}$ by

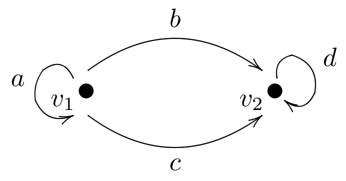
$$\hat{\Delta} = \{\perp\} \cup \bigcup_{i=0}^{\infty} \Delta_i$$

The multiplication in $\hat{\Delta}$ is, for $\gamma \in \Delta_i$, $\delta \in \Delta_j$, defined by $\gamma \cdot \delta = \gamma\delta$, if $t(\gamma) = o(\delta)$ and $\gamma \cdot \delta = \perp$ otherwise. For all $\alpha \in \hat{\Delta}$ we have $\alpha \cdot \perp = \perp \cdot \alpha = \perp$. For k a commutative ring, we may now form the semigroup-algebra $k\hat{\Delta}$, and then we may view $k\Delta$, the *path* or *quiver* algebra on Δ, as the quotient algebra $k\hat{\Delta}/(\perp)$. The benefit of this definition is that $k\Delta$ becomes a $\hat{\Delta}-$graded algebra. There is also an $\mathbb{N}-$grading on $k\Delta$, where $\deg_{\mathbb{N}} \alpha = n$, if $\alpha \in \Delta_n$. The paths of length 0, Δ_0 generate a subalgebra $k\Delta_0$ of $k\Delta$; hence $k\Delta$ is a $k\Delta_0-$bimodule. The identity element in $k\Delta$ is given by the sum of vertices. The class of algebras studied here are quotients of path algebras.

Definition 2. A *quadratic monomial algebra* A is a quotient of a quiver algebra $k\Delta$, $A = k\,\Delta/I$, where $I = (\alpha_1, \dots, \alpha_n)$ is a two sided homogeneous ideal generated by a set of paths of length two in Δ.

Since I is a two sided homogeneous ideal with respect to the standard grading, A is a graded algebra and we may describe a canonical $k-$basis for A and denote it $B(A)$. Such a basis consists of all paths that do not contain a path from I. A typical basis element in A may be written as $a_1 \ldots a_n$, where $a_i a_{i+1} \notin I$ for $1 \leq i \leq n-1$.

Example 1. Let k be any field and Δ the following quiver:



We may construct a quadratic monomial algebra which we shall denote by A by choosing an ideal I, generated by paths of length two and then forming the quotient $A = k\,\Delta/I$. For instance if we let $I = (aa, dd)$ then the following is an example of multiplication in A:

$$v_1 \cdot v_1 = v_1,\ \ v_1 \cdot v_2 = 0,\ \ v_1 \cdot b = b,\ \ a \cdot b = ab,\ \ b \cdot a = 0, \quad \text{etc.}$$

The products

$$a \cdot a = aa, \quad d \cdot d = dd$$

are equal to 0 in A, since both of these paths are contained in I.

3. PROJECTIVE MODULES AND PROJECTIVE RESOLUTIONS

A central theme in the world of homological algebra is the exploration of the structure of rings and modules. Projective modules are a basic tool which are used extensively in this examination, since any module may be viewed as an epimorphic image of a projective module - just choose a set of generators $\{g_i\}$ for M and map a projective module on a corresponding set of generators $\{e_i\}$ to M by sending e_i to g_i. In this way it is easy to compare any module to a projective module: if $d : P \longrightarrow M$ is an epimorphism, then we may say that P differs from M by the kernel of d. For more on this, see for example [3]. We will explore this idea in the present section. We will assume that we are working with left modules unless stated otherwise. There are several equivalent definitions of a projective module but the one that will be of most interest to us at present will be the following:

Definition 3. Let A be a ring. An A module P is *projective* if it is isomorphic to a direct summand of a free A module.

We will now introduce the following notation with the quadratic monomial algebra A of example 1 in mind.

Let

$$P_{v_1} := \langle \text{ All paths in } A \text{ starting at } v_1, \text{ including the zero path } v_1 \rangle$$
$$P_{v_2} := \langle \text{ All paths in } A \text{ starting at } v_2, \text{ including the zero path } v_2 \rangle$$
$$P_{v_1} \cdot J := \langle \text{ All paths in } A \text{ starting at } v_1 \text{ of length} \geq 1 \rangle.$$

Since a path may only begin at either vertex v_1 or v_2, we have

$$P_{v_1} + P_{v_2} = A \quad \text{and} \quad P_{v_1} \bigcap P_{v_2} = 0$$

and so

$$A = P_{v_1} \bigoplus P_{v_2}$$

P_{v_1} and P_{v_2} are direct summands of the left module A (which is free over itself) and so we may regard them as (indecomposable) projective left A modules. In the following we write $\operatorname{im} d$ to denote the image of the homomorphism d and $\ker d$ to denote the kernel of d.

Definition 4. Let M be an A-module. The sequence P_* of A-modules and A-module homomorphisms

$$P_* : \quad \cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

is called a *complex* if $\operatorname{im} d_{i+1} \subseteq \ker d_i$ for each i. P_* is said to be *exact* if the $\operatorname{im} d_{i+1} = \ker d_i$ for each i. P_* is called a *projective resolution* of M over A, if it is exact for all $i \geq 0$ and each P_i is a projective A-module.

For now, P_* will denote a projective resolution of M and when we write *deleted* projective resolution, we shall mean P_* with the M term removed. We will be mainly interested in the case when $M = A$. Projective resolutions are utilised in homological algebra as a way of approximating a module by using "well behaved" projective modules. It is well known that any A-module admits a projective resolution, see for instance [3] and we illustrate this below for the

$A-$module $P_{v_1}/P_{v_1}\cdot J$ taking the quadratic monomial algebra A in example 1 on page 41, as a demonstration.

We begin by considering the epimorphism ε in the following exact sequence:

$$P_{v_1} \xrightarrow{\varepsilon} P_{v_1}/P_{v_1}\cdot J \longrightarrow 0$$

P_{v_1} differs from $P_{v_1}/P_{v_1}\cdot J$ and this difference is recorded in $\ker\varepsilon$.

We may now form the exact sequence:

$$\ker\varepsilon \hookrightarrow P_{v_1} \xrightarrow{\varepsilon} P_{v_1}/P_{v_1}\cdot J \longrightarrow 0 \tag{1}$$

but in the above, $\ker\varepsilon = P_{v_1}\cdot J$ is not projective as an $A-$module, since it is not a direct summand of the free module A. We may correct this blemish however by finding a projective resolution of $P_{v_1}\cdot J$. Consider the following sequence

$$\ker d_1 \hookrightarrow\ P \xrightarrow{d_1} P_{v_1}\cdot J \longrightarrow 0$$

We would now like to construct a projective module P and epimorphism d_1, making the aforementioned sequence exact. Note any path in $P_{v_1}\cdot J$ can be expressed using the generating set $\{\, v_1a\alpha,\ v_1b\beta,\ v_1c\gamma\,\}$, where $\alpha\in P_{v_1}$, $\beta\in P_{v_2}$, and $\gamma\in P_{v_2}$ (see quiver on page 41). If we replace P with the direct sum of projective modules $P_{v_1}\oplus P_{v_2}\oplus P_{v_2}$ (which again results in a projective module), then a typical element in $P_{v_1}\oplus P_{v_2}\oplus P_{v_2}$ is of the form

$$v_1\alpha + v_2\beta + v_2\gamma$$

and we may define a surjective homomorphism d_1 on a generating element as follows:

$$\begin{aligned} d_1(v_1\alpha + v_2\beta + v_2\gamma) &= d_1(v_1\alpha) + d_1(v_2\beta) + d_1(v_2\gamma) \\ &= v_1a\alpha + v_1b\beta + v_1c\gamma \end{aligned}$$

It is easy to see that d_1 is surjective and $\operatorname{im} d_1 = \ker\varepsilon = P_{v_1}\cdot J$ and so replacing $\ker\varepsilon$, with $P_{v_1}\oplus P_{v_2}\oplus P_{v_2}$ in (1), the projective resolution of $P_{v_1}/P_{v_1}\cdot J$ now becomes:

$$\ker d_1 \hookrightarrow P_{v_1}\oplus P_{v_2}\oplus P_{v_2} \xrightarrow{d_1} P_{v_1} \xrightarrow{\varepsilon} P_{v_1}/P_{v_1}\cdot J \longrightarrow 0$$

Again as before, the $\ker d_1$ is not projective as an $A-$module. We may again tackle this situation using the same approach as previously: forming a projective resolution of $\ker d_1$. The kernel of d_1

is generated by $v_1 \cdot a$. We need to define a homomorphism and projective module that maps onto $\ker d_1$:

$$P_{v_1} \xrightarrow{d_2} \ker d_1 \longrightarrow 0 \qquad \text{with} \quad v_1 \xrightarrow{d_2} v_1 a$$

and now the projective resolution of $P_{v_1}/P_{v_1} \cdot J$ takes the form:

$$\cdots \xrightarrow{d_3} P_{v_1} \xrightarrow{d_2} P_{v_1} \oplus P_{v_2} \oplus P_{v_2} \xrightarrow{d_1} P_{v_1} \xrightarrow{\varepsilon} P_{v_1}/P_{v_1} \cdot J \longrightarrow 0$$

If we delete $P_{v_1}/P_{v_1} \cdot J$ from the above exact complex, we get a projective resolution $P_{v_1}/P_{v_1} \cdot J$. We may view this deleted resolution as an approximation of the simple module $P_{v_1}/P_{v_1} \cdot J$.

4. Hochschild cohomology

The appropriate cohomology theory for the class of associative $k-$algebras was first described by Gerhard Hochschild in [6]. Before continuing any further, we shall introduce the following notation: Given an arbitrary associative $k-$algebra A with unit, we shall write $A^e = A \otimes_k A^{op}$ to denote the enveloping algebra of A. Here we write A^{op} to denote the *opposite* algebra of A; as vector spaces A and A^{op} are isomorphic but A^{op} is endowed with the opposite multiplication of A:

$$a^{op} b^{op} = (ba)^{op}, \text{ where } a,\, b \in A, \quad a^{op},\, b^{op} \in A^{op}$$

By $\mathrm{Hom}_{A^e}(M, N)$ we shall mean the set of all A^e-homomorphisms from M to N. This set may be endowed with the structure of an abelian group, where for $f,\, g \in \mathrm{Hom}_{A^e}(M, N)$ it may be shown that $f + g$ defined by $(f+g)(m) = f(m)+g(m)$ is an A^e-homomorphism for all $m \in M$.

$\mathrm{Ext}_{A^e}(A, M)$ may be defined as the Hochschild cohomology group $H^*(A, M)$ of A with coefficients in the $A-$bimodule M. When $M = A$, it may be shown that $\mathrm{Ext}_{A^e}(A, A)$ also possesses a rich multiplicative structure:

$$\mathrm{Ext}^m_{A^e}(A, A) \otimes_k \mathrm{Ext}^n_{A^e}(A, A) \longrightarrow \mathrm{Ext}^{m+n}_{A^e}(A, A)$$

turning $\mathrm{Ext}_{A^e}(A, A)$ into a *graded commutative* algebra. An algebra A is graded commutative (or *supercommutative*) with homogeneous elements a and b of degree m and n in A respectively if

$$ab = (-1)^{mn}\, ba.$$

To compute the aforementioned cohomology groups of A, we begin by first applying the left exact functor $\mathrm{Hom}_{A^e}(\,\cdot\,, M\,)$ to the deleted *standard* projective resolution of A, where the nth projective module has the form $P_n = A \otimes_k A^{\otimes n} \otimes_k A$ and so the resolution may be written:

$$\cdots \longrightarrow A \otimes A^{\otimes n} \otimes A \xrightarrow{d_n} \cdots \xrightarrow{d_2} A \otimes A \otimes A \xrightarrow{d_1} A \otimes A \longrightarrow 0$$

The differential d_n is given by

$$\begin{aligned} d_n(a_0 \otimes a_1 \cdots a_n \otimes a_{n+1}) &= a_0 a_1 \otimes a_2 \cdots a_n \otimes a_{n+1} \\ &+ \sum_{i=1}^{n-1} (-1)^i a_0 \otimes a_1 \cdots (a_i a_{i+1}) \cdots a_n \otimes a_{n+1} \\ &+ (-1)^n\, a_0 \otimes a_1 \cdots a_{n-1} \otimes a_n a_{n+1} \end{aligned}$$

Now using the isomorphism $\mathrm{Hom}_{A^e}(A \otimes A^{\otimes n} \otimes A, M) \cong \mathrm{Hom}_k(A^{\otimes n}, M)$, we get the so called *Hochschild complex*:

$$0 \longrightarrow A \xrightarrow{\delta^0} \mathrm{Hom}_k(A, M) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta} \mathrm{Hom}_k(A^{\otimes n}, M) \xrightarrow{\delta} \cdots$$

The nth Hochschild cohomology module of A with coefficients in M is given by

$$H^n(A, M) \cong \ker \delta^n / \mathrm{im}\, \delta^{n-1} \cong \mathrm{Ext}_{A^e}(A, M).$$

In particular, we shall be interested in the case when $M = A$ and we shall write $HH^*(A)$ instead of $H^*(A, A)$. In this instance the differential δ has the following form:

$$\delta^0 : A \longrightarrow \mathrm{Hom}_k(A, A) \quad \text{with} \quad (\delta^0 b)(a) = ab - ba \quad \text{for } a,\, b \in A.$$

and for $n \geq 1$, $\quad \delta^n : \mathrm{Hom}_k(A^{\otimes n}, A) \longrightarrow \mathrm{Hom}_k(A^{\otimes(n+1)}, A)$;

$$\begin{aligned} (\delta^n f)(a_1 \otimes \cdots a_{n+1}) &= a_1 f(a_2 \otimes \cdots \otimes a_{n+1}) \\ &+ \sum_{1 \leq j \leq n} (-1)^j f(a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{n+1}) \\ &+ (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) a_{n+1} \end{aligned}$$

4.1. **Interpreting the 0th and 1st cohomology groups.** In this subsection we highlight some aspects of the cohomology groups in dimensions ≤ 2. In particular the Hochschild cohomology groups may be interpreted as providing an insight into the structure of an algebra. We begin by considering $HH^0(A)$. This group is isomorphic to the kernel of δ^0 and hence consists of all those elements in A that commute with all elements in A, that is $HH^0(A)$ is the centre of A:

$$HH^0(A) \cong \{b \in A |\, ab = ba \text{ for all } a \in A\}.$$

Definition 5. A *derivation* of A to A is an $k-$module homomorphism $f : A \longrightarrow A$ that satisfies Leibnitz's rule:

$$f(ab) = af(b) + f(a)b \quad \text{ for all } a, b \in A$$

A derivation f of A to A is an *inner derivation* if there exists $b \in A$ such that:

$$f(a) = ab - ba \quad \text{ for all } a \in A$$

Now returning to the coboundary δ and setting

$$\delta^1(f)(a \otimes b) = af(b) - f(ab) + f(a)b$$

equal to zero, we observe that a 1-cocycle (an element in the kernel of δ^1) is also a linear map $f : A \longrightarrow A$ which satisfies Leibnitz's condition. Similarly for each $b \in A$,

$$\delta^0(b)(a) = ab - ba \quad \text{for all } a \in A$$

and so there is a one-to-one correspondence between the coboundaries lying in $\operatorname{im} \delta^0$ and the inner derivations of A. The $k-$module $HH^1(A)$ may be interpreted as the space of all bimodule derivations of A modulo the inner derivations of A.

There are also connections to algebraic geometry. In [4], Murray Gerstenhaber introduced a deformation theory for rings and algebras based on formal power series. A formal deformation of an associative algebra (A, μ) is an associative algebra $A[[t]]$ with a multiplication μ_t defined by

$$\mu_t(p, q) = \mu(p, q) + t\mu_1(p, q) + t^2\mu_2(p, q) + \cdots$$

where $p, q \in A$. The algebra is said to be *rigid* if every formal deformation is isomorphic to a trivial deformation. One may show that separable semi-simple algebras are rigid. In the same paper

Gerstenhaber observed that algebras A which satisfy $HH^2(A) = 0$ are *rigid.*

4.2. **A Projective Resolution of a Quadratic Monomial Algebra.** In general a module may have several projective resolutions. When one wishes to compute (co)homology, a *minimal* projective resolution is best. In the following we shall state a minimal projective A^e- resolution for a quadratic monomial algebra. This minimal projective resolution was constructed by Emil Sköldberg in[8]. In order to write down this resolution the following definition will be required:

Definition 6. Let $A = k\Delta/I$ be an algebra such that I is an ideal generated by quadratic monomials. Define the ideal J to be generated by all quadratic monomials that do not lie in I; then the *Koszul dual* of A, denoted by $A^!$, is defined by $A^! = k\Delta/J$.

Example 2. For the given quadratic monomial algebra A in example 1 on page 41, we have $A^! = k\Delta/J$, with $J = (ab, ac, cd, bd)$. $A^!$ has the same generators of A, with the following as an example of multiplication in $A^!$:

$$v_1 \cdot v_1 = v_1,\ \ v_1 \cdot v_2 = 0,\ \ v_1 \cdot b = b,\ \ a \cdot a = aa,\ \ d \cdot d = dd, \quad \text{etc.}$$

The products

$$a \cdot b = ab, \quad d \cdot d = dd \quad a \cdot c = ac, \quad c \cdot d = cd$$

are equal to zero in $A^!$, since all products are contained in J.

The projective modules in the minimal projective graded A^e-resolution of A have the following description. From here on, when we write P_* we shall be referring to the following minimal A^e-resolution.

Lemma 1. *If $A = k\Delta/I$ is a quadratic monomial algebra, then a minimal projective resolution of A given as a left A^e-module is*

$$P_i = A \otimes_{k\Delta_0} A^!_i \otimes_{k\Delta_0} A$$

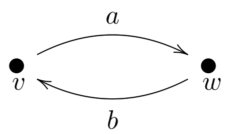
and the A^e-linear differential is defined on the basis elements by

$$d_i(1 \otimes\ a_1 \cdots a_i \otimes\ 1) = a_1 \otimes\ a_2 \cdots a_i \otimes\ 1 + (-1)^i\, 1 \otimes\ a_1 \cdots a_{i-1} \otimes a_i$$

Proof. See [8].

□

Example 3. Consider $A = k\Delta/I$, where

$\Delta =$ 

$I = (ab, ba)$ and $J = (\emptyset)$. The projective modules P_i in P_*, have the form, where we write $\otimes = \otimes_{k\Delta_0}$:

$$\cdots \xrightarrow{d_4} \begin{array}{c} A\otimes aba\otimes A \\ \oplus \\ A\otimes bab\otimes A \end{array} \xrightarrow{d_3} \begin{array}{c} A\otimes ab\otimes A \\ \oplus \\ A\otimes ba\otimes A \end{array} \xrightarrow{d_2} \begin{array}{c} A\otimes a\otimes A \\ \oplus \\ A\otimes b\otimes A \end{array} \xrightarrow{d_1} \begin{array}{c} A\otimes v\otimes A \\ \oplus \\ A\otimes w\otimes A \end{array} \xrightarrow{\varepsilon} A \to 0$$

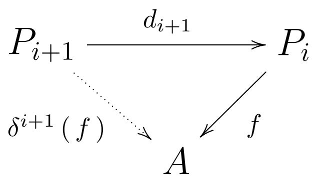
Notice that the module P_0 records the vertices of the quiver, P_1 records the arrows that generate A, P_2 records the relations of A, P_3 records the relations $(ab)a = a(ba)$ among relations of A, etc.

5. Cohomology of a Quadratic Monomial Algebra

In a forthcoming paper with Emil Sköldberg, the cohomology groups for a quadratic monomial algebra are explicitly described. In this section we illustrate this theory by calculating the Hochschild cohomology groups of the given quadratic monomial algebra A in example 3 on page 48. We begin with the following complex:

$$0 \longrightarrow \mathrm{Hom}_{A^e}(P_0, A) \xrightarrow{\delta^0} \cdots \xrightarrow{\delta^{n-1}} \mathrm{Hom}_{A^e}(P_n, A) \xrightarrow{\delta^n} \cdots \qquad (2)$$

The coboundary δ is induced by the differential d on P_*:



$$\begin{aligned} \delta^{i+1} f(1\otimes a_1\cdots a_{i+1}\otimes 1) &= f(d_{i+1}(1\otimes a_1\cdots a_{i+1}\otimes 1)) \\ &= f(a_1 \otimes a_2\cdots a_{i+1}\otimes 1) \\ &\quad + (-1)^{i+1} f(1\otimes a_1\cdots a_i\otimes a_{i+1}) \end{aligned}$$

where $f \in \mathrm{Hom}_{A^e}(P_i, A)$. As we have seen earlier, the nth Hochschild cohomology module of A with coefficents in A, may be found by computing

$$HH^n(A) \cong \ker\delta^n/\mathrm{im}\,\delta^{n-1} \cong \mathrm{Ext}^n_{A^e}(A, A)$$

We will use the following lemma to simplify our calculation of the cohomology groups.

Lemma 2. *The map* $\phi : \mathrm{Hom}_{A^e}(P_i, A) \longrightarrow \mathrm{Hom}_{k\Delta_0^e}(A_i^!, A)$ *defined by*

$$\phi(f)(a_1 \cdots a_i) := f(1 \otimes_{k\Delta_0} a_1 \cdots a_i \otimes_{k\Delta_0} 1),$$

$f \in \mathrm{Hom}_{A^e}(P_i, A)$ *is a chain map and vector space isomorphism for each* i.

Proof. This result is proved in the authors thesis. □

We have established through lemma 2 that calculating the cohomology of the cochain complex at (2), will yield the same results as computing the cohomology of the following cochain complex:

$$0 \longrightarrow \mathrm{Hom}_{k\Delta_0^e}(A_0^!, A) \xrightarrow{\bar{\delta}^0} \cdots \xrightarrow{\bar{\delta}^{n-1}} \mathrm{Hom}_{k\Delta_0^e}(A_n^!, A) \xrightarrow{\bar{\delta}^n} \cdots \qquad (3)$$

We shall now take a moment to describe the coboundary homomorphism $\bar{\delta}$ and the $k-$module $\mathrm{Hom}_{k\Delta_0^e}(A^!, A)$ in a little more detail. An element $f \in \mathrm{Hom}_{k\Delta_0^e}(A^!, A)$ is a $k\Delta_0^e-$ linear homomorphism from $A^!$ to A. For $\alpha \in B(A^!)$ and $\beta \in B(A)$, we shall use the notation (α, β) for the morphism $\alpha \stackrel{f}{\longmapsto} \beta$, and $\gamma \mapsto 0$ for all other basis elements $\gamma \in B(A^!)$. We shall write $o(\alpha) = v$ and $t(\alpha) = w$. Since $f \in \mathrm{Hom}_{k\Delta_0^e}(A_i^!, A)$ is linear over the vertices, we have

$$f(\alpha) = f(v \cdot \alpha) = v \cdot f(\alpha) = v \cdot \beta$$

and

$$f(\alpha) = f(\alpha \cdot w) = f(\alpha) \cdot w = \beta \cdot w$$

and so for an $f \in \mathrm{Hom}_{k\Delta_0^e}(A^!, A)$, we shall write (α, β) such that $o(\alpha) = o(\beta)$ and $t(\alpha) = t(\beta)$. It is shown in [7] that the coboundary operator $\bar{\delta}$ on these basis elements is given by

$$\bar{\delta}(\alpha, \beta) = \sum_{a \in \Delta_1} (a\alpha, a\beta) + (-1)^{|\alpha|+1} \sum_{b \in \Delta_1} (\alpha b, \beta b)$$

From here on we shall write δ in of place of $\bar{\delta}$. We are now in a position to calculate the cohomology groups of the quadratic monomial algebra given in example 2, page 48.

Example 4. We begin with the following deleted projective resolution of A:

$$\cdots \xrightarrow{d_3} \begin{matrix} A\otimes ab\otimes A \\ \oplus \\ A\otimes ba\otimes A \end{matrix} \xrightarrow{d_2} \begin{matrix} A\otimes a\otimes A \\ \oplus \\ A\otimes b\otimes A \end{matrix} \xrightarrow{d_1} \begin{matrix} A\otimes v\otimes A \\ \oplus \\ A\otimes w\otimes A \end{matrix} \longrightarrow 0$$

Applying $\mathrm{Hom}_{A^e}(\cdot,\, A\,)$ and then using the isomorphism $\mathrm{Hom}_{A^e}(P_i, A) \cong \mathrm{Hom}_{k\Delta_0^e}(A_i^!, A)$,we get the resulting cochain complex :

$$0 \longrightarrow \begin{matrix} \mathrm{Hom}_{k\Delta_0^e}(v,\, A\,) \\ \oplus \\ \mathrm{Hom}_{k\Delta_0^e}(w,\, A\,) \end{matrix} \xrightarrow{\delta^1} \begin{matrix} \mathrm{Hom}_{k\Delta_0^e}(a,\, A\,) \\ \oplus \\ \mathrm{Hom}_{k\Delta_0^e}(\,b,\, A\,) \end{matrix} \xrightarrow{\delta^2} \begin{matrix} \mathrm{Hom}_{k\Delta_0^e}(\,ab,\, A\,) \\ \oplus \\ \mathrm{Hom}_{k\Delta_0^e}(\,ba,\, A\,) \end{matrix} \xrightarrow{d^3} \cdots$$

We would now like to calculate the cohomology in each degree of this complex but before doing this we note a k-basis for $\mathrm{Hom}_{k\Delta_0^e}(A^!, A)$ is given by all (α, β) with $o(\alpha) = o(\beta)$ and $t(\alpha) = t(\beta)$. Hence we may simplify the notation in the aforementioned complex by rewriting it as:

$$0 \longrightarrow \begin{matrix} k\cdot(v,\, v) \\ \oplus \\ k\cdot(w,\, w) \end{matrix} \xrightarrow{\delta^0} \begin{matrix} k\cdot(a,\, a) \\ \oplus \\ k\cdot(b,\, b) \end{matrix} \xrightarrow{\delta^1} \begin{matrix} k\cdot(ab,\, v) \\ \oplus \\ k\cdot(ba,\, w) \end{matrix} \xrightarrow{\delta^2} \cdots$$

We may now calculate the cohomology groups in each degree. We begin by computing $HH^0(A) \cong \ker \delta^0$:

$$\begin{aligned} \delta^0(\lambda(v,v) + \mu(w,w)) = \lambda((b,\, b) - (a,\, a)) \\ + \mu((a,\, a) - (b,\, b)) \end{aligned}$$

which is equal to zero $\Leftrightarrow \lambda = \mu$ for $\lambda,\, \mu \in k$.

Hence the kernel of δ^0 is one dimensional and is generated by $(u,\, u) + (v,\, v)$ and so $HH^0(A) \cong k$.

Next we compute $HH^1(A)$. Since $\dim(\ker \delta^0) \cong k$ and δ^0 is a map from a two dimensional vector space, we have $\dim(\mathrm{im}\, \delta^0) \cong k$. Now

$$\delta^1(\lambda(a,a)) = \lambda(ba,\, ba) + \lambda(ab,\, ab) = 0$$

since $ba \in J$ or $ba \in I$ and $ab \in J$ or $ab \in I$. For the same reason we also have

$$\delta^1((\mu(b,b))) = \mu(ab,ab) + \mu(ba,ba) = 0$$

Hence $\ker \delta^1 \cong k \oplus k$ and so we have $HH^1(A) \cong k$.

Finally we compute $HH^2(A)$. Since the kernel of δ^1 is two dimensional, this means the dimension of the image $\operatorname{im}\delta^1$ is trivial. Computing as before, it is easy to show that the kernel of δ^2 is generated by $(ab, v) + (ba, w)$ and so $\dim(\ker\delta^2) \cong k$. Hence $HH^2(A) \cong k$.

□

6. Interpreting the 0th & 1st cohomology groups of A

As we have already seen, the zeroth cohomology group $HH^0(A)$ coincides with the centre of A. We shall illustrate this with an example.

Example 5. Let $A = k\Delta/I$ be the quadratic monomial algebra obtained from the quiver

$$\underset{u}{\bullet} \xrightarrow{a} \underset{v}{\bullet}$$

where in this instance $I = J = \{\emptyset\}$. Note the centre of an algebra A consists of all those x of A such that $xa = ax$ for all a in A.

We shall first compute the centre of the given quadratic monomial algebra:

$$(\lambda u + \varphi v + \psi a) \cdot u = u \cdot (\lambda u + \varphi v + \psi a)$$
$$\Leftrightarrow \lambda u = \lambda u + \psi a \quad \Rightarrow \psi = 0$$
$$(\lambda u + \varphi v) \cdot a = a \cdot (\lambda u + \varphi v)$$
$$\Leftrightarrow \lambda a = \varphi a \quad \Rightarrow \lambda = \varphi$$

and so the centre of A is a one dimensional vector space spanned by $\langle u + v \rangle$. Computing $HH^0(A)$ from the following complex:

$$0 \longrightarrow k \cdot (u, u) \oplus k \cdot (v, v) \longrightarrow k \cdot (a, a) \longrightarrow 0$$

it is easy to show that it is a one dimensional vector space generated by $(u, u) + (v, v)$. Hence the centre of A and $HH^0(A)$ are isomorphic as vector spaces.

□

Example 6. Let $A = k\Delta/I$ be obtained from the following quiver

$$\Delta := \quad \underset{u}{\bullet} \underset{b}{\overset{a}{\rightrightarrows}} \underset{v}{\bullet}$$

We write $f \in \operatorname{Hom}_k(A, A)$ as

$$\begin{aligned} f(u) &= \lambda_1 u + \lambda_2 v + \lambda_3 a + \lambda_4 b \\ f(v) &= \tau_1 u + \tau_2 v + \tau_3 a + \tau_4 b \\ f(a) &= \varphi_1 u + \varphi_2 v + \varphi_3 a + \varphi_4 b \\ f(b) &= \psi_1 u + \psi_2 v + \psi_3 a + \psi_4 b \end{aligned}$$

for all λ_i, τ_i, φ_i, $\psi_i \in k$. The derivations of A are defined as those $k-$module homomorphisms $f : A \longrightarrow A$ that satisfy

$$f(ab) = af(b) + f(a)\,b\ , \quad \text{for all} \quad a, b \in A.$$

We shall denote this group of derivations by $\mathrm{Der}(A, A)$ and we begin now by computing all the derivations of the given algebra A:

$$\begin{aligned} \lambda_1 u + \lambda_2 v + \lambda_3 a + \lambda_4 b = f(u) = f(u \cdot u) &= u \cdot f(u) + f(u) \cdot u \\ &= 2\lambda_1 u + \lambda_3 a + \lambda_4 b \end{aligned}$$

This implies $\lambda_1 u - \lambda_2 v = 0$ or $\lambda_1 = \lambda_2 = 0$.

$$0 = f(u \cdot v) = u \cdot f(v) + f(u) \cdot v = \tau_1 u + \tau_3 a + \tau_4 b + \lambda_2 v + \lambda_3 a + \lambda_4 b$$

This means $\tau_3 = -\lambda_3$, $\tau_4 = -\lambda_4$, $\tau_1 = 0$, $\lambda_2 = 0$. Continuing in the same way and computing the remaining 14 derivations:

$$f(u \cdot a),\ f(u \cdot b),\ f(v \cdot u),\ f(v \cdot v),\ f(v \cdot a)\ f(v \cdot b),\ f(a \cdot a),$$

$$f(a \cdot v),\ f(a \cdot b),\ f(a \cdot u),\ f(b \cdot u),\ f(b \cdot a),\ f(b \cdot b),\ f(b \cdot v)$$

we also have $\lambda_1 = \tau_2 = \psi_1 = \psi_2 = \varphi_1 = \varphi_2 = 0$ Substituting $\tau_3 = -\lambda_3$, $\tau_4 = -\lambda_4$, the k-module of all derivations from A to A is spanned by:

$$\begin{aligned} f(u) &= \lambda_3 a + \lambda_4 b, \\ f(v) &= -\lambda_3 a - \lambda_4 b = -f(u), \\ f(a) &= \varphi_3 a + \varphi_4 b \\ f(b) &= \psi_3 a + \psi_4 b \end{aligned}$$

for λ_3, λ_4, φ_3, φ_4, ψ_3, $\psi_4 \in k$. Hence $\mathrm{Der}(A, A)$ is a 6-dimensional vector space, with basis , given by the set of all derivations $f_i : A \longrightarrow A$, $1 \leq i \leq 6$;

$$f_1(x) = \begin{cases} a, & \textit{if} \quad x = u, \\ -a, & \textit{if} \quad x = v, \\ 0 & \textit{otherwise.} \end{cases} \qquad f_2(x) = \begin{cases} b, & \textit{if} \quad x = u, \\ -b, & \textit{if} \quad x = v, \\ 0 & \textit{otherwise.} \end{cases}$$

$$f_3(x) = \begin{cases} a, & if \quad x = a, \\ 0 & otherwise. \end{cases} \qquad f_4(x) = \begin{cases} b, & if \quad x = a, \\ 0 & otherwise. \end{cases}$$

$$f_5(x) = \begin{cases} a, & if \quad x = b, \\ 0 & otherwise. \end{cases} \qquad f_6(x) = \begin{cases} b, & if \quad x = b, \\ 0 & otherwise. \end{cases}$$

where x a basis element in A. The inner derivations of A are computed next.

$$\begin{array}{ll} u \longrightarrow u \cdot u - u \cdot u = 0 & u \longrightarrow u \cdot v - v \cdot u = 0 \\ v \longrightarrow v \cdot u - u \cdot v = 0 & v \longrightarrow v \cdot v - v \cdot v = 0 \\ a \longrightarrow a \cdot u - u \cdot a = -a & a \longrightarrow a \cdot v - v \cdot a = a \\ b \longrightarrow b \cdot u - u \cdot b = -b & b \longrightarrow b \cdot v - v \cdot b = b \\ = -f_3 - f_6 & = f_3 + f_6 = -(-f_3 - f_6) \end{array}$$

$$\begin{array}{ll} u \longrightarrow u \cdot a - a \cdot u = a & u \longrightarrow u \cdot b - b \cdot u = b \\ v \longrightarrow v \cdot a - a \cdot v = -a & v \longrightarrow v \cdot b - b \cdot v = -b \\ a \longrightarrow a \cdot a - a \cdot a = 0 & a \longrightarrow a \cdot b - b \cdot a = 0 \\ b \longrightarrow b \cdot a - a \cdot b = 0 & b \longrightarrow b \cdot b - b \cdot b = 0 \\ = f_1 & = f_2 \end{array}$$

The inner derivations form a 3-dimensional subspace of Der(A,A), and so the quotient space of derivations modulo inner derivations is:

$$\frac{k \oplus k \oplus k \oplus k \oplus k \oplus k}{k \oplus k \oplus k} \cong k \oplus k \oplus k$$

On the other hand, suppose we calculate the Hochschild cohomology of the complex associated with the given algebra:

$$0 \longrightarrow k(u,u) \oplus k(v,v) \xrightarrow{\delta^0} k(a,a) \oplus k(b,b) \oplus k(b,a) \oplus k(a,b) \xrightarrow{\delta^1} 0$$

we then have

$$HH^1(A) \cong \frac{k \oplus k \oplus k \oplus k}{k} \cong k \oplus k \oplus k$$

Hence as vector spaces, the space of outer derivations modulo the space of inner derivations and $HH^1(A)$ are isomorphic (as expected from section 4.1).

□

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