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The aim of the *Bulletin* is to inform Society members, and the mathematical community at large, about the activities of the Society and about items of general mathematical interest. It appears twice each year. The *Bulletin* is supplied free of charge to members; it is sent abroad by surface mail. Libraries may subscribe to the *Bulletin* for 30 euro per annum.

The *Bulletin* seeks articles written in an expository style and likely to be of interest to the members of the Society and the wider mathematical community. Short research papers are also welcome. All areas of mathematics will be considered, pure and applied, old and new. The *Bulletin* is typeset using LaTeX. Authors must submit their articles in one of the formats of TEX. See the inside back cover for instructions.

Correspondence concerning the *Bulletin* should be sent, in the first instance, by e-mail to the Editor at

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Further information about the Irish Mathematical Society and its *Bulletin* can be obtained from the IMS webpage

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EDITORIAL

In assuming the rôle of Editor of the Bulletin, I am conscious of the high standard maintained by my predecessors, and particularly by Martin Mathieu over the past ten years. The Society owes him a substantial debt for his dedicated service and his achievement.

I hope to remain true to the objectives that the Society has set for the Bulletin. This will be possible as long as the membership continues to support it, as in the past, by contributing interesting material of a suitable standard.

The Research Notes section attracts many submissions of indifferent quality, on rather niche or special subjects from people with no particular Irish connection. There is no point in encouraging this kind of submission, and I have determined to take a rather severe approach to all submissions for the Notes. I think that other categories of paper have been much more successful, and aligned with the aims of the journal, and I would like to encourage more submissions along the hitherto successful lines:

1. *informative surveys of active research areas, written for the general mathematically-literate reader,*
2. *biographical and historical articles related to Irish mathematics, including obituaries and interviews with senior figures. The recent series of interviews by Gary McGuire was very well received.*
3. *informative and factual articles, and letters with views, about important developments and events in Irish mathematics,*
4. *thesis summaries or abstracts from Irish schools and departments in the mathematical areas*
5. *book reviews.*

The Research Notes section will continue, but I am looking for well-written material likely to be of wide interest, preferably by someone with some obvious connection to Ireland.

Of course, if you feel the above policy does a disservice to the membership, I would be glad to listen to your views, and indeed the Bulletin is open to publishing letters on maths-related policy

Happily, Ian Short has undertaken to manage a Problem Section, and this is launched with this issue. It is really a re-launch: we had such a section for a period in the past, run by Tom Laffey (from Newsletter #1 (1978) to #4 (1981)) and by Phil Rippon (from Newsletter #5 (1982) to #15 (1985) and Bulletin #16 (1986) to #23 (1989)), and we hope that it will again prove a popular and useful feature.

This issue also sees the launch of a Classroom Notes section, dedicated to innovative teaching ideas.

Finally, we include a survey article on paradoxes, written in Irish. This is the first article in Irish since Jim Flavin's paper in Bulletin 54 (2004), pp. 53-62. It includes a glossary, and we hope members find this useful. People reading the pdf online can take advantage of the popup glosses that appear when the mouse is passed over terms that occur in the glossary.

—AOF

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1. The Irish Mathematical Society has reciprocity agreements with the American Mathematical Society, the Irish Mathematics Teachers Association, the New Zealand Mathematical Society and the Real Sociedad Matemática Española.

2. The current subscription fees are given below:

Institutional member	160 euro
Ordinary member	25 euro
Student member	12.50 euro
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   The subscription fees listed above should be paid in euro by means of a cheque drawn on a bank in the Irish Republic, a Eurocheque, or an international money-order.

3. The subscription fee for ordinary membership can also be paid in a currency other than euro using a cheque drawn on a foreign bank according to the following schedule:

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4. Any member with a bank account in the Irish Republic may pay his or her subscription by a bank standing order using the form supplied by the Society.

5. Any ordinary member who has reached the age of 65 years and has been a fully paid up member for the previous five years may pay at the student membership rate of subscription.

6. Subscriptions normally fall due on 1 February each year.

7. Cheques should be made payable to the Irish Mathematical Society. If a Eurocheque is used then the card number should be written on the back of the cheque.

8. Any application for membership must be presented to the Committee of the I.M.S. before it can be accepted. This Committee meets twice each year.

9. Please send the completed application form with one year's subscription to:

   The Treasurer, I.M.S.
   Department of Mathematics
   St Patrick's College
   Drumcondra
   Dublin 9, Ireland

Irish Math. Soc. Bulletin **67** *(2011), 5–25*

Alexei V. Pokrovskii

02.06.1948 – 01.09.2010

Alexei Vadimovich Pokrovskii, an outstanding mathematician, a scientist with broad mathematical interests and a pioneer in the mathematical theory of systems with hysteresis, died unexpectedly on September 1, 2010, aged 62. For the last nine years he was Professor and Head of Department of Applied Mathematics at University College Cork.

The main body of Alexei's work belongs to the areas of nonlinear dynamical systems (including systems with hysteresis, discontinuous and nonsmooth systems), control theory, nonlinear functional analysis and applied mathematical modelling. However, the remarkable diversity of his research was broader and included, at different stages of his work, contributions to game theory, stochastic systems, complexity and general functional analysis.

Alexei was born and reared in Voronezh, a city in Central Russia about 500 kilometres south of Moscow. His family came from a medical background. His paternal grandfather, Alexei Ivanovich Pokrovskii (1880-1958), was a professor and Chair of Ophthalmology at Voronezh Medical Academy, the author of more than 90 research publications. Alexei's father, Vadim Alexeevich Pokrovskii, was a professor and Chair of Hygiene in Voronezh Medical Academy; his uncle, Alexei Alexeevich Pokrovskii, was an academician, a vice-president of the Academy of Medical Science of the USSR and the director of the Institute of Nutrition in Moscow. Alexei's mother, Angelina, was a teacher of English. His daughter Olya continuing the family tradition graduated from University College Cork (UCC) with a primary medical degree in 2009; the same year, his son Alexei Jr. received MSc degree in mathematics from University of Cambridge.

Alexei, his wife Natasha, daughter Olya and son Alexei Jr. at the conferring ceremony on the day of Olya's graduation from UCC.

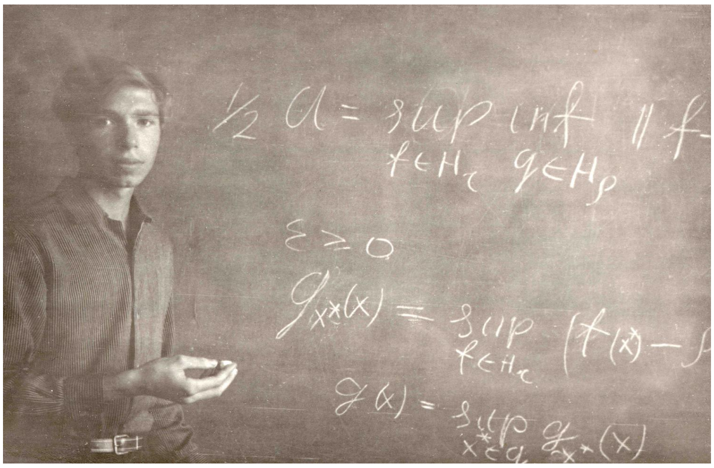
Alexei attended Voronezh State University in 1966-1971, where he received his BSc and MSc degrees in mathematics. He was a student of Mark Alexandrovich Krasnosel'skii, one of Russia's foremost mathematicians of the last century and the founder and the leader of the famous mathematical school of Nonlinear Functional Analysis. Mark Krasnosel'skii received his PhD under the direction of Mark Grigorievich Krein in 1948, and his Dr. Sci. (Dr. habil.) in 1950 at Kiev State University. He was invited to Voronezh by Vladimir Ivanovich Sobolev, a renowned expert in functional analysis, and was offered and accepted the Chair of Functional Analysis at Voronezh State University in 1952 at the age of 32. Later, they, together with Selim Krein, the younger brother of Mark Krein, who moved to Voronezh simultaneously with Mark Krasnosel'skii, organised the Mathematical Institute at Voronezh State University.

Mark Krasnosel'skii (left) and Vladimir Sobolev (right) in Voronezh.

By the time Alexei started his degree, Voronezh had become an important centre of mathematical and applied mathematical research. Alexei's early academic career was notable for the fact that he started publishing original mathematics while still a teenager. Professor Petr Petrovich Zabreiko recalled that a paper of A. F.

Timan on approximation theory was discussed at the Nonlinear Functional Analysis Seminar series (the renowned Krasnosel'skii's seminar). It was a big paper of more than 100 hundred pages. At some point, Alexei, a second year undergraduate student at the time, stepped forward to the blackboard and presented a simple but nontrivial equation. After a brief discussion it became clear that most of the results of Timan's paper follow from this formula. Alexei published this result in his first paper [1], which was frequently cited afterwards.

Alexei's outstanding talent and keen interest in mathematics became widely recognised in the University as he engaged in research with senior colleagues while still an undergraduate. Remarkably, the range of work he published as an undergraduate student included approximation theory, bifurcation theory, positive almost periodic functions, game theory and hysteresis operators. The latter topic is of special importance.



Alexei at Nonlinear Functional Analysis Seminar in Voronezh.

From 1969, when Alexei was yet only a 4^{th} year student (but already an author of 4 publications), Krasnosel'skii's group began to discuss the phenomenon of hysteresis, first presented at the Nonlinear Functional Analysis Seminar series by a physicist, Boris Darinskii. The term 'hysteresis' was coined by the physicist James Alfred Ewing in his paper on electromagnetism published in 1881. See [2] for the history of the question. Many phenomenological models of hysteretic relationships between physical variables have been known since the beginning of the last century, including the so-called models with local memory, such as Prandtl's ideal plastic element (also known as *stop*) and the *non-ideal relay*, and complex models with non-local memory such as the Preisach model of magnetic hysteresis, similar models of capillary hysteresis based on domain theory, the Prandtl-Ishlinskii model of plasticity and others. Hysteresis effects were vaguely associated with memory and multivalued functions. This memory was, however, different from the memory modelled by convolution operators or delayed systems. The main characterisation of the memory manifested through hysteresis was a permanent effect of certain events in the past on the future; permanent magnetisation of a ferromagnetic material resulting from a single fluctuation of an external magnetic field would be a typical example.

A group of participants of Krasnosel'skii's seminar, including Alexei, made the first step towards the mathematical treatment of this phenomenon by introducing a new class of operators related to Prandtl's model. These operators, now known as *hysterons*, have a simple definition on the class of piecewise monotone continuous functions of time (inputs), which they map to outputs from the same class. A continuity argument was used to extend these operators to the whole space of continuous functions [3]. In a subsequent paper, Krasnosel'skii and Pokrovskii proposed a new class of differential control equations, which include hysterons as a particular case, and found the conditions that ensure the continuity of the input-output relationships defined by these equations with respect to the uniform norm [4].

In the early 1970s, Alexei moved with Mark Krasnosel'skii and a part of his group to Moscow to the Institute for Control Problems of the Russian Academy of Sciences. Here, Alexei completed his PhD under the direction of Mark Krasnosel'skii in 1974. By the time of completion, his published work numbered 16 articles.

The Institute for Control Problems was founded in 1939 with the active participation of Alexander Alexandrovich Andronov, the author of classical results in Nonlinear Oscillations theory and the

A part of Krasnosel'skii's group (Research Laboratory of Mathematical Methods for Analysis of Complex Systems) at Moscow Institute for Control Problems. From left to right, upper row: A. V. Pokrovskii, V. I. Opoitsev, A. Sobolev, M. A. Krasnosel'skii; below: N. A. Bobylev.

founder of the famous Nonlinear Dynamics school in Lobachevsky State University of Nizhny Novgorod where he moved later. The new research environment, which included applied mathematicians, physicists and engineers, stimulated the interest of Krasnosel'skii and Pokrovskii in problems of control as well as reinforcing their research in hysteretic systems and providing a new perspective on them. They formulated a research programme aimed at developing a rigorous mathematical theory, which should deliver efficient mathematical tools for modelling systems with hysteresis, simultaneously making them amenable to the study by methods of differential equations, operator theory and nonlinear functional analysis. In particular, it should resolve the ambiguity about the nature of the permanent memory associated with hysteresis phenomena and the means of modelling it. Moreover, the theory should have the means of describing systems where some relationships between the variables were formulated in terms of differential equations, while other relationships were hysteretic and could be described by the hysteron operator or the like. A typical motivating example is Maxwell's equations coupled to a hysteretic constitutive relationship between the magnetic induction B and the magnetic field H such as the relationship assumed in the Preisach model of ferromagnetic media.

The realisation of this programme took a decade. Publications of Krasnosel'skii and Pokrovskii tell the story of the evolution of their views on the subject over this time. The language and paradigm of systems theory fused with the ideology of nonlinear analysis to create the fundamentals of the theory of systems with hysteresis. A hysteretic relationship with nonlocal memory was represented by a composition of the input-state operator and the state-output function (functional) with an infinite-dimensional (or multi-dimensional) state encoding the memory of the system. In many phenomenological models the output is obtained as the superposition of outputs of infinitely many hysterons such as non-ideal relays with varying parameters in the Preisach model, stops in the Prandtl-Ishlinskii model etc. Krasnosel'skii and Pokrovskii proposed a mathematical formalism of parallel connections and cascades of hysterons which allows one to extend the input-state-output operators from the class of piecewise monotone inputs to all continuous inputs. They studied the regularity properties of these operators using an alternative geometrical description of the evolution of memory states of the Preisach model [5]. A further important step was to extend these ideas to models/operators with vector-valued inputs and outputs [6]. At the same time, they began to study dynamical systems with hysteretic components starting from the example of an oscillator described by a second order differential equation coupled with the Prandtl-Ishlinskii and Preisach operators [7, 8].

The range of Alexei's interests was continuously growing. Together with Mark Krasnosel'skii and other colleagues, he addressed a number of control problems such as, for example, the mathematical formalism of the method of block diagrams [9]; the effect of small (in uniform norm) perturbations, which have finite quadratic variation (energy), on multicomponent systems and sliding modes [10, 11]; and the problem of absolute stability [12]. For example, the absolute stability of the zero solution was shown to be equivalent to the absence of nonzero uniformly bounded solutions for a wide class of evolutionary systems. Further results included conditions ensuring the positivity of the impulse-frequency response of linear systems, that is the property allowing one to apply the method of monotone operators to the analysis of such systems and their nonlinear extensions.

Together with Mark Krasnosel'skii, Alexei investigated discontinuous systems. Some typical problems associated with discontinuity are illustrated by their example of the "monster" binary function $f(t, x)$, which they showed to exist if the continuum hypothesis is true [13]. For every t, the "monster" function has the value 1 for all x except at at most a countable number of points; at the same time, for every x, this function is zero for almost every t (moreover, one can require the equality $f(t, x(t)) = 0$ to be true almost everywhere in t for every measurable function $x(t)$). In particular, the "monster" is not a measurable function of the two variables. However, the superposition operator defined by this function maps every measurable function $x(t)$ (input) to a measurable function $f(t, x(t))$ of t (output).

Another group of discontinuous problems is related to monotonic systems and discontinuous monotone operators. The Birkhoff-Tarski theorem guarantees the existence of a fixed point within an invariant interval of a monotone operator A, which acts in a Banach space semiordered by a cone. However, such a fixed point can be a discontinuity point of A. Krasnosel'skii and Pokrovskii showed the existence of a fixed point which is a continuity point of A (a regular fixed point) for general classes of monotone operators [14]. In applications to boundary-value problems, such a fixed point is a solution which passes through continuity points of discontinuous terms almost everywhere. In the context of dynamical systems, such points define stable solutions. Later, Alexei proposed a simple iterative algorithm, the so-called shuttle algorithm, for finding regular fixed points [15, 16].

Many results of Krasnosel'skii and Pokrovskii from this period, including some of the above mentioned, had links to the study of hysteresis phenomena. For example, the differential control equations they studied in relation to the hysteron and Duhem's magnetization model proved to be intimately connected to stochastic differential equations and allowed them to obtain a description of individual trajectories of Itō and Stratonovich stochastic equations [17].

In 1983 Alexei was awarded the prestigious Andronov prize by the USSR Academy of Sciences.

The seminal monograph of M. A. Krasnosel'skii and A. V. Pokrovskii "Systems with Hysteresis" appeared the same year [18]. It laid the foundations of the mathematical theory of hysteresis operators on the basis of the analysis of many phenomenological models of hysteresis, thereby forming the modern concepts of mathematical modelling of hysteretic systems, opening the door to the systematic application of mathematical tools to the analysis of dynamical systems with hysteretic components, and paving the way for the research that followed. The theory of Krasnosel'skii and Pokrovskii is essentially the theory of rate-independent operators[1] (the term and concept introduced later by Augusto Visintin). That is, these operators, acting in spaces of functions of time, are invariant with respect to the action of the group of monotone transformations of the time scale. This general definition entails a set of non-trivial properties of hysteresis operators which are sufficient for developing formal concepts with various applications. In particular, such operators are never differentiable, i.e., either nonsmooth or discontinuous. Rate-independent input-state-output operators have been constructed for models of hysteresis proposed in diverse disciplines.

The density of new ideas and concepts in Krasnosel'skii and Pokrovskii's book is remarkable. Literally every chapter of the book and sometimes even a section or remark pioneered a branch of the mathematical theory of systems with hysteresis. Topics in the book, to mention just some, include the regularity of hysteresis operators; identification theorems; composition and inversion of hysteresis operators and construction of compensators (a topic of importance for engineering and control applications); representation theorems; vibrostable differential equations; links with the theory of sweeping processes and Skorokhod problems; discontinuous transducers; and, the geometrical interpretation of dynamics of states for complex hysteresis models (such as Preisach and Prandtl-Ishlinskii operators).

The publication of the book in 1983 in Russian and of its extended English translation in 1989 [19] was followed by an explosion of interest in the mathematical tools it offered to the applied mathematics community, resulting in a raft of publications in the 1990s

[1]The theory of rate-independent hysteresis operators has also been extended to some classes of rate-dependent models of hysteresis.

continuing into the first decade of this century. Several groups of researchers in Europe, the USA and Japan contributed to the development of the mathematical theory. Important monographs were written by Isaak Mayergoyz, Augusto Visintin, Pavel Krejčı, Martin Brokate and Jürgen Sprekels [20–24]. The convergence of mathematicians with researchers in hysteretic systems from different fields (electromagnetism, phase transitions, mechanics, engineering, economics and others) on the basis of common mathematical language and common understanding of hysteresis phenomena enriched the theory by new problems and methods, leading to the versatile science of hysteresis [25], which has many faces depending on the application. Links with thermodynamics, statistical physics (including the Ising model), multi-rate systems, stochastic systems, optimal control as well as to the mathematical disciplines such as variational inequalities and queueing theory have been discovered and are being explored. The interaction of researchers in hysteresis with different backgrounds benefited from regular interdisciplinary meetings such as the series of conferences in Trento (continued in Berlin), the US-European Hysteresis Modelling and Micromagnetics symposium series and, later, the MURPHYS conference series which Alexei organised in Cork.

Until 1992 Alexei remained in Moscow. He obtained his Dr. Sci (Dr. habil) at the Institute for Control Problems in 1989 and next year became head of the centre responsible for developing mathematical methods in Control at the Institute for Information Transmission Problems of the Russian Academy of Sciences, an institution renowned for many excellent mathematicians working there including three Fields medalists. His mathematical universality became apparent and his collaboration with others was wide-ranging and prolific.

He published a few works on game theory, Kolmogorov's complexity and predictability [26–28]. In particular, he introduced measures of unpredictability of binary sequences using hierarchies of sets of predictors. The measure of unpredictability of a sequence is dependent on a particular choice of a set of predictors (finite automata, Turing machines etc.); however, as Alexei showed, it satisfies certain universal relationships. This work was highly valued by Andrei Nikolaevech Kolmogorov.

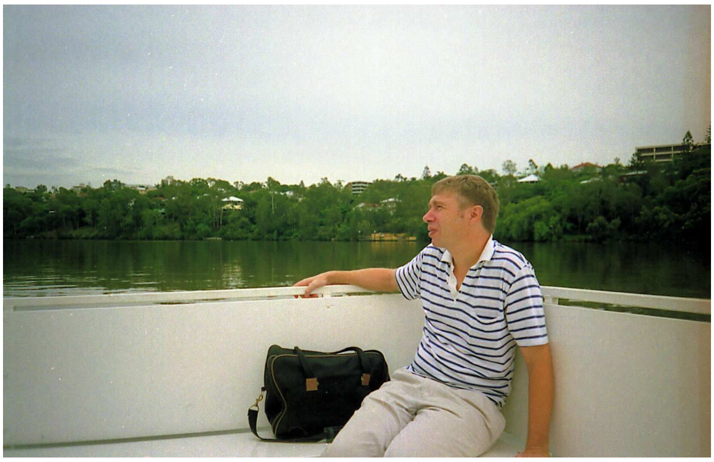
Another body of work addressed linear systems and included methods of identification of a linear system on the basis of a few tests [29];

limit norm of linear operators and its application to control [30]; and, stability of asynchronous systems [31]. The latter problem is related to infinite products of matrices selected from a finite set and the theory of switching systems.

His research of hysteretic systems continued apace with a focus on dynamics of closed systems with hysteretic components, including problems of stability, dissipative properties, oscillations, method of averaging and dynamics of distributed systems of parabolic type [32–35].

From the beginning of 1990s, Russian scientists started to travel abroad. In 1991 Alexei attended the "Models of Hysteresis" meeting organised by Professor Augusto Visintin in Trento. Here, for the first time, he met colleagues from the western hysteresis community and initiated collaboration and friendship, which continued for many years afterwards.

In 1992 Alexei accepted a research position in Australia, where he worked until 1997 dividing his time between being an Adjunct Professor at Deakin University, Geelong, and Director for European Operations at the Centre for Applied Dynamical Systems, Mathematical Analysis and Probability, at the University of Queensland, Brisbane. The Australian period was very productive for him with his research output numbering around 50 papers. The focus of his research during this time shifted towards nonlinear dynamics, specifically, dynamics of discretisations of chaotic systems. The starting point of this research can be illustrated by the observation that the dynamics of the logistic map $x_{n+1} = \mu x_n(1 - x_n)$ are qualitatively different from those of its discretised version (computer realisation) no matter how accurate the discretisation may be. The understanding of this effect was achieved by Alexei and his colleagues Phil Diamond, Peter E. Kloeden and Victor Sergeevich Kozyakin when it became clear that the discretisation acts as a randomising factor. As a result, they developed phenomenological models based on the theory of a class of random maps which were capable of an accurate description of the effect of discretisation on the original system dynamics. In order to ensure that computer models accurately mimicked the dynamics of a map (i.e., robustness to discretisation), they developed new mathematical tools such as split-hyperbolicity and bi-shadowing on the basis of topological degree theory. These ideas have been summarised and further developed in the monograph [36], most part of which was written when the authors lived already far from each other in Ireland, Australia, Germany[2], and Russia. It is now in press and should be published soon.



Alexei on a ferry to the University in Brisbane-apparently, absorbed in another mathematical problem.

Alexei cultivated and enjoyed a collective way of doing research, which is apparent from his publications. He had a talent to identify and consolidate interests of his colleagues and involve them in joint research projects; at the same time, he was invariably interested in the research done by others and truly enthusiastic about their achievements and success. Given his method of work, the huge number of collaborators he had comes as no surprise. Professor Phil Diamond, Professor Peter E. Kloeden and Alexei organised a large scale collaboration with many mathematicians worldwide who came to visit the University of Queensland and Deakin University for varying periods of time. In particular, many Russian colleagues of Alexei visited the research center in Australia and enjoyed the welcoming warm hospitality of Pokrovskii's family during their stay-among

[2]Professor Kloeden moved to the Johann Wolfgang Goethe University in Frankfurt am Main in Germany where he was appointed to the Chair of Applied and Instrumental Mathematics in 1997; simultaneously, Alexei moved to Cork.

them, N. A. Bobylev, V. A. Bondarenko, M. L. Kleptsyna, V. S. Kozyakin, A. M. Krasnosel'skii, N. A. Kuznetsov, B. N. Sadovskii, A. A. Vladimirov and I. G. Vladimirov.

Even though Alexei didn't have security of tenure in Australia, nevertheless, the time spent there was a happy time for his family, who retain fond memories of their stay.[3] However, Alexei probably wanted to be closer to European centres of mathematica research and especially centres of active 'hysteretic life'. Still in Australia, he tested the applicability of the split-hyperbolicity technique to analysis of complex dynamical systems with hysteresis [37].

Alexei first came to Cork in the Spring of 1997 to be interviewed for the Chair of Applied Mathematics at UCC. By that time his research output ran to over 100 papers. As part of the interview process he delivered a lecture about his research interests, during the course of which he paid tribute to the work of his mentor and teacher, Mark Krasnosel'skii, who had just passed away. While his bid for the Chair wasn't successful on that occasion, his exceptional ability as a research scientist of the first rank was recognised.[4] Alexei was offered and accepted a research position, which was created in the Institute of Non-linear Science in UCC with the support of Professor Michael P. Mortell, the UCC President, and Professor John McInerney, Head of the Department of Physics. This post, though not a permanent one, provided a modicum of security for Alexei and his family for the next three or four years, which they used to good effect to settle in Cork. During that period, his expertise in several branches of mathematics, and his capacity to interact productively with a range of experts working in fields outside pure and applied mathematics, such as computer science, physics, engineering and economics, became known within the College. Accordingly, when the Chair of Applied Mathematics became vacant again, and Alexei was an applicant, it came as no surprise that his star-quality was acknowledged by the College, and in 2001 he was appointed to this Chair.

Following his appointment, while continuing to work with former colleagues elsewhere in different parts of the world, he developed

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[3]Indeed, Alexei's daughter Olya returned to Australia for a while after graduating from UCC to commence her medical career.

[4]Professor Michael P. Mortell recollects that the Extern of the Selection Committee strongly advised to use every opportunity to retain Alexei.

fruitful collaborations with people in his own department, and in the UCC Departments of Computer Science, Civil, Electrical and Food Engineering, Mathematics, Microbiology, Statistics, Physics and Zoology, which led to a raft of joint publications. In doing so, he displayed a commanding knowledge of several branches of science which enabled him to appreciate the relevance of mathematical advances to the world around us, and the ability to apply them in a host of different areas.

The applied aspect of mathematical research clearly drove Alexei's interests during the Irish period of his career. He enjoyed modelling as much as analysis and immensely enjoyed and valued collaboration with colleagues from other disciplines, finding it most interesting and stimulating. The research themes of the last cohort of his PhD students included modelling hysteresis in macroeconomics (Hugh McNamara), soil-water hysteresis in hydrology (Denis Flynn, Andew Zhezherun), epidemics and seasonal dynamics of wild bird populations (Suzanne O'Reagan), canard solutions and chaos in nonsmooth singularly perturbed systems (Andrew Zhezherun), bifurcations and chaos in systems with Preisach hysteresis operator (Oleg Rasskazov).

He adapted the split-hyperbolicity concept and other techniques developed in Australia to analyse complex dynamics and chaos in laser systems, wave patterns, models of epidemiology and other applied problems using rigorous computer-based proofs [38–42].

Hysteresis, the subject which was always close to Alexei's heart, became again central to his interests. His research was now inspired by challenges of modelling hysteresis in economics, hydrology, epidemiology, biology (population dynamics) and multi-rate systems. He aspired to make the theory available to, and useful for, problems in these new areas of application, in the same way, as it has already proved to be successful in more traditional fields such as magnetism, plasticity, material science, mechanical engineering and control design. He had a very productive collaboration focusing on modelling hysteresis in hydrology [43–46] with J. Philip O'Kane, Professor and Head of Department of Civil & Environmental Engineering in UCC with whom he co-supervised two PhD students; and with Rod Cross, Professor of Economics at University of Strathclyde with whom he developed models of hysteresis in macroeconomics [47–50]. Their memoirs [2, 51] reflecting on Alexei's impact on these subjects are available in open access.

One means Alexei used to establish the UCC Department of Applied Mathematics as a centre of research excellence in applied mathematics and dynamics of hysteretic systems on the international scene was the Multi-rate Processes and Hysteresis conference series. The conference originated from his idea to explore the links between the methods of the theory of multi-rate systems and the theory of systems with hysteresis [52,53]; Van der Pol relaxation oscillations is one classical fundamental example of such links. After initial discussions with Michael P. Mortell and with Robert E. O'Malley and Vladimir Andreevich Sobolev, two world renowned experts in the analytic and geometric theory of singularly perturbed systems, with whom Alexei met at the Industrial Mathematics Congress in Edinburgh in 1999, a pilot workshop was organised in Cork in 2001. From 2002 it grew to a series of regular bi-annual meetings and gradually acquired the acronym MURPHYS, 'coincidentally' the name of an Irish stout brewed in Cork. These successful and truly multi-disciplinary meetings unifying the efforts of the singular perturbation and hysteresis communities for solving new interesting mathematical and applied problems attracted specialists from many places across the world and provided a stimulating forum for researchers in mathematics, applied mathematics, engineering, control, physics, hydrology, combustion processes, economics, financial mathematics, biology, epidemiology and, even, history. Proceedings of the conference, which was enjoyed by all the participants, were edited by the four co-chairmen.

During the last few years of his headship, Alexei put a lot of effort into cultivating biologically oriented mathematical research in his Department. With his usual contagious enthusiasm and energy he set up a regularly meeting working group between the UCC Departments of Applied Mathematics and Zoology, involving Professor Michael J. A. O'Callaghan, Dr. Tom C. Kelly, Dr. Sarah Culloty, Dr. Ruth Ramsay and others. He led several productive research projects [54–57], built collaborations with mathematical biology researchers in Ireland and the USA, co-supervised two PhD theses in environmental science and epidemiology, increased the presence of biological modelling theme in BSc and MSc applied mathematics degrees and launched a mathematical module for Systems Biology students. In one of the latest works he formulated principles of modelling hysteretic response of human population to epidemics [58].

Naturally, wherever Alexei worked, he was a centre and an attractor of excellent research, a leading mind, most esteemed by colleagues

Michael O'Callaghan, Alexei and Jim Grannel, organisers of MURPHYS conference in UCC.

and beloved by students. His charisma was irresistible. Incredibly imaginative and infinitely rich in ideas, he was absolutely generous in sharing them with others. Quoting Professor Finbarr Holland, "He had a child-like curiosity and wonderment for the scientific world, a deep knowledge of several disparate areas which, combined with a penetrating mind, enabled him to make significant progress in whatever problem that took his interest. But he also took a keen interest in other people's work, and whenever somebody shared a surprising new fact with him, his countenance would alter, his eyes would sparkle with delight, and one would get the 'thumbs up', signifying his pleasure. Such a response was very encouraging to the person sharing the information, especially to a young researcher, still unsure of his or her own ability. He was immensely generous with his time and talents, and warm-hearted in attributing to others ideas that were very often his alone, qualities which endeared him to his students. In truth, he was a polymath of the first rank." He was a truly kind and wise man, caring for people, always willing to help, responsive and discreet; and, the best colleague and mentor you could wish to have, invariably supportive and incredibly encouraging. All his countenance and manner radiated comforting friendliness. "Alexei had a most distinctive manner of speaking. Instead of using full stops, he would punctuate his sentences with a variety of smiles, ranging from the rueful to the exuberant. When we talked on the telephone I could always picture the type of smile on his face. I will miss those conversations." (Rod Cross).

Several papers, which commemorate Alexei and pay tribute to his mathematical work, and his full list of publications can be found on the web page

http://euclid.ucc.ie/pages/staff/pokrovskii/alexeipokrovskii.htm

The article by Finbarr Holland in *The Irish Times* and a few memoirs of colleagues and friends published in the open access *Journal of Physics: Conference Series* in the volume of Proceedings of conference MURPHYS'2010 are also available online [59, 60]. A tribute to Alexei has been paid at the International Symposium on Hysteresis Modelling and Micromagnetics in Levico, Italy, in May and at the Nonlinear Dynamics Conference organised in his memory in UCC in September [61]. A special volume dedicated to him will be published in *Discrete and Continuous Dynamical Systems B* next year.

There is a lot of unfinished business; the work Alexei initiated and research he developed is being continued by students whom he taught and mentored and his colleagues whom he continues to inspire.

Alexei was an outstanding mathematician, with a special way with people. He was loved by everyone who knew him; he is sadly missed.

His wife Natasha works in Tyndall National Institute. His daughter Olya is working in Cork University Hospital and studying ophthalmology. Alexei's son Alexei Jr. is doing a PhD in graph theory in the London School of Economics.

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Mathematics Education and Reform in Ireland: An Outsider's Analysis of Project Maths

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Abstract. Project Maths is an ambitious reform of Irish, post-primary education. In this paper, a U.S. Fulbright scholar reports on her impressions of Project Maths, based on interviews with leaders and teachers, observations in pilot school classrooms, attendance at a teacher workshop, and analyses of materials and textbooks. The paper highlights several aspects the author found particularly impressive, including the phased, collaborative approach to Project Maths implementation, Irish mathematics teacher engagement, and the impact of the reform on some pilot school teachers. The author also raises questions about the Irish exam system, mathematics textbooks, the clarity of Project Maths' vision, and the challenge of teacher change. The paper concludes with lessons the U.S. could learn from Project Maths' example of policy development and implementation.

Introduction

I came to Ireland in Fall, 2010 to learn more about Project Maths. My 4-month visit was supported by a Fulbright fellowship and the Center for the Advancement of Science Teaching and Learning (CASTeL). During my relatively short stay, I was fortunate to have several windows into Irish mathematics education – as an instructor in Dublin City University's (DCU) School of Mathematical Sciences, as a parent of two children in Irish schools, and as a researcher who was generously granted access to several schools and offices involved with Project Maths.

I write this article as an outsider to the Irish education system. There are definite limits to what I can contribute to the discourse on Irish education as an interloper. However, I hope there might also be the benefit of fresh eyes, with my U.S. mathematics education research and experience serving as a backdrop to my analyses.

I became intrigued with Project Maths while considering potential Fulbright opportunities in Ireland. The goals of Project Maths appeared strikingly similar to the goals of the reform movement led by the National Council of Teachers of Mathematics (NCTM) in the U.S.[11, 12, 13]. In particular, both call for more student sense making, problem solving, engagement in the classroom, and conceptual understanding to accompany procedural skill. Both NCTM and Project Maths call for more real world connections and the use of instructional technology, and both promote an increased emphasis on statistics and probability, multiple representations in algebra, and geometric reasoning. Indeed, as I scoured the Project Maths website, I was often struck by the familiarity of the quotes I encountered, such as the following:

> *Project Maths . . . involves changes to what students learn in mathematics, how they learn it and how they will be assessed . . . Much greater emphasis will be placed on student understanding of mathematical concepts, with increased use of contexts and applications that will enable students to relate mathematics to everyday experience. The initiative will also focus on developing students problem-solving skills* [14].

However, despite the similarity in reform rhetoric, the two reform movements were occurring in very different educational contexts. The U.S., due in part to its larger size, has a tradition of decentralized education, with decisions about curriculum and instruction left to each state. NCTM has made valiant efforts to reach teachers through publications and conferences, but actual implementation of the NCTM *Standards* is completely voluntary and highly uneven across states, schools, and classrooms. U.S. researchers have found that, despite professional development efforts and teachers' best intentions, changing mathematics teachers' practice is extremely difficult. Surface changes often occur, such as the use of manipulatives or technology, but there is less often substantial movement toward student reasoning and sense-making [3, 1].

As one who has studied the NCTM reform movement, I was impressed by the thoughtful, ambitious plan for Project Maths, scheduled to roll out to all Irish schools in 2010. With its government mandate and support, I wondered whether this reform movement, would, indeed, be implemented as planned, and what that implementation might look like.

Data Collection and Analysis

During my brief, 4-month stay, I was in no position to conduct a large-scale, randomized study of the implementation and impact of Project Maths. However, I did collect data about Project Maths from a variety of sources, both formal and informal. To protect informants, I am limiting the details I provide about the Project Maths leaders, teachers, and schools that participated in this study (and pseudonyms are used).

I began my research by reading dozens of documents related to Project Maths and talking informally with Irish professors of mathematics and mathematics education, who generously helped orient me to the Irish system of maths education. I also attended several relevant conference presentations, as well as a day-long workshop for teachers that was part of the Project Maths national roll-out. I conducted formal interviews (lasting 1-3 hours each) with several Project Maths leaders, including members of the Project Maths Development Team and the National Council for Curriculum and Assessment (NCCA). The interview protocol contained 18 questions, probing many different aspects of Project Maths, including its history, goals, design, teacher support, curriculum materials, criticisms, obstacles, and surprises. At the end of each interview, I asked for recommendations of pilot schools to visit.

I then conducted 1-day visits to three recommended pilot schools, spanning a broad socioeconomic spectrum. I observed mathematics lessons taught by 2-4 teachers in each school, for a total of 10 teachers. I talked informally with principals and a variety of teachers at these schools. I also conducted formal interviews with two of the observed teachers in each school, asking about their teaching backgrounds, curriculum, role in Project Maths, and implementation struggles and successes. When observing maths lessons, I utilized a classroom observation protocol that prompted me to code various aspects of the lesson (e.g., lesson design, discourse, student sense making, classroom climate)[1]. This protocol served as a tool to focus my attention during observations, but I do not present statistics

[1]The protocol was developed by the Sense Making in Mathematics and Science Project, directed by Barbara Hug and Sarah Lubienski at the University

about these aspects due to the small, non-representative nature of the sample.

Although not formally part of my research, my teaching experiences at DCU were very illuminating. I taught two modules — one for prospective teachers and one for experienced maths teachers — in which we discussed Project Maths and compared texts with the Project Maths label. I am especially grateful to the teachers in my graduate course for engaging in those assignments so enthusiastically and answering my many questions about Irish maths education.

After collecting the data, I audio-recorded and transcribed all formal interviews. Although I began with initial categories of interest when I created the interview protocol, I analyzed the data primarily inductively, looking for themes that recurred and stood out, particularly as I considered Project Maths in light of mathematics education reform in the U.S.

Although rooted in data, my findings are presented more tentatively than traditional research results because of the limits of both the data I collected and my understanding as an outsider. I organize the findings into three sections. In Section 1, I comment on five aspects of Irish education that I found to be particularly impressive. In Section 2, I raise questions about a variety of issues that struck me as curious or concerning. Finally, in Section 3, I discuss a few of the many aspects of the U.S. system that I now consider more critically in light of my experiences in Ireland.

1. Notable Highlights

1.1. Ambitious, Collaborative Planning for Project Maths.

When I first learned about Project Maths from afar, I was impressed by the coherence of the implementation plan, with its phased-in approach involving 24 pilot schools. Although things always look messier on the ground than they do on paper, I have maintained my admiration for the plan and the collaboration behind it, even after seeing Project Maths up close.

There are several groups that have played a role, including:

- National Council for Curriculum and Assessment (NCCA)
- Department of Education and Skills (DES)

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of Illinois. The protocol drew from the Local Systematic Change (LSC) and the Oregon Mathematics Leadership Institute (OMLI) protocols.

- Teacher Education Section (TES) (includes the Project Maths Development Team)
- Maths Inspectorate
- State Exams Commission (SEC)

Through my interviews with Project Maths leaders, I learned that traditionally, these various bodies have worked relatively independently, with a linear progression as follows: NCCA prepares the syllabus, DES implements, and SEC examines.

However, according to the leaders I interviewed, there has been unprecedented collaboration in the creation and implementation of Project Maths, with these groups working together toward common goals:

> *The collaboration with the exams committee is huge — for the first time, there are many feedback loops . . . there hasn't been an initiative like this before that is so widespread, and so complicated — its revolutionary!*
>
> — A Project Maths Leader

> *When we first design the [teacher] workshops, we have someone from the inspectorate, SEC, TES, and NCCA — we show them what we intend to do, ask them if they have any advice, because at the end of the day, we want to get the best product out there.*
>
> — A Project Maths Leader

The fact that 95% of workshop attendees have expressed satisfaction on post-workshop surveys indicates that this collaboration has, indeed, resulted in relevant, useful experiences for teachers. Overall, the plan for Project Maths is impressive, and the collaboration propelling Project Maths forward is indeed striking.

1.2. **Implementation as Scheduled.** It is one thing to create a plan for reform, but quite another to actually stick to that plan and implement it. In exploring the origins of the plan, I learned that those funding the plan wanted tangible results sooner rather than later. This pressure led to the unpopular decision to begin Project Maths implementation at both 1st and 5th years simultaneously — the subject of the majority of complaints I heard from Irish teachers.

But the unpopularity of that decision should not overshadow the accomplishments of those responsible for implementing Project

Maths. Given the political time pressures, there were only a few months between the approval of Project Maths in the Spring of 2008, and the beginning of implementation with the pilot schools in August 2008. During those few months, the Project Maths Development Team was assembled, with Regional Development Officers (RDOs) hired to support the schools (at a 1:4 ratio). Additionally, pilot schools were recruited, and a stratified sample of 24 schools was selected from the 230 schools that volunteered. The NCCA and the Project Maths Development Team quickly began drafting a new syllabus, sample lesson plans, teacher workshops, and other support materials.

Given the major overhaul of the system that Project Maths entails, the 24 schools wondered if Project Maths really would move forward as promised, after the 2-year pilot:

> *There was such distrust of the system. Now the 24 schools have the full picture, they have seen all 5 strands, and they have seen that all the schools nationally are starting . . . they are not going to be stranded . . . they're over the hump of the unknown.*
>
> — Project Maths Leader

Indeed, after the pilot phase, the national rollout began as scheduled, in August 2010. From my U.S. standpoint, the scope of work initiated and accomplished over the past 3 years is, indeed, remarkable.

1.3. **Responsiveness to Feedback from Pilot Schools.** During the two-year pilot, there was substantial give-and-take between Project Maths leaders and the pilot schools. Teachers were initially upset by the lack of curricular guidance available to them, and it took some time for them to understand Project Maths' intention to collaboratively develop resources with the schools during the pilot phase.

During that time, the pilot school teachers made several requests. First, the teachers asked for release time to meet with other maths teachers and their RDO, and the NCCA supported this request. Second, the pilot teachers expressed frustration at the lack of ready-made student materials available to them. In response, NCCA scrambled during the Summer of 2009 and created additional resources for the teachers and their students. One Project Maths leader explained this as follows:

> *It was a small little question (teachers asking for student materials), but the consequences were an entire summer of work in getting those finalized.*
>
> — Project Maths Leader

A third major concern among the pilot teachers was that they did not know what the new maths questions would look like on the Junior and Leaving Certificate exams. The leaders of Project Maths, in collaboration with the SEC, sought to address this concern:

> *We told them that the exams commission will create a sample paper and will trial it in the schools . . . We weren't in the position in the first year to give them a paper, but we gave them sample questions.*

Teachers also expressed concern about the statistics strand, arguing that it was too long and difficult, particularly for 5th and 6th year students who had not had this strand in prior years. This concern prompted two temporary adjustments, with the first being the deferment of some statistics-related material. The second was the provision of a temporary exam option:

> *We gave them a choice between the further end of the statistics, or a section of the geometry, and we said we will guarantee you a choice on the exam between those. So we strove to accommodate the concerns that were there by putting in a choice that originally we hadn't intended. . .*
>
> — Project Maths Leader

Finally, many teachers indicated that they, themselves, needed help with the content they were expected to teach, especially in the area of statistics and probability. Hence, a 3-day summer course was offered for teachers in the pilot schools, and DVDs of this course were made for all teachers in Ireland. Additionally, evening workshops focusing on maths content are now available during the school year for all Irish teachers, to supplement the more pedagogical-focused workshops offered during the school day.

Overall, the pilot school teachers I talked with were pleased at the level of support they received from their RDO and other Project Maths leaders (although pilot teachers told me that experiences varied across the schools, depending upon the assigned RDO). According to an NCCA survey administered at the end of the first pilot year, the majority of pilot schools reported being happy they were involved. Still, this same survey indicated that Project Maths placed extra demands on teachers time. Indeed, the prevailing sentiment of the pilot teachers I interviewed was that involvement as a pilot school was extra work but worth it:

> *I really think the students are getting a better feel for maths, and even though we had an awful lot of work to do the first year, we had a great experience from it.*
>
> — Nancy, pilot school teacher
>
> *What it's trying to do is beyond reproach — I think it's the only way to go if we're going to get students to be math literate and be able to apply maths. But that's not to say that it hasn't had problems... that it hasn't placed a huge workload on teachers.*
>
> — Ned, pilot school teacher

1.4. **Teacher Professionalism.** When I talked with teachers in Ireland, I was often struck by their deep concern for students, their interest in improving instruction, and their involvement in the profession. The quotes from Nancy and Ned above nicely illustrate this commitment to improved mathematics education. On a broader scale, I learned with interest about the active Irish Mathematics Teacher Association, including some heated debates about Project Maths (e.g., particularly at the Dublin branch meetings). Although one might prefer a serene picture of unanimous agreement among the mathematics teachers of Ireland, it is impressive that Irish teachers care enough about their profession to argue about it, without allowing these debates to derail progress toward common goals of improving mathematics instruction.

I am also impressed by the fact that the vast majority of teachers are attending the Project Maths workshops nationally. At the time of my visit, 3 of the 10 national, day-long workshops had occurred, with over 80% of Irish Maths teachers attending (according to figures given to me by Project Maths leaders). Additionally, as of Fall, 2010, at least 2000 teachers (1/3 of all mathematics teachers in Ireland) had attended an evening Project Maths workshop to learn more mathematics content or instructional technology. Participation in these evening workshops is not required or compensated in any way — teachers simply want to learn and improve their practice.

In the U.S., there seems less of a tradition of teacher involvement in local maths teacher organizations, and more of a tradition of mandating and paying teachers to attend professional development. Given the differences in incentives and participation, I was indeed impressed by the level of interest and professionalism I found among Irish teachers.

1.5. **Pilot School Teacher Change.** At the beginning of the study, I had wondered if implementing Project Maths in the volunteer pilot schools was somewhat like preaching to the choir, perhaps involving only those teachers who supported and already implemented instruction aligned with Project Maths. However, during interviews, teachers in each of the three schools talked about substantive changes they made in their classrooms. For example, one teacher, Elizabeth, said that her instruction had drastically changed:

> *I'm getting them to think and do problem solving. It used to be rote learning. I'm really excited about the geometry. It used to be taught in 3rd year right before the exam. It was "Here are your theorems, learn it off." ... I'm taking a back seat more now. I'm doing more work at home, but less talking in class.*

Additionally, Ned, told me about his new instructional approaches, particularly in the area of statistics:

> *... I've been surprised at the innovation in some of the approaches of teaching topics. It's been a real eye-opener ... What's really encouraging is that students are getting experience with the "why" of statistics ... and actually applying that to real situations for example in the "Census of Schools" activity, they get data from other schools ... Students are actually making decisions for themselves — if comparing 2 datasets, do they use mean, median, or mode? Before it was just "know how to calculate each one."*

Another teacher, Mike, told me that he was initially against Project Maths and felt that it would be "dumbing down" the curriculum. However, he was not against being a pilot school:

> *Whether we were a pilot school or not, we were going to have to teach the new syllabus — we may as*

> *well be in there at the start — where we can have some input on the changes.*

After two years of piloting Project Maths, Mike still had concerns about the elimination or treatment of some topics (e.g., vectors). However, he also saw important benefits to the Project Maths approach:

> *I feel less reserved about Project Maths now, because the syllabus makes a big effort to be more tangible and practical, so students can see how maths fits in with the real world. Some of the materials . . . are good to motivate students, easing them in to topics, so I do like that approach. And [the new Project Maths] exams are at times quite challenging as well, because students in the past never had to move from a context to actually formulating some mathematical problem that they then had to solve themselves.*

Having no "pre-Project-Maths" data on these teachers, I am in no position to judge the extent to which changes have actually occurred in their classrooms, and I cannot say how typical these teachers are. Still, given the persistent difficulties of changing teachers deeply held beliefs about mathematics teaching, it is, indeed, impressive that fundamental changes in beliefs and practices were described by three of the six pilot school teachers I interviewed. The Project Maths lesson plans, workshops, and RDOs were regularly mentioned as influential.

2. Issues for Ireland to Consider

One of the things that repeatedly struck me about Ireland was its relatively small size — e.g., RDOs from around the country could come together for face-to-face meetings. This close proximity obviously offers benefits in terms of greater communication and coherence. On the flip side, there are fewer human and financial resources from which to draw upon in a smaller country. This issue of country size was important for me to consider as I made sense of the Irish system. At times, I was amazed at all that the Project Maths team was able to accomplish, given the limited staff and funds. At other times, I wondered if my thoughts about what should be or could be occurring in Ireland were too rooted in my experiences in a much larger system. It is against this backdrop that I move to a discussion of questions I have about Irish education in general, and the implementation of Project Maths, in particular. I begin with relatively extended discussions of the Irish exam system and then textbooks, followed by brief observations about Project Maths' vision and the challenges of teacher change.

2.1. **The Leaving Certificate Exam.** The most striking feature of the Irish system to me, coming from the U.S., was the pervasive emphasis placed on the Leaving Certificate exam. I was struck by this emphasis in the lessons I observed, in my interviews with teachers, and in the textbooks I examined. As one small example, I noticed that some Irish textbooks routinely highlight specific questions that appeared on prior exams, along with the number of marks each question was worth.

U.S. students usually take a college entrance exam — either the Scholastic Aptitude Test (SAT) or the American College Test (ACT), or both if students wish. Although some students attend the equivalent of "grinds" for a short time to prepare for the SAT or ACT, there is generally not such a "teach to the college test" focus in U.S. secondary schools. There *are* annual tests given to elementary and secondary students by governing authorities, and teachers are feeling increasing pressure to teach to these tests. However, in most states these tests are high-stakes for teachers and schools — not for the students — thereby weakening public support for focusing on these tests.

Still, the SAT and ACT are high-stakes exams, and while in Ireland I began to wonder why U.S. teachers rarely teach to the college entrance tests. I believe that one major reason is that the exam content in the U.S. is less predictable. That is, the exams are copyrighted, and the questions contained on each test are considered just a small sample of all possible questions, with no pattern in what might occur from one form to the next. The fact that the SAT began as an effort to measure "IQ", traditionally perceived as "raw, mental ability" as opposed to learned content, likely also shapes people's perceptions of these tests today.

A second, related reason for less teaching to the test might be that American students can repeat the exam every few months (for a fee) if they do not like the score they earned the first time. Third, the fact that independent, non-governmental testing organisations create and administer the tests might also create more distance between the tests and the schools. Finally, students' SAT or ACT scores are considered along with students' grades, application essays, extracurricular activities, and teachers' letters of recommendation as colleges make selection and funding decisions.

Despite my concerns about a "teach to the test" emphasis and the consequent promotion of external rather than internal motivation for learning, high-stakes exams do have some benefits. As Conway and Sloane write,

> *One of the advantages of an exam tradition in any educational culture is the very fact that it is reflected in some degree of shared understanding about what knowledge is valued ... how students go about the actual exam (typically, a sit-down paper-and-pencil mode of assessment), and most importantly there is typically a very significant degree of credibility attached to the results in terms of both their validity and fairness.* [5, pp. 234-5]

In weighing the benefits and drawbacks of the "exam tradition" in Ireland, I discerned several themes in my notes and interviews, including the role of the teacher, impact on classroom instruction, the purpose of exams, the role of exams in education reform, and the credibility attached to the exams.

2.1.1. *Teacher as exam coach.* Irish teachers seem to play the role of "exam coach" with students, where its "us against the exam." Even though the Junior or Leaving Cert was months – or years – away, I noticed several Project Maths pilot teachers routinely giving their students hints about how to score points with exam graders – e.g., "Don't color in the bars [on the bar chart]. It won't count–it's a waste of time."

On the one hand, this nicely places the teacher and student on the same team, more so than is the case in the U.S. Indeed, many teachers seemed to view their role as getting students to succeed on the exam, and I was touched by the concern that teachers expressed about their students. Several teachers told me about the tradition in their school to wait outside the exam room door (on the teachers' day off), so they could talk to their exiting students and find out, "Did I prepare them on the right stuff?" Project Maths pilot teachers faced additional anxiety about the new questions on the maths exam. One such teacher, Mike, told me:

> *I couldn't sleep the night before the Leaving Cert — it's the first time I ever had that kind of worry in me.*

Despite the benefits of this "us against the exam" approach to learning, one potential drawback is that this tends to place mathematical authority with the exam that looms in the future instead of with the teacher, perhaps providing less motivation for Irish secondary students to engage in daily classroom activities not directly linked to exam preparation. As I was teaching undergraduates at DCU, I wondered if some of the exam-focused mindset also impacts Irish students' approaches to learning in college. I was surprised at the number of my students who seemed content with a 40% average (and I was surprised that 40% instead of 60% was the passing cut-off). I had always been annoyed with my anxious, grade-conscious students in the U.S., who flood my email inbox with questions about assignments and are unhappy with scores below 95%. But I began to feel a little nostalgia for my neurotic students back home, who view me – as opposed to a distant exam grader – as the final authority to be feared.

2.1.2. *Exams constraining instruction.* In my observations and interviews, I noticed four ways in which exams might constrain Irish mathematics instruction:

A. Gaming the system.
While some Project Maths pilot teachers spoke specifically about resisting the urge to teach strictly to the test, other teachers had a beat the exam mindset when making curricular decisions:

> *In the old system with ordinary math, with geometry constructions and theorems, there would be a question, but students would avoid it. So over the years I didn't do it. So it was a whole area that I had to get back into [now with Project Maths]. There were 6 questions to be done out of 8. Learning the formal proofs was something they didn't like, so then you tended to say, "Why invest time in a question that they won't end up doing?"*
>
> — Eliza, Project Maths pilot school teacher

Project Maths is now causing Eliza to return to teaching theorems, given that the new exam will not allow students to avoid such questions. However, this "beat the exam" mindset can also affect instruction in other ways.

B. Time pressures.
Shortage of time can inhibit teachers' willingness to involve students in mathematical problem solving and sense-making, particularly as the Leaving Cert draws nearer and teachers feel increasing pressure to cover the curriculum. Indeed, an NCCA survey administered to pilot schools after their first year of Project Maths revealed that over 80% of schools reported greater student engagement with mathematics among 1st year students, while this percentage was less than 60% for 5th year classrooms. As one leader explained, "*The idea you have to cover the course is a big thing here. It's harder to get teachers to focus on what the students are learning, or understanding.*"

As a side note, coming from the U.S. where daily maths classes of 45-50 minutes are common, I was surprised at how short the class periods were that I observed in the pilot schools (typically 35-40 minutes). The difference in time available for maths appears to be due, at least in part, to the inclusion of religious education and Irish language classes — two subjects not required in U.S. public schools.

C. Form over substance?
A third issue arose as I observed the Project Maths day-long workshop and pilot school classrooms. I noticed that some teachers placed great emphasis on the format of student responses, with the focus on writing answers in a way that would maximize marks awarded on the Junior and Leaving Cert exams. For example, at the Project Maths workshop, the leader introduced white boards as one tool for problem solving, allowing students to work on ideas and easily erase and start again as needed. While some teachers remarked that they liked this non-threatening way for students to approach difficult problems, another teacher (with nods of support from several colleagues) said, "*I don't want my kids to rub out mistakes, because examiners want to see their work.*" I also noticed the "form over substance" emphasis when the teacher (mentioned above) told the third-year student not to "waste time" coloring in the bars of his chart. In truth, the student's colored chart showed the pattern of interest better than the non-colored chart, yet the teacher did not consider this. Hence, I began to wonder how often this emphasis on "proper exam form" interferes with substantive learning goals.

D. Distinction between instruction and assessment?
When I made my first public presentation in Ireland, I flippantly stated that "real-world problems" are important for students to encounter in mathematics classrooms, but they do not always make the best exam problems. I received several surprised looks and was later told, "We don't teach problems that aren't on the exam."

However, I stand by my statement that some problems are better for instruction than for assessment. Some mathematics problems can help students understand the exploratory nature of mathematics, including the beauty of mathematical patterns. For example, secondary school students can enjoy being exposed to unsolved mathematics problems, or figuring out how many squares or rectangles are on a chessboard. Other problems help students understand a concept more deeply, such as the locker problem, which illuminates why a number has an odd number of factors if and only if it is a square number[2]. And other problems can help students grapple with messy, real-world applications of mathematics, where there are many constraints to be considered and there is rarely a single right answer. However, these problems are not necessarily good exam questions.

Exam questions are designed to assess particular content and processes, and exams are designed to sort students along a continuum for college admission purposes. Exam questions must be succinctly worded, unambiguous, solvable in a short time period and generally should have a correct answer that is easy to mark. Hence, despite the many benefits of having content alignment between instruction and assessment, I propose that there should not be too tight a coupling between the maths problems used in instruction and on exams. That is, teachers should be free to use messy, engaging mathematics problems that can teach students interesting things about mathematics, regardless of whether those particular problems will show up on the Leaving Cert.

2.1.3. *What is the purpose of the exams?* Of course, the above discussion raises the question of what, exactly, are the purposes of the exams, and this is something I began to wonder about as I heard the following remarks from teachers:

[2] `http://connectedmath.msu.edu/CD/Grade6/Locker/index.html (Accessed 1-4-2011)`

> *I can't for the life of me figure out why there's so much guesswork as to what's going to be on the exams [with Project Maths]. The results aren't going to be as good.*
>
> *It is unfair on students. The very first day the students start the Leaving Cert, they should know the structure of the exam, and what types of questions they will need to answer.*
>
> *It's not fair to not allow them to use a calculator on a test if they have been using it as a resource in school (Regarding a university placement exam given at orientation)*
>
> *Students would freak out if the questions were out of order.*

Indeed, I was surprised at the predictability of the Irish exams, and the common expectation that students should be able to predict what questions will be asked and in what order. I began to wonder what the purposes of the Leaving Cert are — to reward students' memorization skills? To assess student understanding of mathematical ideas? To assess students' problem solving and reasoning skills? To help evaluate and/or improve teachers or textbooks or schools? To predict students' future success in university programs? To predict student success in future careers? I began to wonder whether the predictability of the Leaving Cert exam is helping or hindering those purposes.

Overall, I suspect it is a good thing that Project Maths is striving to make the maths exam questions less predictable with fewer options for students to omit parts of course content. That said, I became concerned at the Project Maths workshop for teachers, where substantial time was spent on a clinometer problem that was difficult for many students who took the Project Maths trial paper. The discussion of that problem prompted some teachers to ask insightfully whether this was becoming just another form of "teach to the test," where teachers are now being coached how to prepare their students for new, but still predictable problems. This issue of predictability and the purposes of the exam is something for Project Maths leaders and others across Ireland to continue to grapple with.

2.1.4. *Exam as both change lever and barrier.* For Project Maths and likely other Irish education reforms, the exam system presents both a barrier to change and a lever of change:

> *Teachers in schools where students have always done well on the exams are harder to convince.*
>
> *Teachers absolutely wanted to hold on to the old style ... They want things they can teach kids in advance. Teachers need to trust that IF they develop the skills of the students, then they will be able to figure out a novel question ... Students and teachers had a predictable exam, and teachers could train their students to practice for the exams. We removed that predictability ...*
>
> — Project Maths leaders

Indeed, teachers who view their role as preparing students for exams, and who have had success in the past, are understandably reluctant to change their instruction. Additionally, as noted previously, exams can add time pressures for teachers, making them less inclined to try new approaches, particularly those that promote deeper instead of broader content coverage.

But the fact that the Project Maths leaders have worked closely with the SEC to revise the exams has provided a necessary lever of change. The first sample exam containing Project Maths questions made the reformers' intentions "real" to the pilot school teachers:

> *Teachers could see student answers, responses on the problems, and the teachers could see the big picture 'Oh, we see what you want us to do.' This document was powerful.*
>
> — A Project Maths leader

When talking with teachers, I routinely asked if they thought that many teachers would simply ignore Project Maths and hope that it would go away (a fairly common reaction to school reform in the U.S.). However, I was consistently told "No," that teachers would need to get on board with Project Maths because of the changing exams. Indeed, the vast majority of Irish teachers have flocked to workshops in order to receive materials and instruction that will help them align their instruction with Project Maths.

Another reform that occurred while I was in Ireland was the announcement that the Irish Universities Association would give 25 bonus points for students who passed the higher-level maths Leaving Cert exam[3]. This, again, illustrates how the Irish exam system has a powerful lever of change built in, allowing leaders to efficiently address difficult problems, such as a shortage of students pursuing higher-level maths. There is no equivalent policy lever in the U.S.

2.1.5. *Credibility of the exams.* The final question I raise about the exams system stems from teachers' responses to my questions about the probable impact of Project Maths on students' Leaving Cert scores. I received several responses suggesting that the SEC will "just make the results come out," or in other words, that the exam results will show whatever the DES and NCCA want them to show. Hence, I began to wonder what checks and balances there are in the Irish exam system, how much trust the Irish people have in the exam scores, and whether bridging studies would be used to compare students' results on the old and new maths exams. And I was, again, struck by the issue of country size, as I considered the vast amount of specialized testing expertise necessary to create, validate, administer, and analyze results of national exams, not to mention the additional burden placed on the exam system by reforms, such as Project Maths. I did not have the opportunity to speak with those at the SEC during my short stay in Ireland, but I was pleased to learn that the NCCA has funded a group (from the United Kingdom) to conduct research on Project Maths' impact on student learning. I applaud this external involvement in the evaluation of Project Maths, and I hope this group will work closely with the SEC and make full use of all relevant exam data.

Overall, as I talked with people from various walks of life in Ireland, I was struck by their love/hate relationship with the exam system. On the one hand, the exams are the target of much complaint, occasional distrust, and are a source of stress for teachers and students. On the other hand, people were quick to defend the system as the fairest way to allocate university slots (despite the fact that some students can afford grinds more than others). Given the importance of the exams, the SEC has tremendous responsibility and merits ample support as it copes with the changing demands brought on by the Project Maths reforms.

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[3] *Irish Independent*, October 12, 2010.

2.2. **Textbooks.** Despite the lack of any official government mandate for the NCTM reforms in the U.S., in the early 1990s, the federal government's National Science Foundation chose to invest millions of dollars in the development of textbooks aligned with the NCTM Standards. The NSF-funded author teams included scholars with expertise in the teaching and learning of the various relevant mathematical areas (e.g., algebra, geometry, statistics, etc.). Each author team partnered with mathematicians and school teachers as they developed, piloted, assessed and revised their text for publication. The process generally took 4-5 years. The NSF made this major investment because of past lessons learned about the critical importance of textbooks in maths instruction and reform. These lessons are not specific to the U.S., as Conway and Sloane note:

> *The message from the TIMSS textbook study is loud and clear: there is a mismatch in many countries between reform goals in mathematics and the actual mathematics embodied in textbooks Looney (2003), in research with teachers working in the support services for post-primary, found that they believed the textbook was more influential than the curriculum in making decisions about classroom teaching* [5, p.31].
>
> *Given textbooks' function as mediators between curricular intention and implementation, a reform of post-primary mathematics toward a more problem-solving orientation will, it could be argued, necessitate a radical overhaul of mathematics textbooks.* [5, p. 166]

Given the importance of textbooks, there are two issues that I wish to highlight — one specific to Project Maths, and the other more general.

2.2.1. *Textbooks and Project Maths.* First, the issue of textbooks in Ireland seems politically sensitive, with Project Maths leaders seemingly afraid to say anything positive or negative about any particular book. They appear to be circumventing textbooks as opposed to leveraging them, as illustrated by these quotes:

> *I deliberately have not seen any of the textbooks.*
>
> *I haven't seen any [texts] so I don't know what's out there — and the best thing to do is not look*

> *at them, so I can hand on heart say 'I havent seen any'.*
>
> *The teacher needs to be autonomous and say 'Oh, I realize it's all active learning' — they need to be able to develop this [curriculum materials] themselves.*
>
> — Project Maths leaders

Many of the pilot school teachers clearly struggled with the lack of a textbook, both because of the time it took for them to plan lessons, as well as students' difficulty with keeping mountains of handouts organized. However, now that Project Maths is in the national rollout stage, there are several "Project Maths" textbooks emerging on the market, and schools are beginning to adopt these. I looked carefully at two of these texts and was struck by their differences, with one text presenting traditional boxed formulas and examples for students to follow and the other text structuring a sequence of investigations through which students derive the formulas. Clearly, not all texts claiming the "Project Maths" label will help teachers implement the type of instruction that Project Maths envisions. Instead of circumventing textbooks, Project Maths leaders might need to help teachers develop tools to critically analyze these various texts, so that teachers will select texts that help instead of hinder the goals of Project Maths. Professional development activities that ask teachers to compare texts with a focus on the treatment of specific topics, such as the development of the distance formula, can promote valuable analyses and conversation among teachers.

2.2.2. *Textbook development in Ireland.* I was surprised to learn that many Irish maths textbooks are authored by mathematics teachers (often while they are teaching full time), with little or no substantive input from mathematics education scholars or mathematicians. Clearly, teachers have much practical teaching expertise that should inform textbook development. But I want to offer one example of how additional expertise is needed, and this comes, not from Project Maths, but from the 5th class textbook my daughter used while in Ireland.

There has been substantial research on students' learning of geometry, including work demonstrating Van Hiele's theory of how students progress from viewing shapes in informal to more formal ways[6]. The classic example is that young children tend to think that a square is a square because it “looks like a square.” If you turn the square 45 degrees, most youngsters will say it is no longer a square (it’s now a “diamond”). Hence, elementary mathematics education experts know the importance of designing tasks that prompt students to pay attention to the specific properties of shapes and move beyond erroneous assumptions about shape orientation.

Given that backdrop, I was dismayed when my daughter repeatedly came home with assigned problems in which the textbook showed shapes in a traditional “upright” orientation — with a horizontal base — accompanied by the question “How many horizontal (or vertical) lines are in a pentagon (or octagon, hexagon, etc.)?” My daughter soon learned that to get the textbook’s expected answer, one needed to assume that the orientation shown in the picture was the only acceptable orientation for those shapes (i.e., there is 1 horizontal line in a pentagon, 2 in a hexagon and octagon, etc.).

Obviously, mistakes and missed opportunities can be found in textbooks all over the world, and these are just a few examples from one book. However, in talking with teachers, including some who were textbook authors, I grew concerned about several issues. First, it is unclear to me how a teacher can focus on both teaching full time and authoring a textbook, particularly one that needs to be completed within a year (the timetable for some of the new “Project Maths textbooks”). Second, I think both mathematics education scholars and mathematicians have a greater role to play in at least reviewing books, if not actually co-authoring them. Mathematics education scholars would notice the blatant disregard for what is known about geometry learning in the above examples. Mathematicians would likely notice other issues, such as the messages that texts convey about proof. I noticed, for example, that one of the new Project Maths texts uses what might be considered “proof by 2 examples” — e.g., concluding that the midpoint formula holds after working for two cases. Despite my enthusiastic support for the inquiry spirit of this textbook, I think mathematicians might rightly raise cautions about students learning to generalize in this way.

2.3. **Project Maths Vision?** The more I delved into Project Maths, the less sure I became about what, exactly, its instructional vision is. In the U.S. reform movement, the push has been toward problem solving as the primary means of learning mathematics [12, 13]. That is, students are given a problem (or a carefully designed sequence of questions), and through the process of solving and discussing, they gain understanding of intended mathematical ideas. The Project Maths teaching and learning plans I examined were consistent with this approach. However, after interviews with key Project Maths players, I became less sure about the role of problem solving and discovery learning in Project Maths.

> *We're not quite into the Realistic Maths Education approach which is, 'Here's a problem, let's puzzle our way through that.' We want more problem solving, we want to develop those skills, but we can't go whole hog, we're taking a mix and a match. We're putting in some basic maths learning but then seeing it applied through contexts, through problem solving.*
>
> Project Maths leader

> *Project Maths is more about investigational work ... it's directed discovery learning — you give them a path to follow, and if they follow that path, they should get to the conclusion itself. And you're there to maybe jockey them along a little bit. But that's only 5-10 minutes of the class. The rest of it [the lesson] is back down to what's been working for thousands of years — so it's a bit of both.*
>
> Project Maths textbook author

There can be benefits to having room for interpretation of a reform vision, as it allows broader buy-in to that vision. But the downside is that teachers can read what they want to read in reform documents, and too quickly assume that they teach as reformers intend. I observed this phenomenon to varying degrees during some of my pilot classroom visits. Consequently, it might be helpful for Project Maths leaders to discuss distinctions among teaching "about," "for," and "through" problem solving, both with each other and with teachers [7]. More clarity and specificity may help spur Project Maths teachers toward deeper dialogue and more meaningful change in the classroom.

In addition to clarifying the role of problem solving, Project Maths leaders may need to grapple with the question, "What is a problem"? I noticed a tendency to equate "teaching through problem solving" with "Realistic Mathematics," which assumes that problems are set in real world situations. However, there are many good mathematics problems that have no real world context. "Realistic mathematics" is not the only form of problem-centered instruction. Some NSF-funded curricula in the U.S. may offer a slightly different interpretation of "teaching through problem solving," including Core Plus, Math Connections, and Connected Mathematics Project[4].

2.4. **The Challenge of Teacher Change.** Research on U.S. mathematics education reform suggests that effective maths professional development involves intense, sustained contact with teachers and focuses on both the textbooks that teachers use and students' thinking about mathematics [2, 4]. Pilot schools have enjoyed ongoing contact with their RDOs, including regular school visits. The pilot school teachers I talked with agreed that this level of support is necessary, and some wondered how teachers in other schools will cope with only a few 1-day workshops each year:

> *I worry an awful lot about the other schools. All they're getting is a few workshops. They don't have somebody coming into the school helping them like we do.*
>
> — Elizabeth, pilot school teacher

In a letter written in April, 2009, the Dublin branch of the IMTA also expressed their concerns about the scope of change and the need for teacher support:

> *The existing Probability and Statistics option is answered by only 2-3% of students... This means the existing pool of Mathematics teachers to Higher Level will not just have to be trained in new methodologies but will need further Mathematics education. The extent of curricular change is huge as confirmed by the experience of the teachers in the pilot schools.*

Professional development is time-consuming and expensive. With Project Maths, teachers are being asked to teach material they never learned in ways that require more – not less – mathematical confidence. Additionally, almost half of those who teach post-primary maths in Ireland have no mathematics teaching credential [15]. This

[4]More information about these curricula can be found at `http://www.wmich.edu/cpmp`, `http://www.its-about-time.com/math/index.html`, and `http://connectedmath.msu.edu`. (All accessed 1-4-2011).

situation raises several questions, many of which would take additional resources to address. Will the non-pilot teachers receive enough support for Project Maths implementation? Should mathematics content be at least as important as pedagogy in the Project Maths workshops? Is there a way to offer additional content or content/pedagogy hybrid workshops for teachers during school days? Should graduate courses play a larger role?

Despite these questions, I am impressed by recent reports offering recommendations for the improvement of Irish teacher education. For example, the Project Maths Implementation Support Group (2010) argued for the creation of university mathematics content/pedagogy graduate courses to support the implementation of Project Maths, as well as for requiring teachers to regularly participate in professional development as part of their ongoing registration with the Teaching Council. These directions appear promising.

3. Issues for the US to Consider

My experiences in Ireland prompted me to regularly ask why we do things the way we do in the U.S.. Given that this article is directed toward an Irish audience, I will not dwell on all of my thoughts pertaining to the U.S., but will offer a few as points of contrast with the Irish system.

3.1. **Scholarly discourse leading to policy.** The U.S. has many warring interest groups that issue reports and counter-reports about education policy. There is a tendency for these groups to demonize opponents, obscure facts, and use "crisis rhetoric" as a means of persuasion. It was a breath of fresh air to read Irish mathematics education research reports, which tended to use both previous reports and the latest evidence to build toward arguments for more effective policies. Similarly, it was refreshing to have my questions about public relations campaigns for Project Maths met with confusion (due to the absence of such campaigns) and to hear remarks from Project Maths leaders, such as, "Positive and negative reactions have contributed to very useful debate." Although I'm sure there are ugly politics in Irish education just as anywhere else, my experiences there have made me more concerned about the state of education discourse in the United States.

As just one example that contrasts with what I typically see in the U.S., I could clearly identify a tight, 3-year progression of research and discussion leading up to Project Maths, as I read through the following four reports:

- *Inside Classrooms: The Teaching and Learning of Mathematics in Social Context* [8] concluded that much of Irish mathematics instruction centers around teacher lecture, memorization of procedures, and drill.
- *International Trends in Post Primary Maths Education: Perspectives on Learning Teaching and Assessment* [5] discussed a variety of international initiatives, highlighting the trend toward more problem-centered mathematics instruction.
- *A Discussion Paper: Review of Mathematics in Post Primary Education* [9] was a companion to the International Trends report and outlined the current state of mathematics education in Ireland as a way of fostering discussion. This was distributed to all schools and colleges as the start of a consultation process that involved an online survey and focus group meetings with parents, the IMTA, etc.
- *Review of Mathematics in Post-Primary Education: Report of the Consultation* [10] summarized the results of the discussion process.

Unlike many reports in the United States, these Irish reports presented evidence and raised questions to spur discussion, progressively building toward a collective understanding. Even more impressive is the fact that the reports culminated in the creation and implementation of a major national reform.

3.2. **Thoughtful approaches to policy implementation.** Project Maths teachers and leaders impressed me with their wise, long-term perspectives on reform, often expressed in response to my queries about how they will know if Project Maths "is working:"

> *I think it's far too soon to say that it's working. From my own experience, I'm only teaching Project Maths full time this year, and that's to students who have no history learning in this way. So in 2 years time, I would expect to see differences — or in 3 years time, when things get more smooth.*
>
> Ned, Project Maths pilot teacher

> *The proof in the pudding will be after students have been through the entire program — students*

> *won't have emerged from the system until 2017 — we said at the start that it will be 7-10 years before the whole system is in place. But the benefit of staggering in like this is that it gives time for the system to settle in — it would have been a nightmare to shift everything over all at once.*
>
> Project Maths leader

Project Maths' staggered implementation, beginning with 24 volunteer pilot schools and introducing 1-2 mathematics strands per year, seems a smart approach, allowing 2 years in which to work through major issues before scaling up to the rest of the country. Similarly, I was impressed at the thoughtful way in which a maths bonus points reform was introduced, with an announcement made in 2010 that the policy would begin in 2012 on a four-year trial basis[5].

In the U.S., unreasonable expectations of quick results have too often led to the rapid abandonment of policies, along with pendulum swings in education rhetoric and reform. This has created the popular "wait for it to go away" response among educators. Indeed, reform implementation takes time, and seeing the effects of that implementation takes even longer. The U.S. could learn much from these Irish examples of thoughtful policy introduction.

3.3. **Exams.** My time in Ireland gave me an appreciation for some aspects of the U.S.' relatively low-stakes exam system, yet, I did begin to grapple with several questions. First, I began to wonder why the U.S. allows private testing groups — as opposed to a government body — to determine the content of college entrance exams. These exams could provide a focal point for public discourse, as well as a key policy lever that is missing in the U.S. (although I do fear that this lever could be over-used if subject to the whims of U.S. policy makers).

Second, I began to question the timing of U.S. exams, which generally occur before students' final year of high school, making "senioritis" (students having little regard for their senior year) a problem among U.S. students. In fact, the college testing, application and selection process occurs on a completely different timeline, often spanning two or more years in the U.S., as opposed to only the summer months following secondary school in Ireland (however,

[5]Donnelly, K. (October 12, 2010). 25-point bonus for passing honours maths, *Irish Independent*.

there is no "transition year" in the U.S.). I am still considering the trade-offs of each approach.

Finally, as I talked with Irish teachers who expressed great concern for their students' performance and future opportunities, I began to wonder about the current situation in the U.S., which tends to place teachers in a very difficult position of teaching to state-level tests that are high-stakes for themselves, but low-stakes for their students. This situation is intensifying amid proposals linking U.S. teachers' salary to their students' performance on such tests. Watching teachers and students who are so clearly on the "same team" in Ireland has made me think more critically about the U.S. approach to high-stakes testing.

Parting Words

Overall, I am grateful to the Project Maths leaders and teachers in Ireland who welcomed me warmly and shared their thoughts with me openly. I am also grateful to DCU and CASTeL for hosting my visit. I was consistently impressed by the professionalism and collegiality of Ireland's mathematics educators, at the elementary, secondary and university levels. I look forward to seeing the fruits of the Project Maths team's labours as I continue to watch with keen interest from the other side of the pond.

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Message from the EMS

You may be interested to know that the European Mathematical Society has created a multi-lingual mathematical website with the aim of raising public awareness (RPA) of Mathematics:

`www.mathematics-in-europe.eu`

It provides information and help for everyone interested in Mathematics. Visitors to the site can find articles on various aspects of the subject including history, philosophy, mathematical professions, and research. The next step is to prepare a database for teachers and to collect helpful information for schoolchildren.

The EMS has requested that those interested in collaborating on this project, to continue to successfully develop and maintain the site, email the chair of their RPA committee:

`behrends@math.fu-berlin.de`

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A Hilbert space analogue of Heron's reflection principle

FINBARR HOLLAND AND ANCA MUSTAŢĂ[1]

Abstract. This note is concerned with the simultaneous approximation of two vectors in a Hilbert space by an element in one of its closed subspaces. The corresponding problem in elementary plane geometry admits a short and elegant solution based on reflection, probably due to Heron. Our discussion of the Hilbert space analogue follows a similar line, displaying a one-parameter family of non-linear isometries which fix the chosen subspace, and enjoy other properties possessed by linear reflections. A natural choice of parameter then yields the required minimum.

1. Introduction

Every secondary-school student learns the technique of dropping a perpendicular from a point to a straight line, thereby establishing, via the theorem of Pythagoras, the existence of a unique point on the line that is closer to the given point than any other point on the line. This is arguably the most influential theorem to emerge from elementary Euclidean Geometry, giving, as it does, prominence to the concept of perpendicularity, which is fundamental throughout Mathematics. It is also very likely the first instance of an approximation problem that dealt with existence, uniqueness and construction of a solution all at once.

This classical result opened up the Theory of Approximation in Banach spaces, and, in particular, it has a direct analogue in Hilbert space, one version of which we recall here for convenience [2]: Suppose M is a closed subspace of a Hilbert space H, with inner-product

[1] Support from Science Foundation Ireland through the Research Frontiers Programme is gratefully acknowledged.

$< \cdot, \cdot >$, and $x \in H$. Then there is a unique point $Px \in M$ such that

$$||x - Px|| = \inf\{||x - t|| : t \in M\},$$
$$< x - Px, t >= 0, \ \forall t \in M,$$

and

$$||x||^2 = ||x - Px||^2 + ||Px||^2.$$

It turns out that P is a *projection operator* whose range is M, i.e., P is a bounded self-adjoint linear mapping on H to M with $P^2 = P$. This is a key result in Hilbert space. As is well-known, not only do many profound facts about the space flow from it, such as, for instance, a description of its dual space, and a description of the space as a direct sum of one-dimensional subspaces, it also has wide applicability.

In this note we address the possibility of simultaneously approximating two or more vectors in a Hilbert space by an element in one of its closed subspaces. Given a finite subset F of vectors in a Hilbert space H, and a closed subspace M of H, can we determine an element $m \in M$ for which the elements in $\{||m - x|| : x \in F\}$ are simultaneously small? Any meaningful answer of this will of necessity involve a measure of "smallness", and we have several such measures to choose from. One natural such measure leads to the following simple result, whose proof we leave for the reader.

Theorem 1.1. *Let M be a closed subspace of a Hilbert space H. Let $x_1, x_2, \ldots, x_n$ be distinct vectors in H. Then there is a unique vector $m \in M$ such that*

$$\sum_{k=1}^{n} ||m - x_k||^2 = \inf\Big\{\sum_{k=1}^{n} ||x - x_k||^2 : x \in M\Big\}.$$

But what's the answer if we select the ℓ_1-norm rather than the ℓ_2-norm as our measure? Is the infimum of the set

$$\Big\{\sum_{k=1}^{n} ||x - x_k|| : x \in M\Big\}$$

attained? If so, what is its value?

These questions appear to be much more complex if $n > 2$. But, fortunately for us, the case $n = 2$ has a paradigm in elementary plane geometry which led to Fermat's principle of least time in optics. A preliminary version of this principle appears to have been first mooted by Heron (or Hero of Alexandria) who is thought to have lived in the first century, between 10–70 AD. Heron is probably best known to students of mathematics for his formula for the area of a triangle in terms of its side-lengths, but he is also renowned for his ingenious inventions of, for instance, precursors of the steam engine and vending machines, and his work on surveying and optics. (A concise account of Heron and his works is given in [1]; more detail about him can be elicited from the World Wide Web.) Seemingly, he discovered his area formula when he attempted to show that the "angle of incidence" in optics is equal to the "angle of reflection". So, for these reasons, it seems fair to ascribe the following result to him.

Theorem 1.2 (Heron)**.** *Suppose a, b are two complex numbers and L is a line in the complex plane. Then*

$$\max\big(|a-b|, |a-R_L b|\big) = \inf\big\{|z-a|+|z-b| : z \in L\big\},$$

where $R_L b$ denotes the reflection of b in L. Moreover, unless $a, b \in L$, the infimum is attained at a unique point in L.

Of course, the interesting case of this theorem is when a, b are on the same side of L.

In the next section we attempt to present a direct analogue of this result in a Hilbert space setting.

2. Is there a direct analogue of Heron's theorem in Hilbert space?

Given a closed subspace M of a Hilbert space H and distinct points $a, b \in H$, is the infimum of the set

$$\{||x-a|| + ||x-b|| : x \in M\}$$

attained by some point in M? If so, what is the value of the infimum? In what circumstances, if any, is the infimum attained by a unique point in M?

Can we imitate Heron's method to settle these questions? The latter question immediately raises another: What's meant by the reflection of a point in M?

If, as above, P denotes the orthogonal projection on M, and $x \in H$, $2Px - x$ presents itself as an obvious candidate for what might be termed *the reflection Rx of x in M*. It's easy to see that R is a linear isometric involution on H that fixes every element of M. In other words, it has all the characteristic properties of what is meant in plane geometry by a reflection in a line that passes through the origin. In particular, it follows that if $a, b \in H$ and $x \in M$, then

$$||a-Rb|| \leq ||a-x||+||x-Rb|| = ||a-x||+||R(x-b)|| = ||a-x||+||b-x||,$$

and, of course,

$$||a-b|| \leq ||a-y|| + ||b-y||, \ \forall y \in H.$$

Hence,

$$\max\left(||a-b||, ||a-Rb||\right) \leq \inf\left\{||a-x|| + ||b-x|| : x \in M\right\}.$$

However, this inequality is strict, in general, as the following simple example shows.

Example 2.1. Suppose M is the subspace spanned by the unit vector $(0,0,1) \in \mathbb{R}^3$, so that the (suggested) reflection of $x = (x_1, x_1, x_3)$ in M is given by $Rx = (-x_1, -x_2, x_3)$. Let $a = (3,1,1), b = (1,2,1)$. Then

$$\inf\left\{||m-a||+||m-b|| : m \in M\right\} = \sqrt{10}+\sqrt{5} > \max\left(||a-Rb||, ||a-b||\right).$$

Proof. Clearly,

$$\begin{aligned} & \inf\left\{||m-a|| + ||m-b|| : m \in M\right\} \\ = \ & \inf\left\{\sqrt{3^2+1^2+(t-1)^2} + \sqrt{1^2+2^2+(t-1)^2} : -\infty < t < \infty\right\} \\ = \ & \sqrt{10}+\sqrt{5}, \end{aligned}$$

whereas

$$||a-b|| = \sqrt{2^2+1^2} = \sqrt{5}, \ ||a-Rb|| = \sqrt{4^2+3^2} = 5,$$

and $\max(\sqrt{5}, 5) = 5 < \sqrt{10}+\sqrt{5}$. □

Thus, the approach adopted so far is inadequate to answer the opening question of this section. In order to obtain a complete solution we find it convenient to introduce a family of nonlinear norm-preserving operators in the next section, which may be of independent interest.

3. A one-parameter family of non-linear isometries on H

From now on, M will denote a closed subspace in a Hilbert space H, and P will stand for the orthogonal projection from H to M. Let $M^\perp$ stand for the orthogonal complement of M and let $Q =$

$I - P$, the orthogonal projection associated with $M^{\perp}$. With each unit vector $u \in M^{\perp}$, define R_u on H by

$$R_u x = Px - ||Qx||u, \ x \in H.$$

Note the following properties of this non-linear operator.

- $||R_u x - Px|| = ||Qx|| = ||x - Px||, \ \forall x \in H$;
- $R_u x = x, \ \forall x \in M$;
- $||R_u x||^2 = ||Px||^2 + ||Qx||^2 = ||x||^2, \ \forall x \in H$;
- If $z \in M$ and $x \in H$, then $||z - R_u x|| = ||z - x||$;
- $||QR_u x|| = ||Qx||, \ \forall x \in H$;
- $R_u(R_u x) = R_u x, \ \forall x \in H$;
- If v is a unit vector in $M^{\perp}$, then

$$||R_v u - v|| = ||R_u v - u||.$$

In particular, R_u is an isometry that fixes the elements of M, and enjoys other properties possessed by a linear reflection.

4. A Hilbert space analogue of Heron's theorem

Given $y \notin M$, set $\hat{y} = Qy/||Qy||$. Then $\hat{y}$ is a unit vector in $M^{\perp}$ and generates the non-linear isometry $R_{\hat{y}}$ by

$$R_{\hat{y}} x = Px - \frac{||Qx||Qy}{||Qy||}, \ x \in H.$$

Lemma 4.1. *Let $y \notin M$. Then*

(1)
$$||z - x|| = ||z - R_{\hat{y}} x||, \ \forall z \in M;$$

(2)
$$||x - y|| \leq ||R_{\hat{y}} x - y||, \ \forall x \in H,$$

with equality if and only if $R_{\hat{y}} x = x$.

(3) *If also $x \notin M$, then*

$$||y - R_{\hat{y}} x|| = ||x - R_{\hat{x}} y||.$$

Proof. Part 1 was noted above. Part 2 is equivalent to the inequality

$$\begin{aligned} -2\Re < x, y > &\leq -2\Re < R_{\hat{y}} x, y > \\ &= -2\Re \left(< Px, y > - \frac{||Qx||}{||Qy||} < Qy, y > \right), \end{aligned}$$

i.e.,

$$\begin{aligned} 0 &\le \Re\left(< x, y > - < Px, Py > + ||Qx||\,||Qy||\right) \\ &= \Re\left(< Qx, Qy > + ||Qx||\,||Qy||\right), \end{aligned}$$

which holds by the Cauchy-Schwarz inequality. Moreover, the equality holds if and only if $Qx = -||Qx||\hat{y}$, i.e.,

$$R_{\hat{y}}x = Px - ||Qx||\hat{y} = Px + Qx = x,$$

as claimed.

Part 3 follows from the fact that

$$\begin{aligned} < y, R_{\hat{y}}x > &= < y, Px > - \frac{||Qx||}{||Qy||} < Qy, y > \\ &= \underline{< Py, Px >} - ||Qx||\,||Qy|| \\ &= \overline{< x, R_{\hat{x}}y >}. \end{aligned}$$

□

Theorem 4.2. *Suppose $x, y \in H$. Then*

$$\inf\left\{||y-z||+||z-x|| : z \in M\right\} = \begin{cases} ||x-y||, & \text{if } x \in M \text{ or } y \in M, \\ ||y - R_{\hat{y}}x||, & \text{if } x, y \notin M. \end{cases}$$

Moreover, unless $\{x, y\} \subset M$, the infimum is attained by a unique element in M.

Proof. By the triangle inequality,

$$||x-y|| \le ||z-x|| + ||z-y||, \ \forall z \in H,$$

with equality if $z = x$ or $z = y$. This covers the first possibility.

Suppose $y \notin M$. Then, if $z \in M$, by Lemma 1,

$$\begin{aligned} ||y - R_{\hat{y}}x|| &= ||(y-z) + (z - R_{\hat{y}}x)|| \\ &\le ||y-z|| + ||z - R_{\hat{y}}x|| \\ &= ||y-z|| + ||z-x||. \end{aligned}$$

Thus

$$||y - R_{\hat{y}}x|| \le \inf\left\{||y-z|| + ||z-x|| : z \in M\right\}.$$

To show that the equality sign holds here, select $z_t = (1-t)y + tR_{\hat{y}}x$, where

$$t = \frac{||Qy||}{||Qx|| + ||Qy||}.$$

Claim: $z_t \in M$. Equivalently, $Pz_t = z_t$, i.e., $0 = (1-t)Qy + tQR_{\hat{y}}x$. But

$$\begin{aligned}(1-t)Qy + tQR_{\hat{y}}x &= (1-t)Qy + t\left(QPx - \frac{Q^2y||Qx||}{||Qy||}\right)\\ &= (1-t)Qy - \frac{tQy||Qx||}{||Qy||}\\ &= \left(1 - t - \frac{t||Qx||}{||Qy||}\right)Qy\\ &= \left(\frac{||Qy|| - t(||Qy|| + ||Qx||)}{||Qy||}\right)Qy\\ &= 0,\end{aligned}$$

as stated. Finally, since $z_t \in M$,

$$||z_t - y|| = t||y - R_{\hat{y}}x||,\ ||z_t - x|| = ||z_t - R_{\hat{y}}x|| = (1-t)||y - R_{\hat{y}}x||,$$

so that

$$||z_t - y|| + ||z_t - x|| = ||y - R_{\hat{y}}x||.$$

Hence, if $y \notin M$,

$$||y - R_{\hat{y}}x|| = \min\left\{||y - z|| + ||z - x|| : z \in M\right\}.$$

Of course, if $x \in M$, then $R_{\hat{y}}x = x$, and we capture the first case; and if $x \notin M$, then, by the lemma, $||y - R_{\hat{y}}x|| = ||x - R_{\hat{x}}y||$. So, this disposes of the second possibility.

We proceed to examine the cases of equality.

Case A: Both $x, y \in M$. In this case it's easy to see that

$$||z-x||+||z-y|| = ||x-y|| = \min\{||z-x||+||z-y|| : z \in M\},\ (1)$$

for every z in the line segment $[x, y]$. Conversely, if $||x - y|| = ||w - x||+||w-y||$, for some $w \in M\setminus[x, y]$, then, with $p = x-w, q = w-y$, we have $p \neq 0, q \neq 0$ and $||p + q|| = ||p|| + ||p||$. Equivalently, $\Re < p, q >= ||p||\,||q||$, so that, by the case of equality in the Cauchy-Schwarz inequality, $||q||p = ||p||q$. This now means that

$$w = \frac{||q||x + ||p||y}{||q|| + ||p||},$$

which conflicts with our hypothesis. In other words, there is equality in (1) if and only if $z \in [x, y]$. In particular, there is equality for infinitely many points in M unless $x = y$.

Case B: $x \in M, y \notin M$ and there is some $w \in M$ with $w \neq x$ such that

$$||x - y|| = ||w - x|| + ||w - y||.$$

An argument similar to the one just given implies that $w \in [x, y]$, which means that $y \in M$, which is impossible. Hence, the minimum is uniquely attained in this case.

Case C: $x, y \notin M$. Suppose

$$||y - R_{\hat{y}}x|| = ||w - x|| + ||w - y|| = ||w - R_{\hat{y}}x|| + ||w - y||,$$

for some $w \in M$. Again, $w \notin \{x, y\}$. This time, put $p = R_{\hat{y}}x - w$, $q = w - y$, so that, as before, $||p + q|| = ||p|| + ||q||$, whence $||q||p = ||p||q$, i.e.,

$$w = \frac{||q||R_{\hat{y}}x + ||p||y}{||q|| + ||p||} \equiv (1 - a)R_{\hat{y}}x + ay,$$

say. Since $Qw = 0$, $(1 - a)QR_{\hat{y}}x + aQy = 0$, i.e,

$$0 = -(1 - a)\frac{Qy||Qx||}{||Qy||} + aQy = (-(1 - a)\frac{||Qx||}{||Qy||} + a)Qy.$$

But $Qy \neq 0$, by hypothesis. Hence, $(1 - a)||Qx|| = a||Qy||$, so that

$$a = \frac{||Qx||}{||Qx| + ||Qy||} = 1 - t,$$

whence $w = z_t$. Thus, in Case C, the minimum is attained at a unique point in M.

In summary: unless both of x and y belong to M, the minimum is attained at a unique point in M. □

What this means is that, using the ℓ_1-norm, we can approximate simultaneously to two points by an element in M; and the approximating member of M is unique unless the given points both belong to the subspace.

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The Two-Child Paradox: Dichotomy and Ambiguity

PETER LYNCH

Abstract. Given that one of the children in a two-child family is a boy, what are the chances that the other is also a boy. The intuitive answer is 50 : 50. More careful investigation leads us to a 1-in-3 chance. We investigate circumstances under which these answers are correct. The imposition of further conditions yields some very surprising results.

To my sons Owen and Andrew, both born on Tuesday, one on Christmas Day.

1. Introduction

At the Ninth *Gathering 4 Gardner Conference* in March, 2010 [1], Gary Foshee presented a probabilistic puzzle, the solution of which was quite counter-intuitive. It has generated intensive discussion on the internet, with some intriguing contributions, and others that may charitably be described as misleading.

The problem raised by Foshee is simple to state. We consider only families having two children. We are told that one of the two children in a family is a boy born on a Tuesday, and are asked "what is the probability that there are two boys?" On first acquaintance, it seems that the information about the day of birth is irrelevant and cannot affect the result. As we shall see, things are not so straightforward. Foshee presented an answer that astounded his audience and that appeared to defy intuition [2].

We make the usual assumptions that boys and girls are born with equal probability, that the sex of each child is independent of that of the other, that each day of the week is equally probable, likewise each month, each star-sign, etc. These assumptions can be challenged, but we are not concerned here with genetic subtleties, chronological quirks or astrological aberrations.

The problem originally posed by Martin Gardner [3] was: "Given that there is at least one boy, what is the probability that there are

Problem	*Condition* (K)	*Probability* (P)
1	None	
2	The first-born child is a boy	
3	At least one of the children is a boy	
4	At least one is a boy born on a Tuesday	

TABLE 1. Probability of two boys in a two-child family.

two?" (No mention of Tuesdays here). Even this simpler problem led to extensive correspondence. In particular, cognitive psychologists have taken an interest in it from the point of view of human perception [4]. Interesting as this may be, it will not concern us here.

2. The Two-Child Paradox

We will confine attention to the following question: "Under stated conditions, what is the probability that, for a two-child family, there are two boys?" To motivate the discussion, why don't you start by completing Table 1. In each case, enter the value P that you think is the probability of two boys. Unless you have had previous exposure to Problem 4, is is unlikely that you will anticipate the correct answer.

We consider the simplest problem first: there are two children; what is the probability that there are two boys? There are no further conditions. There are four possible family configurations $\Omega = \{BB, BG, GB, GG\}$ where, for example, BG signifies that the first-born child is a boy and the second a girl. We arrange these in an array:

$$\begin{bmatrix} \underset{[1]}{BB} & \underset{[1]}{BG} \\ \\ \underset{[1]}{GB} & \underset{[1]}{GG} \end{bmatrix} \tag{1}$$

As each of the four possibilities is equally likely, we can assign equal relative frequencies or weights [in brackets] to all. In particular, two boys occur with probability $P = \frac{1}{4}$.

Now consider the second problem: the first-born child is a boy. The sample space is now $\Omega = \{BB, BG\}$, *i.e.*, we consider only the first row of (1). As both events are equally likely, the probability of two boys is $P = \frac{1}{2}$.

For the third problem, where the condition is that at least one of the children is a boy, the sample space is $\Omega = \{BB, BG, GB\}$; we retain the first row *and* the first column of (1). As each event is equally likely, the probability of two boys is $P = \frac{1}{3}$. This is the problem that Martin Gardner popularized; it is often called the *Two-Child Paradox*. The intuitive answer is $P = \frac{1}{2}$, whereas mathematical reasoning leads us to the answer $P = \frac{1}{3}$.

3. Tuesday's Child

We now introduce a dichotomy: we assume that all children fall into one of two categories, denoted by subscripts 1 and 2, with relative frequencies L and M respectively, and write $N = L + M$. Furthermore, we assume that these frequencies are independent of the sex of the child. There are now sixteen possibilities for the configuration of a two-child family, which we can arrange, with obvious notation, in an array:

$$\begin{bmatrix} \begin{matrix} B_1B_1 \\ {\scriptstyle [L^2]} \end{matrix} & \begin{matrix} B_1B_2 \\ {\scriptstyle [LM]} \end{matrix} & \begin{matrix} B_1G_1 \\ {\scriptstyle [L^2]} \end{matrix} & \begin{matrix} B_1G_2 \\ {\scriptstyle [LM]} \end{matrix} \\ \begin{matrix} B_2B_1 \\ {\scriptstyle [LM]} \end{matrix} & \begin{matrix} B_2B_2 \\ {\scriptstyle [M^2]} \end{matrix} & \begin{matrix} B_2G_1 \\ {\scriptstyle [LM]} \end{matrix} & \begin{matrix} B_2G_2 \\ {\scriptstyle [M^2]} \end{matrix} \\ \begin{matrix} G_1B_1 \\ {\scriptstyle [L^2]} \end{matrix} & \begin{matrix} G_1B_2 \\ {\scriptstyle [LM]} \end{matrix} & \begin{matrix} G_1G_1 \\ {\scriptstyle [L^2]} \end{matrix} & \begin{matrix} G_1G_2 \\ {\scriptstyle [LM]} \end{matrix} \\ \begin{matrix} G_2B_1 \\ {\scriptstyle [LM]} \end{matrix} & \begin{matrix} G_2B_2 \\ {\scriptstyle [M^2]} \end{matrix} & \begin{matrix} G_2G_1 \\ {\scriptstyle [LM]} \end{matrix} & \begin{matrix} G_2G_2 \\ {\scriptstyle [M^2]} \end{matrix} \end{bmatrix} \tag{2}$$

We have indicated [in brackets] the relative frequencies for each case. The total weight is $4N^2$, with the total for each of the four 2×2 blocks being N^2. Using this array to address Problem 3 (in which at least one child is a boy), we must consider the sample space comprising all cases except those in the bottom right-hand 2×2 block. We find immediately that $P = \frac{N^2}{3N^2} = \frac{1}{3}$, as before.

Consider next the probability of two boys given that one child is a boy in Category 1, *i.e.*, that B_1 occurs. The sample space now comprises the first row and first column of the array (2). The total weight is $4LN - L^2$. The weights for the three cases having two boys sum to $2LN - L^2$. Thus, the probability is

$$P = \frac{2LN - L^2}{4LN - L^2} = \frac{2-p}{4-p}\,, \qquad (3)$$

where $p = L/N$ is the relative frequency of Category 1. We note the two limits

$$\lim_{p\to 0} P = \tfrac{1}{2} \qquad\qquad \lim_{p\to 1} P = \tfrac{1}{3}\,.$$

Thus, the sharper the condition (the smaller L compared to N) the higher the probability of a two-boy family given that condition.

Now let us consider Problem 4 in Table 1: at least one of the children is a boy born on a Tuesday. Then $L = 1$ and $N = 7$ so $p = 1/7$. Thus, by (3), the probability of two boys is

$$P = \frac{2 - \frac{1}{7}}{4 - \frac{1}{7}} = \frac{13}{27}\,.$$

The surprise here is not the particular numerical value, but the fact that the condition of being born on a Tuesday has *any influence whatsoever* on the result!

Let us consider another question: "Given that one child is a boy born on a Christmas Day that falls on a Tuesday, what is the probability of two boys?" (we ignore leap years). Then $p = 1/(7\times 365) \approx 0.00039$ and

$$P = \frac{2-p}{4-p} \approx 0.49995\,.$$

For practical purposes, $P = \frac{1}{2}$. A sharper condition has increased P.

4. Paradox or Ambiguity?

It certainly seems at first amazing that the weekday of birth of one child can influence the probability of the sex of the other. The critical factor is that the information on one boy being born on a Tuesday is used at the outset to determine the sample space: we are really considering the question: "Among all two-child families for which at least one child is a boy born on a Tuesday, for what fraction of these families are there two boys?" To simplify matters, let us return to Gardner's problem: "Among all two-child families for which at least one child is a boy (born on *any* day of the week), for what fraction of these families are there two boys?" We have found above that the answer is $P = \frac{1}{3}$.

Now consider this scenario: you are strolling on Dun Laoghaire pier and meet an old school-chum, Pat, whom you have not seen since your youth. He is accompanied by a boy, and introduces him thus: "This is Jack, one of my two children". What are the chances that his other child is a boy? The answer is $P = \frac{1}{2}$; Pat's family has not been pre-selected from those having at least one boy. Similarly, if Pat had said "This is one of my two children, Jack, who was born on a Tuesday", it would have changed nothing: the chance of his other child being a boy is still 50 : 50. We will demonstrate this now.

5. Bayes' Theorem

We examine the probability that there are two boys in a two-child family X. The possibilities are $X \in \{BB, BG, GB, GG\}$. We denote by H the hypothesis $X \in \{BB\}$. Now we introduce a further condition, K, that "at least one child is a boy". Bayes' Theorem [5] implies

$$P(H|K) = \frac{P(H)P(K|H)}{P(K)} \,. \tag{4}$$

The prior, or unconditional, probability of H is $P(H) = \frac{1}{4}$. Clearly $P(K|H) = 1$, as "two boys" implies "at least one boy". Everything now hangs on the value of $P(K)$.

We consider all two-child families with one or more boys. Since there are three equally likely outcomes out of four that at least one child is a boy, we have $P(K) = \frac{3}{4}$. Using this value in (4) we have

$$P(H|K) = \frac{\frac{1}{4}\cdot 1}{\frac{3}{4}} = \frac{1}{3} \,.$$

However, when you meet your old friend Pat on the pier with his son Jack, you must assume — in the absence of any other information — that he has randomly selected one of his two children to accompany him. The condition K now is that "Pat has brought his son (or one of them) along for a stroll". If Pat has two boys, he must choose one of them; if he has two girls, the chance of a boy is zero; if he has a boy and a girl, the chance of his choosing a boy is $50 : 50$. Thus,

$$P(K|BB) = 1 \,, \quad P(K|BG) = P(K|GB) = \tfrac{1}{2} \,, \quad P(K|GG) = 0 \,.$$

The probability of the condition K may be partitioned as

$$\begin{aligned} P(K) &= P\big(K \cap (BB \cup BG \cup GB \cup GG)\big) \\ &= P(K \cap BB) + P(K \cap BG) + P(K \cap GB) + P(K \cap GG) \\ &= P(BB)P(K|BB) + P(BG)P(K|BG) + \\ &\qquad + P(GB)P(K|GB) + P(GG)P(K|GG) \end{aligned}$$

Substituting the numerical values in this gives

$$P(K) = \tfrac{1}{4} \cdot 1 + \tfrac{1}{4} \cdot \tfrac{1}{2} + \tfrac{1}{4} \cdot \tfrac{1}{2} + \tfrac{1}{4} \cdot 0 = \tfrac{1}{2}\,.$$

Finally, using this value in (4) we have

$$P(H|K) = \frac{\frac{1}{4} \cdot 1}{\frac{1}{2}} = \tfrac{1}{2}\,.$$

Pat's other child is equally likely to be a boy or girl! Moreover, you may ask Jack his birthday, whether he was born on a Tuesday, if he is a Gemini or likes bananas. It doesn't matter. None of this information has any influence on the probability of his sibling being a boy.

Of course, if you ask Jack does he have a brother . . . ?!!!

6. Conclusion

At the G4G Conference, Gary Foshee posed the question: "I have two children. One is a boy born on a Tuesday. What is the probability I have two boys?" He gave the answer $P = \frac{13}{27}$, and we have seen how this arises. But the question that Foshee actually answered was: "Of all two-child families with at least one child being a boy born on a Tuesday, what proportion of those families have two boys?" The correct answer to the question he actually posed is $P = \frac{1}{2}$.

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Paradacsaí

GEARÓID Ó CATHÁIN

Achoimre. Is féidir le paradacsaí deacrachtaí a léiriú go gonta nuair a chuirtear na míreanna d'argóint (tosach, réasúnaíocht, críoch) le chéile, go háirithe nuair a ritheann an chonclúid i gcoinne ár n-iomas. Ní bhíonn réiteach éasca ann i gcónaí. Bhí áit lárnach ag paradacsaí i stair na matamaitice nuair a tugadh aghaidh ar bhunchloch an ábhair – go háirithe trí réasúnaíocht loighce. Tá roinnt de na paradacsaí tógtha isteach anois sa chóras matamaiticiúil; uaireanta tríd an bparadacsa a réiteach agus uaireanta eile tríd an bparadacsa a sheachaint. Bhí deacrachtaí faoi leith leis an éigríoch agus féintagairt. Pléifear réimse leathan de pharadacsaí san alt seo.

1. Éagsúlachtaí de Pharadacsaí

Tagann an focal paradacsa ón nGréigis le ciall 'thar (*para*) chreidiúint (*doxa*)'. Is minic a úsáidtear an téarma nuair a bhíonn sórt ionaidh orainn faoin gconclúid i ndiaidh argóinte réasúnta. Cloímid, go ginearálta, le téarmaíocht W. V. Quine [1]. Dar leis, tá trí saghas:

- veridical
- falsidical
- antinomy

Veridical
Uaireanta, ní bhíonn ach cuma pharadacsúil ann, mar nuair a mhínítear dúinn cad atá cearr, leanann réiteach iomlán ar an scéal. Mar shampla, tosaíonn Quine [1] a chuntas ar pharadacsaí le tagairt don cheoldráma 'Pirates of Penzance' ina ndeirtear 'Is paradacsa é. Is paradacsa é'. Táthar ag rá faoi Frederic, a bhfuil bliain is fiche d'aois ach nach raibh ach *cúig* lá breithe aige. Réitítear an scéal ina iomláine nuair a mhínítear dúinn go bhfuil rud annamh ag tarlú anseo; rud nach dtarlaíonn ach uair amháin i ngach 1,460 lá – rugadh

é ar an 29 Feabhra! Níl aon bhréagadóireacht ag baint le paradacsa mar seo agus nuair a thuigimid an casadh sa scéal, imíonn an t-iontas a bhí ann go tapa.

Falsidical

Sa chás thuas, níl aon locht san argóint. San argóint seo, ó Augustus de Morgan (1806–1871) tá coimhlint ann.

$$\begin{aligned} x &= 1 \\ x^2 &= x \\ x^2-1 &= x-1 \\ (x+1)(x-1) &= x-1 \\ x+1 &= 1 \\ 2 &= 1 \end{aligned}$$

Is léir nach bhfuil anseo ach briseadh rialach (roinnt faoi náid) agus is minic a úsáidtear an téarma 'fallás' ina leith siúd. Úsáideann Quine 'falsidical' nuair a bhíonn bréagadóireacht sa réasúnaíocht agus sa chonclúid. Go ginearálta tá an córas matamaitice bunaithe ar rialacha agus ní féidir glacadh le frithrá mar loiteann sé an córas uile – féach Fíor 1 le tábla fírinne Wittgenstein. Tá go leor falláis mhatamaitice ann – beagnach ceann i leith gach riail. Is áiseanna foghlamtha iad chun a léiriú chomh seafóideach is a bhíonn cúrsaí nuair a bhristear na rialacha. Ach, ar an taobh eile de, léiríonn siad go bunúsach an fáth go bhfuil na rialacha ann. Is foinse dheas é Northrop [2] d'fhalláis.

Antinomy

An tríú sórt paradacsa a luaitear go minic ná 'antinomy' (*anti*–i

1.	p&¬p	tugtha, an frithrá
2.	p	ó 1.
3.	¬p	ó 1.
4.	p∨q	ó 2. p fíor, cuma faoi q
5.	q	ó 3. p bréagach, caithfidh q fíor

.i. Tá q fíor, ach is ráiteas ar bith é q.

p	q	¬p	p&q	p∨q
1	1	0	1	1
1	0	0	0	1
0	1	1	0	1
0	0	1	0	0

Fíor 1. Leanann gach rud ó Fhrithrá – Loighic Chlasaiceach

gcoinne an dlí–*nomy*) – atá níos doimhne ná na cinn eile. Bíonn bunrudaí i gceist agus is minic go n-athraímid ár meon faoi mhír éigin san argóint (tosach, réasúnaíocht nó críoch). Is léir ó chárta gnó an mhatamaiticeora Shasanaigh P. E. B. Jourdain(1913) [3, Caibidil 1] cén sórt deacracht atá againn (Fíor 2). Tá an dá thaobh in éadan a chéile – mar mhadra sa tóir ar a eireaball féin agus ní féidir leis an dá ráiteas bheith fíor ag an am céanna.

Tá an abairt ar an taobh thall den chárta seo fíor.	Tá an abairt ar an taobh thall den chárta seo bréagach.

FÍOR 2. Cárta gnó Jourdain

2. STAITISTIC

Glactar go forleathan le haicsímí Kolmogorov mar bhunchloch na matamaitice i leith dóchúlachta. Ach níl aontas i measc na staitisteoirí maidir leis an léirmhíniú ar dhóchúlacht. Dar le dream amháin is minicíocht choibhneasta fadtéarmach atá i gceist. Dar le dream eile is tomhas suibiachtúil atá ann a leasaítear trí Theoirim Bayes nuair a thagann breis eolais chun solais.

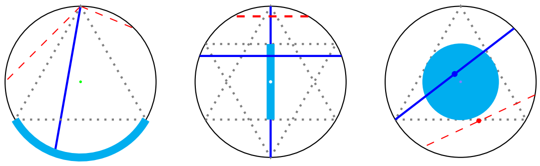
Braitheann an sainmhíniú clasaiceach ar dhóchúlacht,

$$p \equiv \frac{\#\,\text{torthaí fabhracha}}{\#\,\text{torthaí féideartha}}, \quad \left[\text{m.sh. Pr}[\clubsuit] = \tfrac{13}{52}\right],$$

ar an seans céanna a bheith ag gach rogha. Ní sainmhíniú i ndáiríre í seo mar tá an coincheap céanna sa mhíniú is atá sa choincheap atá á mhíniú. Úsáidtear Prionsabal Neafaise (Laplace) anseo. Sé sin, gan eolas roimh ré, níl aon chúis go dtoghfaí rogha áirithe thar cheann eile.

Pléimis le trí mhír

- *Paradacsa Bertrand*, a chuir ionadh ar na staitisteoirí ag an am,
- *Prionsabal Neafaise* a thugann casadh eile suimiúil ar an bprionsabal seo, agus
- *Paradacsa Simpson*, a thugann tacaíocht d'fhocail cháiliúla 'Lies, damn lies and statistics', a chuirtear i leith Benjamin Disraeli (1804–1881) uaireanta.



Trí mhodh chun corda le fad áirithe a phiocadh le dóchúlachtaí éagsúla. Tá na cordaí a théann trí achar scáthaithe fabhrach agus na cordaí eile ró gairid.

Fíor 3. Paradacsa Bertrand

Paradacsa Bertrand

Bhí an Francach Joseph Bertrand (1822–1900) ag plé, thart ar 1888, leis an dóchúlacht go bhfuil fad chorda, a phioctar gan aird i gciorcal, níos mó ná fad slios an triantáin chomhshleasaigh inmheánaigh.
Ag tagairt d'Fhíor 3:

Fíor 3 - I: *Pioc pointe ar an imlíne gan aird.* Tá fad gach corda ón bpointe go dtí an stua tiubh níos faide ná fad taobh an triantáin inmheánaigh. Is trian den imlíne é an stua tiubh: $p = \frac{1}{3}$.

Fíor 3 - II: *Pioc trastomhas gan aird* agus uaidh sin corda dronuillinneach leis. Tá gach corda a thrasnaíonn an trastomhas in áit tiubh níos faide ná fad taobh an triantáin. Leanann sé go bhfuil $p = \frac{1}{2}$.

Fíor 3 - III: *Pioc corda gan aird.* Má thiteann lárphointe an chorda sa chiorcal inmheánach, is corda fabhrach í. An cóimheas idir líon na bpointí sa chiorcal inmheánach (ga = leath ga an chiorcail mhóir) agus an ciorcal mór ná $\frac{\pi r^2}{\pi (2r)^2} = \frac{1}{4}$.

Tá trí fhreagra difriúil ann, agus níl aon locht sa réasúnaíocht. Braitheann an dóchúlacht ar an modh roghnaithe – ní raibh sé sin ar eolas go dtí gur léirigh Bertrand an paradacsa seo.

Prionsabal Neafaise

Tá meascán d'fhíon agus uisce i ngloine. An t-aon réamheolas atá againn ná go bhfuil ar a laghad an méid chéanna d'fhíon agus d'uisce

I - fíon : uisce	
réamheolas	1:1 – 1:2
luach airmheánach	$1{:}1\frac{1}{2}$
II - uisce : fíon	
réamheolas	$\frac{1}{2}{:}1 - 1{:}1$
luach airmheánach	$\frac{3}{4}{:}1$
inbhéarta go ***fíon : uisce***	$1{:}\frac{4}{3}$
$\neq$	$1{:}1\frac{1}{2}$

Fíor 4. Prionsabal Neafaise

ann agus an t-uasmhéid ná go bhfuil dhá oiread d'uisce ann. Féach Fíor 4. De réir an Phrionsabail Neafaise, de dheasca aon eolais bhreise, tá an dóchúlacht chéanna ag gach meascán. I Modh I, tá dóchúlacht 50% ag baint leis an meascán bheith idir $1{:}1 - 1{:}1\frac{1}{2}$ agus sa raon $1{:}1\frac{1}{2} - 1{:}2$. Sé sin, is é $1{:}1\frac{1}{2}$ an luach airmheánach. Mar an gcéanna i Modh II(uisce:fíon) is é $\frac{3}{4}{:}1$ an luach airmheánach. Ach nuair a aistrítear go dtí an cóimheas fíon:uisce is ionann an luach sin agus $1{:}\frac{4}{3}$.

Níl an Prionsabal Neafaise ag teacht slán ar an dá mhodh roghnaithe.

Paradacsa Simpson

I ngach scoil san ollscoil tá ráta pas níos fearr ag na cailíní ach ar an iomlán tá ráta pas níos fearr ag na buachaillí. Cinnte tá cuma pharadacsúil ar an ráiteas sin. Féach Fíor 5.

Rinne 600 buachaill an scrúdú sa Stair ach theip ar 480 díobh ag tabhairt ráta pas de 80%. Ach d'éirigh le 90% de na cailíní sa scrúdú céanna. Arís, tá na cailíní (33%) níos fearr san Fhisic. Ar an iomlán áfach is a mhalairt atá fíor – tá na buachaillí níos fearr (70% v 56%). Tá an uimhríocht i gceart ach fós tá cuma pharadacsúil ag baint leis.

An fhadhb anseo ná, i gcomparáid leis na buachaillí, d'éirigh go maith le líon beag de chailíní i scrúdú éasca agus sa scrúdú deacair

	Pas	*Teip*	*Iomlán*	*Ráta Pas*
Stair				
Buachaillí	480	120	600	80%
Cailíní	180	20	200	90%
Fisic				
Buachaillí	10	90	100	10%
Cailíní	100	200	300	33%
Stair + Fisic				
Buachaillí	490	210	700	70%
Cailíní	280	220	500	56%

Fíor 5. Paradacsa Simpson

bhí líon na gcailíní i bhfad níos mó. Tá cóimheas de 3:1 (600:200) i bhfabhar na mbuachaillí sa scrúdú éasca agus an cóimheas de 3:1 (300:100) i bhfabhar na gcailíní sa scrúdú deacair.

Cosúil leis an bparadacsa i leith an Pirates of Penzance, tá rudaí neamhghnácha ag tarlú.

3. An Éigríoch

Tá áit faoi leith ag an éigríoch sa mhatamaitic. Pléimis le trí pharadacsa:

- *Zeno* a scríobh faoi chainníochtaí deimhneacha bídeacha beaga – an éigríoch ag dul i laghad, nó go neamhfhoirmiúil $\frac{1}{\infty}$.
- *Cantor* a thaispeáin go bhfuil níos mó ná saghas amháin d'éigríoch ann agus nach bhfuil aon éigríoch is mó ann.
- *Burali-Forti* a phléigh an ábhar céanna a bhí ag Cantor trí mhodh difriúil.

Paradacsa Zeno

Tá an-cháil ar Zeno (timpeall 300 RC) a bhí ag plé le rudaí gan teorainn. Bhí argóint bhunúsach aige agus an chríoch air ná nárbh fhéidir aon slí a thaisteal. Seo a leanas mar a léirigh sé.

Má theastaíonn uait an seomra a fhágáil caithfidh tú leath an tslí a shiúl i dtosach. Ansin caithfidh tú leath an tslí atá fágtha a shiúl, agus arís leath an tslí atá fós fágtha a shiúl. Agus mar sin de. Ar

Céim	Fad Céime	Fad Siúlta
1	$\frac{1}{2}$	$\frac{1}{2} = 0.5$
2	$\frac{1}{4}$	$\frac{1}{2} + \frac{1}{4} = 0.75$
3	$\frac{1}{8}$	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 0.875$
$\vdots$	$\vdots$	$\vdots$
20	$\frac{1}{1{,}048{,}576}$	$\frac{1}{2} + \ldots + \frac{1}{2^{20}} = 0.9999999046$
$\vdots$	$\vdots$	$\vdots$
n	$\frac{1}{2^n}$	$\sum_{i=1}^{n} (\frac{1}{2})^i = 1 - (\frac{1}{2})^n$
$\vdots$	$\vdots$	$\vdots$

FÍOR 6. Paradacsa Zeno i nodaireacht an lae inniu

deireadh bíonn ort suim na bhfad uile sin a thaisteal. Dar le Zeno, ba chóir go mbeadh an suim sin d'fhaid gan teorainn – go háirithe mar nach bhfuil teorainn leis an méid faid atá le suimiú. I bhfocail eile, bheadh ort fad gan teorainn a thaisteal agus ní bheithfeá in ann an seomra a fhágáil.

Ba pharadacsa bunúsach é seo ag an am. Níorbh fhéidir glacadh leis an gconclúid agus ní raibh sé soiléir cá raibh an locht sa réasúnaíocht. Níor chuir sé isteach rómhór ar Aristotle [4, ltch 28] mar dar leis, bhí difear idir éigríoch trí shuimiú (sé sin má thógtar fad áirithe agus é a shuimiú leis féin go héigríoch – cinnte ní féidir an fad nua sin a shiúl) agus éigríoch trí dhéroinnt mar a dhein Zeno (mar go bhfuil an fad cuimsithe i dtosach).

Feictear dúinn i bhFíor 6 cad atá ag tarlú Is léir go bhfuil líon na gcéimeanna, n, ag dul i méid agus go bhfuil fad gach céime, $(\frac{1}{2})^n$, ag dul i laghad – níl ach an milliúnú chuid san fhichiú chéim. Is léir freisin go bhfuil an fad iomlán siúlta ag druidim cóngarach do a haon. Faraor, ní raibh an mhatamaitic seo ag na Gréagaigh.

Ba é Karl Weierstrass (1815–1897) a rinne an dul chun cinn maidir le luach feidhme, mar $(\frac{1}{2})^n$, nuair a ligtear n le héigríoch. Thug Weierstrass [5, ltch 161] sainmhíniú beacht ar an mbrí le luach $f(x)$ agus x ag druidim chun a; sé sin $\operatorname*{tr}_{x \to a} f(x) = L$ $(L \in \Re)$.

Dar leis, is cuma cé chomh cóngarach (abair $\pm\epsilon > 0$) go L a bhfuiltear, is féidir luach x atá cóngarach do a (abair $\pm\delta > 0$) a aimsiú sa chaoi is, go bhfuil $|f(x) - L| < \epsilon$. An cleas a bhí aige ná gan tagairt d'aon chainníocht nach réaduimhir é nó don éigríoch.

Sa chás seo tá $\underset{n\to\infty}{\text{tr}}\, n = \infty$ agus $\underset{n\to\infty}{\text{tr}} (\frac{1}{2})^n = 0$, ach tá $(\frac{1}{2})^n$ ag laghdú *níos tapúla* ná mar atá n ag dul i méid. Agus an teorainn le $S_n = \sum_{i=1}^{n} (\frac{1}{2})^i$ ná 1.

Bhí fadhb, cosúil ar shlí, ag Grandi [6, ltch 118] i 1703. Bhí seisean ag plé leis an sraith

$$1 - 1 + 1 - 1 + 1 - 1 \dots \quad .$$

Ag brath ar conas a chuirimid na baill le chéile, faighimid freagraí difriúla, m.sh.

$$1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \dots = 1$$

$$(1 - 1) + (1 - 1) + (1 - 1) + \dots = 0$$

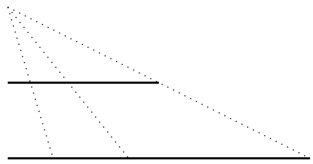
Cé go bhfuil cuma pharadacsúil anseo, d'aontódh matamaiticeoirí na linne seo le Grandi. Is sraith dibhéirseach í agus níl aon suim aige.

I dtéarmaí Quine [1] is paradacsa `falsidical` é paradacsa Zeno, mar (sa lá inniu) níl ann ach briseadh rialach faoi shraith atá coinbhéirseach, agus ní gá dúinn ár dtuiscint i leith na héigríche a leasú. Is dócha gur pharadacsa `antinomy` a bhí ann ag an am agus gur thóg sé breis is 2,000 bliain chun teacht ar réiteach sásúil. B'fhéidir go mbeidh réiteach sa todhchaí ar pharadacsaí na linne seo!

Paradacsa Cantor

De ghnáth bheithfeá ag súil go bhfuil an t-iomlán níos mó ná cuid. Tá sé sin fíor maidir le rudaí críochta. Ach bhí fios le fada nach mar sin a tharla maidir leis an éigríoch. Mar shampla, tá an líon céanna d'uimhreacha cearnacha $(1, 4, 9, 16, \dots)$ agus uimhreacha aiceanta – ar a dtugtaí Paradacsa Galileo (1564–1642) uaireanta [6, ltch 5] – mar is féidir iad a chur le chéile aon-le-haon mar seo,

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \dots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 1^2 & 2^2 & 3^2 & 4^2 & 5^2 & 6^2 & \dots \end{array} .$$



Fíor 7. Tá an méid céanna de phointí i mírlínte éagsúla

Is soiléir leis, go bhfuil líon na bpointí i mírlíne amháin ar aon mhéid le líon na bpointí i mírlíne eile, mar is féidir ceangal aon-le-haon a dhéanamh eatarthu (Fíor 7).

Bé Georg Cantor (1845–1918) a rinne an dul chun cinn san ábhar seo. Thaispeáin sé go raibh éagsúlachtaí d'éigríoch ann; mar shampla go raibh an éigríoch de réaduimhreacha níos mó ná an éigríoch d'uimhreacha aiceanta.

An paradacsa ná nach bhfuil aon uilethacar ann! Thaispeáin sé go bhfuil bunuimhir an tacair chumhachtaigh (sé sin an tacair de na bhfo-thacar) níos mó i gcónaí ná bunuimhir an tacair féin. An chiall le 'níos mó' sa chomhthéacs seo ná nach féidir mapáil aon-le-haon a dhéanamh eatarthu. Tá sé seo soiléir i leith tacair críochta:
$X = \{1, 2, 3\}$, le bunuimhir $|X| = 3$, agus an tacar cumhachtach
$2^X = \{\{\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$ le $|2^X| = 2^3$.

Bhain Cantor úsáid as argóint trasnánach chun a thaispeáint go raibh tacar cumhtachtach de na réaduimhreacha níos mó ná líon na huimhreacha aiceanta, ar ar thug sé $\aleph_0$. Rinne sé iarracht an dá thacar a mhapáil le chéile agus bhí breis baill sa tacar cumhtachtach.

Ag féachaint ar na huimhreacha aiceanta $1, 2, 3, \ldots$ rinne sé iarracht na fo-thacair go léir a liostáil – féach Fíor 8. Theip ar an iarracht sin.

Paradacsa Burali-Forti

Chonaiceamar thuas gur léirigh Cantor na bunuimhreacha i dtéarmaí aicmí coibhéiseacha faoi mhapáil dhétheilgeach. Mar an gcéanna, is féidir orduimhreacha a shainmhíniú mar aicmí coibhéiseacha de tacair dhea-ordaithe faoi mhapáil dhétheilgeach a choiméadann coibhéis

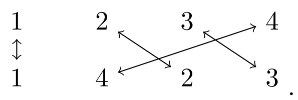
	Na huimhreacha aiceanta					
Liosta fo-thacar	*1*	*2*	*3*	*4*	*5*	...
gach uimhir	✔	✔	✔	✔	✔	...
uimhir a trí	✗	✗	✔	✗	✗	...
corr uimhreacha	✔	✗	✔	✗	✔	...
cearnaithe	✔	✗	✗	✔	✗	...
ré-uimhreacha	✗	✔	✗	✔	✗	...
⋮	⋮	⋮	⋮	⋮	⋮	...
fo-thacar trasnánach	✔	✗	✔	✔	✗	...
fo-thacar sa bhreis	✗	✔	✗	✗	✔	...

Tá liosta de na fo-thacair ar chlé agus iad innéacsaithe leis na huimhreacha aiceanta mar atá ar dheis. Nuair atá an liosta lán tóg an fo-thacar trasnánach agus bíodh fo-thacar nua againn trí athrú gach innéacs san fho-thacar trasnánach. Níl an fo-thacar nua ar an liosta mar tá éagsúlacht idir é agus an $n^ú$ fo-thacar ar an liosta san $n^ú$ innéacs. Leanann nach bhfuil an liosta de na fo-thacair uileghabhálach agus go bhfuil níos mó fo-thacair ann ná mar atá d'uimhreacha aiceanta

Fíor 8. Argóint Thrasnánach Cantor

san ord – is tacar dea-ordaithe $(A, \leq)$ más i leith gach fo-thacar de A, nach bhfuil folamh, go bhfuil íosbhall ann faoin ord sin.

Mar shampla, tá $A = \{1, 2, 3, 4\}$ faoi ord '$<$' agus $B = \{1, 4, 2, 3\}$ faoi ord na haibítre (aon, ceathair, dó, trí) coibhéiseach, san mhíniú seo, mar choiméadann an mhapáil seo an ord:



Is é an t-orduimhir ω den gnáth-ord atá ag dul leis na huimhreacha aiceanta, an t-orduimhir éigríche is lú.

Más orduimhir α, atá léirithe ag tacar dea-ordaithe $(A, \leq)$, agus má roghnaítear rud $t \notin A$, is féidir (de réir dealraimh) dea-ord a chur ar $A \cup \{t\}$ trí t a chur sa suíomh deiridh i ndiaidh le baill go léir de A; agus cuirtear an t-orduimhir comhfhreagrach, ar a dtugtaí an comharba, in iúl mar $\alpha + 1$. Ní comharba é gach orduimhir ach tá comharba ag gach orduimhir. Mar sin níl aon orduimhir is mó.

Ar an lámh eile, má thógaimid an bailiúchán de na tacair dhea-ordaithe go léir, tá ord páirteach so-fheicithe ag dul leis agus is soiléir go bhfuil uasluach ag gach fo-bhailiúchán dea-ordaithe. De réir Leama Zorn [7, Caibidil 7], tá uas-tacar dea-ordaithe ann arbh é a aicme Ω an t-orduimhir is mó.

Seo é an fhrithrá, ar lámh amháin tá $\Omega < \Omega + 1$ agus ar an lámh eile tá uasluach ann, .i.

$$\Omega < \Omega + 1 < \Omega.$$

De réir cosúlachta thuig Cantor é seo i 1895 ach d'fhoilsigh Burali-Forti é i 1897. Ba iad paradacsaí Cantor, Burali-Forti agus Russell (thíos) a chuir srian ar an gcoinceap saonta faoi thacair thart ar chasadh an chéid 1800 go 1900.

4. Ionduchtú

Is minic a dhéantar idirdhealú idir réasúnaíocht déaduchtacht agus réasúnaíocht ionduchtach san eolaíocht. Níl mórán deacrachta le réasúnaíocht déaduchtach – sé sin an rud atá fíor i ngach uile chás, tá sé fíor i gcás faoi leith. I ndáiríre, tá an t-eolas ann cheana féin, agus nílimid ach á luaigh i gcás faoi leith, m.sh.

Neach básmhar is ea gach duine.
Duine is ea Sócraitéas.
Neach básmhar is ea Sócraitéas.

Ar shlí is disciplín déaduchtach í matamaitic mar, i ndiaidh roinnt bunphrionsabail a leagadh síos, leanann gach rud eile – cé gur léirigh Kurt Gödel nach féidir gach ráiteas sa mhatamaitic a aimsiú ó chóras aicsímiteach.

Ach tá an réasúnaíocht ionduchtach conspóideach ó am David Hume (1711–1776) i leith. Seo í an tslí ina bhfaightear breis eolais sna disciplíní fisiceacha – mar shampla, sa staitistic, nuair a leathnaíomar torthaí trí shuirbhé shamplach go pobal níos mó.

Úsáidtear ionduchtú matamaiticiúil mar mhodh cruthúnais, ach i ndáiríre is réasúnaíocht déaduchtach atá ann mar tá an t-eolas ann cheana féin.

Bhí scéal ag Bertrand Russell faoin turcaí a chreid sa mhodh ionduchtach. Bhí sé ag fáil béile ón bhfeirmeoir lá i ndiaidh lae agus ag tnúth le amárach go dtí am roimh Nollaig nuair a chas an feirmeoir muineál an turcaí! [8, ltch 164]

Chas Karl Popper (1902–1994) [9] an fhadhb bunoscionn. Dar leis, ní gá an bhéim a chur ar theoiric a dheimhniú. Déantar tuairimíocht agus sé an aidhm ná é a bhréagnú (`Falsfication`).

Pléimis le dhá pharadacsa cháiliúla maidir le hionduchtú

- *Grue – Goodman*, agus
- *Fiach Dubh – Hempel.*

Grue Goodman

Má iniúchaimid a lán smaragaidí (emeralds) agus má bhíonn dath glas ag baint leo go léir, leanann sé de réir réasúnaíochta (ionduchtaithe) go bhfuil gach smaragaid, fiú na cinn nár iniúchadh, glas. Shamhlaigh Nelson Goodman (1906–1998) dhá thréith i leith smaragaide – glas agus grue, seo a leanas

***glas*:** an gnáth dath glas.

***grue*:** dath glas léi má dhéantar iniúchadh ar smaragaid roimh am t (sa todhchaí) agus dath gorm má dhéantar iniúchadh uirthi i ndiaidh ama t.

Fuair sé an focal 'grue' ó *gruebleen* a chum James Joyce i Finnegan's Wake [8, ltch 68]. De réir mar a ritheann an réasúnaíocht ionduchtach, má bhíonn tréith áirithe ann i leith gach smaragaide ar a ndearnadh iniúchadh go dtí seo, tá an tréith ann i leith gach smaragaide – fiú na cinn nár scrúdaíodh.

Cinnte tá dath glas ag baint le gach smaragaid go dtí seo agus de réir na réasúnaíochta beidh an dath sin ar gach ceann sa todhchaí. Ach, tá sé fíor freisin go bhfuil an tréith *grue* ag baint le gach smaragaid go dtí seo agus de réir na hargóinte céanna, beidh gach smaragaid sa todhchaí *grue*.

Ach, tá fadhb anseo. Roimh am t, tá glas agus grue mar an gcéanna, ach i ndiaidh ama t tá difríocht ann – ciallaíonn grue go bhfuil dath gorm ag baint léi. Mar sin, i ndiaidh ama t má thógaimid smaragaid úr ón talamh, agus má bhíonn sé glas, ní féidir léi bheith grue agus *vice versa*, má bhíonn sí gorm (.i. grue) níl sí glas.

Seo léiriú go bhfuil deacrachtaí le hionduchtú. Tá ionduchtú ríthábhachtach mar is de bharr réasúnaíochta ionduchtaithe a fhaighimid breis eolais.

Réiteach amháin ná béim níos mó a chur ar an tréith 'glas' a bheith ag smaragaid mar tá sé níos nádúrtha agus ag teacht níos fearr faoi mar a bhfuil cúrsaí sa dhomhan nádúrtha.

Fiach Dubh Hempel

Phléigh Karl Hempel (1905–1997) [10, Caibidil 4] leis an modh ionduchtach chomh maith. Chun a pharadacsa a chur in iúl rinne sé tagairt d'éan – an fiach dubh (raven). Chun bheith dílis don téarmaíocht, úsáidfimid an focal 'fiach' don éan úd agus úsáidfimid an focal 'dubh' le feidhm aidiachta. Tá sé seo ag teacht le foclóir an tSeabhaic (Irish-English Pronouncing Dictionary, An Seabhac, Talbot Press, 1959), áit ar thug seisean fiach ar 'raven'.

I ndiaidh breathnú ar a lán fiaigh tugtar faoi deara go bhfuil siad go léir dubh, agus de réir réasúnaíochta ionduchtaithe leanann an hipitéis

$$\text{H: I leith gach fiach, tá sé dubh.}$$
$$(x)(\mathrm{F}x \to \mathrm{D}x)$$

Má fhéachaimid ar fhiach agus má thugaimid faoi deara go bhfuil sé dubh, sin tacaíocht don hipitéis. Ach má bhíonn dath eile aige, diúltaítear an hipitéis. Níl aon fhadhb le sin.

Ach is féidir leagan eile den hipitéis a scríobh atá, de réir rialacha loighce, díreach mar an gcéanna

$$\text{H*: I leith gach rud nach bhfuil dubh, ní fiach é.}$$
$$(x)(\neg\,\mathrm{D}x \to \neg\,\mathrm{F}x).$$

Má iniúchaimid rud mar pheann gorm, is léir nach bhfuil sé dubh agus nach fiach é agus tugann sin tacaíocht do H*. Ach tá H* cothrom le H. An paradacsa ná go bhfuil sé ait go dtugann peann gorm fianaise dúinn go bhfuil dath dubh ar gach fiach.

Níl mórán deacrachta anseo don mhatamaiticeoir má bhíonn cainníocht i gceist. Abair go bhfuil 100 fiach ann agus 100 milliún rudaí eile ann nach bhfuil dubh. Tá dhá shlí ann chun an hipitéis a scrúdú. An tslí is éasca ná an 100 fiach a iniúchadh agus glacadh leis an hipitéis má bhíonn siad go léir dubh. An tslí eile ná an 100 milliún rudaí eile nach bhfuil dubh a iniúchadh agus glacadh leis an hipitéis dá mba nach fiach iad go léir.

Cinnte níl an dara mhodh éifeachtach ach réitíonn sé an paradacsa (maidir le cainníocht).

Ach dar le Quine níl aon bhrí bheith ag plé le rudaí 'neamh-dubh' mar seo mar níl aon nádúr ag baint leo.

5. Féintagairt

Tá trí pharadacsa sa roinn seo ag plé leis an gcoincheap de fhéintagairt:

- Paradacsa an Bhréagadóra,
- Paradacsa Russell, agus
- Paradacsa an Bhearbóra.

Paradacsa an Bhréagadóra

Bhí Naomh Pól dian ar áitreabhaigh oileáin Créit nuair a chuir sé ina leith (sa Bhíobla Naofa, Títeas 1:12-13) gur daoine místuama iad. I ngan fhios dó féin is dócha, bhí Naomh Pól ag tagairt do pharadacsa a bhí ag an nGréagach Epimenides timpeall 600 bliain roimhe sin:

- Is bréagadóirí iad go léir na Créitigh.
- Is Créiteach a dúirt é.

Is féidir leagan gearr den pharadacsa seo a chur amach mar

Ⓐ: 'Tá an abairt seo bréagach.'

Más fíor í leanann sé go bhfuil sí bréagach, agus más bréagach an abairt leanann sé go bhfuil sí fíor. Sé sin, bíonn Ⓐ fíor nuair nach bhfuil sí agus *vice versa.*

Paradacsa den scoth é seo. Níl an réiteach simplí. Tuigimid go bhfuil dhá ní ag teacht le chéile

- fírinne, agus
- féintagairt.

Phléigh Alfred Tarski (1901–1983) [11, ltch 109–123] le brí an fhírinne agus na bréagadóireachta (séimeantaic). Cé gur phléigh Tarski le teangacha foirmiúla, dar leis baineann fírinne leis an gcoincheap de mheta-teanga:

Thit sé. — ráiteas sa bhunteanga L.

Tá 6 litir i 'Thit sé' — sa mheta-teanga L'

Is fíor é "Tá 6 litir i 'Thit sé'" — sa mheta-mheta-teanga L''.

Níl aon fhadhb le fírinne nó féintagairt leo féin:

'Tá sé fliuch' — atá fíor nó bréagach;

'Tá an abairt seo as Gaeilge' – féintagairt.

Ach nuair a chuirtear le chéile iad, mar a rinneadh in Ⓐ thuas táimid i bponc. Dar le Tarski, sainmhínítear fírinne sa mhetatheanga – oibrítear ar chéim níos airde. Ní féidir fírinne a réiteach ina hiomláine sa bhunteanga. Bhí sé díomách an raibh aon réiteach ar fhírinne i dteanga nádúrtha.

Bhí tuairim eile ag Kripke [12, ltch 145–148]. Dar leis tá ráitis ann agus ní féidir a rá an bhfuil siad fíor nó nach bhfuil.

Is é seo an paradacsa ar úsáid Kurt Gödel ina pháipéar clúiteach nuair a chruthaigh sé go bhfuil ráitis sa mhatamaitic agus ní féidir a chruthú an bhfuil siad fíor nó nach bhfuil [nuair a chuirtear matamaitic ar bhunchloch aicsímiteach ar chomhchéim le huimhríocht].

Paradacsa Russell

Phléigh Tarski le paradacsa an bhréagadóra maidir le séimeantaic. Maidir le fírinne sa mhatamaitic, de ghnáth, is comhsheasmhacht le bunphrionsabail a bhíonn i gceist. Mar aon le paradacsa an bhréagadóra, is féintagairt i bhfoirm féinbhallraíocht an coincheap láidir i bparadacsa Bertrand Russell (1872-1970). Measadh, ag an am, go bhféadfaí tacar a shainmhíniú i dtéarma preideacáideacha, cosúil le

$$X = \{x\text{: leanann } x \text{ coinníoll ar bith}\},$$

agus uaidh sin bhí Gottlob Frege (1848–1925) ag iarraidh teacht ar na huimhreacha aiceanta trí rialacha loighce. Cheap sé go bhféadfaí bunchloch na matamaitice a leagadh ar loighic.

Ag baint úsáid as an míniú sin is féidir tacair a roinnt i ndhá grúpa:

- tacair gan féinbhallraíocht, agus
- tacair le féinbhallraíocht.

Mar shampla, tá an tacar seo gan féinbhallraíocht:

$$X_{\text{cúige}} = \{x\text{: is Cúige } x\} = \{\text{Laighin, Mumha, Connachta, Ulaidh}\},$$

Is léir nach bhfuil an tacar $X_{\text{cúige}}$ ina bhall dá féin, mar ní cúige é $X_{\text{cúige}}$, ach tacar.

Ar an taobh eile tá an tacar seo ina bhall dá féin:

$$\begin{aligned} X_{>3} &= \{x\text{: is tacar } x \text{ le níos mó ná 3 bhall }\} \\ &= \{X^4_{\text{cúige}}, X^{52}_{\text{cartaí}}, X^?_{\text{daoine}}, X^{32}_{\text{contae}}, \ldots\}. \\ &= \{X^4_{\text{cúige}}, X^{52}_{\text{cartaí}}, X^?_{\text{daoine}}, X^{32}_{\text{contae}}, X^?_{>3} \ldots\}. \end{aligned}$$

Chuir Russell an cheist faoi

$$X = \{x\text{: is tacar } x \text{ agus } x \notin x \ \}.$$

Tá X ina bhall de X nuair nach bhfuil sé ina bhall agus *vice versa* – paradacsa bunúsach.

I dtéarmaí Quine, is `antinomy` an paradacsa seo mar bhí ar na saineolaithe ag an am a dtuairimí a leasú. An réiteach atá i bhfeidhm ná cosc a chur ar fhéinbhallraíocht – cosúil leis an gcosc ar roinnt faoi náid.

Ba bhuile thubaisteach é an paradacsa sin do Gottlob Frege. Bhí sé ar tí a obair mhór a fhoilsiú nuair a fuair sé scéal ó Russell. Thuig sé láithreach go raibh fadhb dhoréitithe aige.

Paradacsa an Bhearbóra

Dar le Bertrand Russell, tá sráidbhaile sa Rúis inar féidir na fir fásta a roinnt i ndhá grúpa – daoine féinbhearrtha a bhearrann a bhféasóga féin agus daoine nach mbearrann a bhféasóga féin. An tasc atá ag an mbearbóir ná féasóga na bhfear nach mbearrann iad féin a bhearradh agus gan bacaint leis na daoine a bhearrann iad féin.

Ach tá fadhb ann. Is fear fásta an bearbóir é féin agus 'Cé a bhearrann an bearbóir?'. Má bhearrann an bearbóir a fhéasóg féin, is duine féinbhearrtha é agus ní cóir don bhearbóir é a bhearradh. Ar an lámh eile, mura mbearrann an bearbóir é féin, caithfidh an bearbóir é do bhearradh. Táimid i bponc.

Ach tá réiteach simplí ar an gcruachás seo. Ní fhéadfadh sráidbhaile mar sin bheith ann. Ní féidir coinníollacha an tsainmhíniú a tugadh faoin sráidbhaile a chomhlíonadh.

An ceacht sa mhatamaitic atá anseo ná nach leor rud éigin a shainmhíniú, caithfear eiseadh a chruthú chomh maith.

6. Miosúr

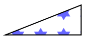
De réir teoiric miosúir má scoiltear corp A (cuimsithe i $\Re^n$) i míreanna scartha, agus iad a chur ar ais le chéile i gcorp eile B, beidh siad ar chomhmhiosúr, $\mu(A) = \mu(B)$. Pléimis le dhá pharadacsa faoin ábhar seo

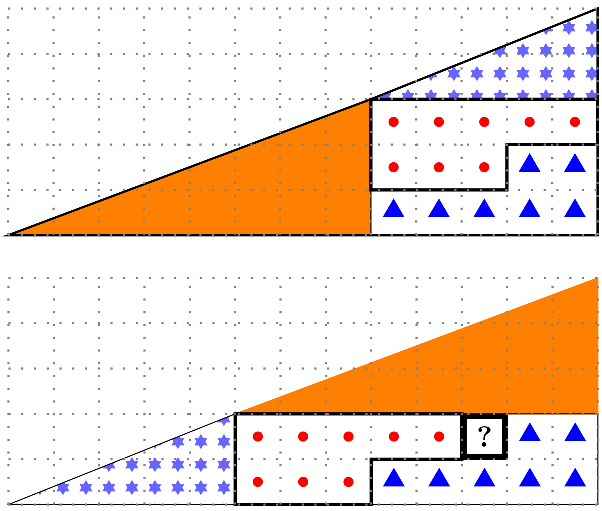
- *Curry*, agus

- *Banach-Tarski.*

Paradacsa Curry

Scoiltear an triantán (ar barr) i bhFíor 9 i gceithre chuid agus cuirtear ar ais iad sa triantán (ar bun), ach tá cillín sa bhreis againn! Is paradacsa na súl an ceann seo mar ní triantáin iad ar chor ar bith ach ceathairshleasáin. Ag úsáid na gcomhordanáidí sa chúlra, is léir nach dtéann an líne (0,0)–(13,5) tríd an bpointe (8,3) nuair nach bhfuil na géaruillinneacha is lú sna triantáin cothrom

 $\tan^{-1}\frac{3}{8} = 20.556^\circ$,  $\tan^{-1}\frac{2}{5} = 21.801^\circ$.

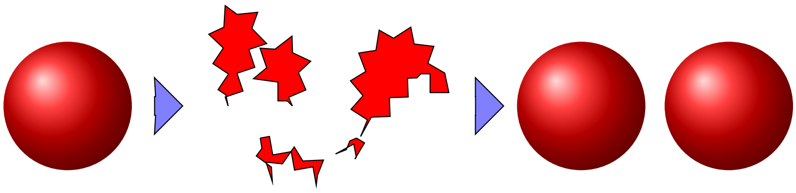


FÍOR 9. Paradacsa Curry – cillín sa bhreis

Tá sé cliste sa chaoi is go mbailítear an difríocht atá idir achair an dá cheathairshleasán i gcillín amháin. Tabhair faoi deara gur uimhreacha leantacha iad na huimhreacha tabhachtacha (5,8,13) sa sraith Fibonacci.

Paradacsa Banach-Tarski

De réir leagan amháin de Banach-Tarski (1924) [13, ltch 366–367], is féidir an sféar aonad a scoilt i gcúig phíosa scartha agus iad a chur ar



FÍOR 10. Paradacsa Banach-Tarski

ais le chéile sa chaoi is go bhfuil **dhá** sféar díreach mar an gcéanna leis an gcéad sféar – féach Fíor 10. Gan dabht tá deacracht anseo. Ach ní paradacsa atá ann – is teoirim é!

Tá a fhios againn cheana go dtarlaíonn rudaí gan coinne nuair a bhímid ag plé leis an éigríoch, m.sh $\aleph_0 = \aleph_0 + \aleph_0$. Is mar seo atá anseo leis. Ní dlúthchoirp iad na píosaí seo – níl iontu ach cur le chéile casta de phointí a roghnaítear leis an Aicsím Rogha (Axiom of Choice). Níor thug Banach-Tarski aon slí chun an scoilt a dhéanamh. Thaispeáin siad gur féidir é a dhéanamh má roghnaítear ball amháin as gach ceann de (méid éigríche) na fo-thacair.

Is cáineadh an toradh seo ar an Aicsím Rogha, ach úsáidtear an aicsím seo chomh minic san go bhfuil leisce ar na matamaiticeoirí é a chur ar leataobh.

Colafan

D'úsáideadh an ríomhchlár clóchuradóireachta LaTeX chun an páipéar seo a chur amach. Mar chuid de sin, d'úsáideadh `TikZ` chun na léaráidí a tharraingt agus `cooltips` chun an t-aistriúchán go Béarla i leith na bhfocal sa Ghluais a nochtadh sa leagan leictreonach, nuair a théann an pointeoir luiche tharstu. (Oibríonn sé seo le AdobeReader – agus bhféidir le léitheoirí PDF eile). Tá an Ghluais bunaithe ar `http://www.focal.ie`. D'úsáideadh An Gramadóir ag `http://borel.slu.edu/gramadoir` chun feabhas a chur ar chruinneas na Gaeilge. Tá na táirgí seo go léir saor ar an idirlíon.

Gluais

aicme choibhéiseach *equivalence class* · **airmheánach** *median* · **aon le haon** *one to one* · **bréagadóireacht** *falsehood* · **bunuimhir**

cardinal number · **cainníocht** *quantity* · **ceathairshleasán** *quadrilateral* · **cóimheas** *ratio* · **coinbhéirseach** *convergent* · **comharba** *successor* · **comhsheasmhacht** *consistency* · **comhshleasach** *equilateral* · **corda** *chord* · **críochta** *finite* · **cuimsithe** *bounded* · **cur le chéile** *assemblage* · **dea-ordaithe** *well-ordered* · **déaduchtú** *deduction* · **détheilgean** *bijection* · **dibhéirseach** *divergent* · **dóchúlacht** *probability* · **éigríoch** *infinity* · **eiseadh** *existence* · **feidhm** *function* · **féinbhallraíocht** *self-membership* · **féintagairt** *self-reference* · **fírinne** *truth* · **fo-thacar** *subset* · **frithrá** *contradiction* · **gan aird** *random* · **hipitéis** *hypothesis* · **iomas** *intuition* · **ionduchtú** *induction* · **íosbhall** *least element* · **minicíocht choibhneasta** *relative frequency* · **miosúr** *measure* · **neamhnitheach** *null (set)* · **orduimhir** *ordinal* · **preideacáid** *predicate* · **Prionsabal Neafaise** *Principle of Indifference* · **réaduimhir** *real number* · **saonta** *naive* · **scartha** *disjoint* · **séimeantaic** *semantic* · **sraith** *series* · **stua** *arc* · **suibiachtúil** *subjective* · **tacar cumhachtach** *power set* · **tacar** *set* · **teorainn** *limit* · **trasnán** *diagonal* · **tuairimíocht** *conjecture* · **uileghabhálach** *exhaustive* · **uilethacar** *universal set* · **uimhir aiceanta** *natural number*

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Gearóid Ó Catháin: Tá céimeanna aige sa Mhata agus sa Staitistic (1973, 1974) as Ollscoil Corcaigh agus Ph.D. (2002) as Ollscoil Luimnigh.

Doibrigh sé sa Phríomh Oifig Staidrimh agus sa Roinn Sláinte. Tá sé fostaithe mar staitisteoir ag Córas Iompair Éireann i Stáisin Heuston i mBaile tha Cliath ó 1982. Is maith leis bheith ag rothaíocht, ag cadhcáil (kayaking) agus ag rith. Is leantóir dóchasach é dfhoireann iomána na nDéise.

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BOOK REVIEWS

LOVING + HATING MATHEMATICS
Challenging the Myths of Mathematical Life

by Reuben Hersh and Vera John-Steiner

Princeton University Press, 2011. ISBN 978-0-691-14247-0. $29.95

Reviewed by Anthony G. O'Farrell

This is an entertaining, useful, and provocative book. It is about mathematicians, rather than mathematics. Its aim is missionary: to rehabilitate mathematicians in the opinion of the general public (GP).

The majority of the GP are more-or-less indifferent to mathematics. A large minority love it, and a larger minority hate it — a hate usually born of schoolday fear. Recently, a number of films and documentaries on mathematicians have attracted wider attention, and these upset professionals because they reinforce the myths. There have been calls for mathematicians to redress the balance, and this book is one attempt in that direction. Professor Hersh has already published an outstanding book about the mathematical experience [1].

The myths of the subtitle are four:

(1) Mathematicians are different from other people, lacking emotional complexity.
(2) Mathematics is a solitary pursuit.
(3) Mathematics is a young man's game.
(4) Mathematics is an effective filter for higher education.

The book is structured as a systematic attempt to debunk these myths. The evidence adduced consists in the main of anecdotes drawn from the increasingly voluminous literature of writing about the lives and foibles of mathematicians, supplemented by some informal survey work by the authors and by reasoned argument.

Professionals will recognise many of the anecdotes, but the authors have trawled well, and I encountered many new gems here.

One of the useful features of the book is its excellent bibliography of sources on writing about mathematicians. However, professionals are not the target readership. The book is written to be accessible to the general public. There is essentially no mathematical content. The remarks about the substance of various mathematical achievements will not enlighten anyone who does not already understand them. There are just three equations: the quadratic equation and its standard solution, and a Rogers-Ramanujan identity (p.92). The latter seems to be there just for show, and has a misprint. No doubt the misprint will be corrected in future printings, but it might be a better idea to drop all three. The solution to the quadratic is typeset in an odd way, using (+ or −) instead of ±, (as though a reader who can understand $\surd$ will not understand ±), and the comment about the solution — "not beautiful" — is debatable. *De gustibus non disputandum est,* but I distinctly recall being bowled over when I was eleven by the trick of completing the square, when Br. Kevin Skehan showed it to us. Besides, according to a well-known publishing principle, the elimination of three equations should have the effect of multiplying the prospective sales figure by eight!

Myth (1) is challenged in the first four chapters, which examine anecdotally how people grow up to become mathematicians, how mathematical culture operates, the role of mathematics as a solace in terrible times, and mathematics as addiction and obsession. Included are the touching stories of J.-V. Poncelet and José Luis Massera, and the more troubling tales of Grothendieck, Gödel, and the murderers Bloch and Kaczynski — the Unabomber (Irish readers, familiar with the events of 1649, will be interested in the authors' idea that some words of Cromwell — of all people — might be used to urge Kaczynski to reconsider his murderous conclusions).

Taken together, these stories support the view that competent mathematicians come in various personality types, and many exhibit emotional complexity, but in the reviewer's opinion they also support the case that mathematicians are different. The story of how J.H. Conway re-invented the filing cabinet is typical. Mind you, the book would be far less entertaining if they were the same as everyone else.

There is proven interest among the GP in anecdotes and biographies of highly eccentric mathematical geniuses. Whether there will ever be much interest in the lives of the many perfectly sane mathematicians who obtain wonderful but generally-incomprehensible results is open to question. This is the problem about divorcing the account of the people from their work. We may just have to face the reality that the GP will never understand us. The brutal truth is that even most of our scientific colleagues don't understand what we do, or why it should matter.

By the way, the illustrations, consisting of mediocre-quality black-and-white photographs of mathematicians, will do little to dispel any stereotypes. Quite a few are of women, but that's about it.

Myth (2) is successfully demolished in two chapters, one on friendships and partnerships, including marriages, and one on famous mathematical communities, ranging from Göttingen in the 1890's to the Association of Women in Mathematics and the online community built by Gowers and flourishing today. The pocket accounts here will stimulate readers to pursue the original sources for fuller accounts. The reviewer was particularly taken by the accounts of mathematical friendships and community life in the former Soviet Union. Of course, his interest in these stories is coloured by his knowledge of the technical achievements of the participants, and it is hard to judge how the same stories will strike a reader to whom their names are just names of men and women, as opposed to the names of demigods. That Kolmogoroff and Alexandroff spent many a sunny March day skiing across country in their underpants gains an interest it might not otherwise have, if you know something of what these men created.

Myth (3) is perhaps not a myth of the GP, but rather of mathematical enthusiasts. It is challenged on two grounds.

First, cases of successful women are given, starting with the usual Germain-Kovalevskaya-Noether trio. More interesting is some witness on the somewhat improved contemporary scene. A reasonable summary would be that mathematics is a man's game, but it doesn't have to be.

Second, an impressive list is given of mathematicians who maintained or even began productivity in old age, and this is supplemented by an account of responses to a survey conducted by Hersh. The results are interesting, but hardly altogether encouraging. A reasonable summary would be productivity can be maintained, but only if appropriate steps are taken to compensate for declining energy, memory and computational ability, and that the most reliable recipe is the combine your accumulated technique and cunning with the energy of a younger collaborator.

Myth (4) is about the rôle of mathematics in education. The related questions are: what mathematics should a given person learn, and what people are capable of learning a given area of mathematics?

Chapter 9 contrasts two extreme approaches to teaching mathematics at university: that of R.L. Moore, and the *Potsdam model* invented by C.F. Stephens of SUNY Potsdam. This chapter is very interesting, but a bit frustrating. Most readers of this Bulletin will be familiar with Moore's method, designed for elite students, rarely used, but it was fascinating to read of Moore's implacable bigotry. Stephen's method, spectacularly successful, is based on the idea that *by lovingly and patiently nurturing students* one can teach mathematics to *any student who wishes to learn.* The frustrating part is the absence of any real detail on how this striking idea is actually carried into practice.

The last chapter addresses the problem of fear and loathing of mathematics among school-children, and advocates as part of a solution that we eliminate "abstract" mathematics (including algebra) as a universal component in secondary education. The point is made that children are not born hating mathematics; they *learn* to hate it in school. There would be no reason to fear it if it could be avoided easily. It is also pointed out that many professional people, such as doctors, make no use of algebra and trigonometry in their work. These facts are not in dispute, but many will dispute the wisdom of the proposed solution. In particular, the authors may underrate the rôle of mathematical studies in developing reasoning skills, which, once developed, may be transferred to other domains. There is also evidence [2] that patients would be better served if many doctors had more, rather than less exposure to mathematics.

Apart from the usual indices and notes, the book includes a useful appendix giving thumbnail biographies of hundreds of mathematicians.

No-one who hates mathematics will pick up this book. Realistically, the most likely reader already belongs to the minority who are positively-disposed. Younger readers of this kind will find support for the view that a reasonable person might find happiness and fulfilment in the pursuit of mathematics, and will be stimulated to pursue further the lives, achievements, and problems mentioned. I recommend this book for school and university libraries, and for prizes. It is priced to be affordable by the public, and worth owning.

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THE BEST WRITING ON MATHEMATICS 2010

Mircea Pitici, Editor
Princeton University Press, 2011. ISBN 978-0691148410. $19.95

Reviewed by Stephen Buckley

Compiling a good anthology is no easy task, but here Mircea Pitici has succeeded in putting together a wonderful and varied bouquet of texts related to mathematics.

The editor says that in putting together this book, he aimed *to make accessible to a wide audience texts originally printed in publications that are often not available outside the scientific community or have limited distribution even inside it.* He also aimed *contribute to the dispersion of thinking on mathematics*, to *illustrate the growing* *presence of mathematical subjects in the mass media*, and to *encourage even more and better writing of a similar sort.* All selected texts were published in 2009, and all are texts about mathematics rather than mathematical texts: in particular, there are no formal proofs and very few mathematical formulae.

A successful anthology requires a clear set of aims and selection criteria such as the above, but it also requires that the editor pores over a large number of candidate texts and chooses wisely. Pitici's considerable efforts have certainly succeeded: the chosen 35 texts are mostly of a very high standard and consistent with the selection criteria. Some are broad surveys of certain areas of mathematics, while others are discussions of mathematical culture, philosophy, or history. Of course it is in the nature of anthologies that the reader will find some selections much more appealing than others.

The book is divided into six sections, although there are no clear delineations between several of these sections. The Section *Mathematicians and the Practice of Mathematics* includes an interesting report by Timothy Gowers and Michael Nielson on massively collaborative mathematics, and the essay *Birds and Frogs* by Freeman Dyson, a written version of Dyson's AMS Einstein Lecture. Here he discusses two types of mathematicians: *birds fly high in the air and survey broad vistas of mathematics*, while *frogs live in the mud...[and] delight in the details of particular objects.* Dyson maintains that Mathematics needs both frogs and birds. The wonderful set of anecdotes about a variety of famous mathematicians, each of whom Dyson classifies as a bird or a frog, is reason enough to recommend this book to all professional mathematicians.

The Section *History and Philosophy of Mathematics* also contains several articles likely to be of considerable interest to the professional mathematician, including a discussion of why Lagrange attempted to prove the Parallel Postulate, and a discussion of Kronecker and constructive mathematics.

In this same section, there is an interesting survey by Philip Bowers on circle packing. He contrasts two branches of circle packing. The first, focusing on the relationship between circle packing and classical complex analysis, is guided by a grand vision given by major conjectures and "revered texts". The second, relating circle packing to the discretization of geometry, gets its impetus from a variety of applications, from minimal surfaces to computer vision, medical imaging, and manufacturing design. This contrast ties in nicely with Dyson's birds and frogs essay.

Other survey texts in this book deal with financial mathematics, models of the Internet, a discussion on how to represent numbers in a computer (including the *level-index system* which curiously is arithmetically closed despite containing only a strictly bounded subset of the real line), and a discussion of certain games of chance. Surveys such as these are likely to be of particular interest to prospective mathematicians.

There are several texts on the nature of truth and proofs in mathematics in the first section of the book. These are particularly suitable for the non-mathematical reader to get a sense of what mathematics is all about, although the survey *An enduring error* by Branko Grünbaum is also likely to be of interest to many mathematicians. This survey examines the various treatments of Archimedean polyhedra and in particular traces a certain error in their enumeration that has been reproduced in many texts.

Other texts of interest include separate articles on the attitude of Einstein and Darwin to mathematics, a report on the Kervaire invariant problem, and newspaper articles on the mathematics of love and on mathematics in the movies (including discussions of zombie movies, a Batman movie, and "Reservoir Dogs").

Overall, I highly recommend this book to everyone with an interest in mathematics, whether they are professional mathematician, graduate or undergraduate students, teachers, or enthusiastic amateurs.

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Problems

IAN SHORT

I was asked the first problem during an interview at a bank in London several years ago.

Problem 67.1. Prove that there does not exist a differentiable function $f : \mathbb{R} \to \mathbb{R}$ that satisfies

$$f'(x) \geqslant 1 + [f(x)]^2$$

for each real number x.

I believe the second problem once appeared in The American Mathematical Monthly. I include it here because it is the only problem from Maynooth's current problem board that remains unsolved.

Problem 67.2. Suppose that $x_1, x_2, \ldots, x_n$, where $n \geq 3$, are non-negative real numbers such that

$$x_1 + x_2 + \cdots + x_n = 2$$

and

$$x_1x_2 + x_2x_3 + \cdots + x_{n-2}x_{n-1} + x_{n-1}x_n = 1.$$

Find the maximum and minimum values of

$$x_1^2 + x_2^2 + \cdots + x_n^2.$$

The last problem is apparently a classic. I first encountered it whilst working for the mathematics outreach project NRICH.

Problem 67.3. There are m gold coins divided unequally between n chests. An enormous queue of people are asked in turn to select a chest. Each member of the queue knows how many coins there are in each chest, and also knows the choice of those ahead in the queue who have selected already. In choosing a chest, each person considers the (possibly non-integer) number of gold coins he would receive were the coins in that chest to be shared equally amongst all those, including him, who have selected that chest so far. He then chooses the chest that maximises this number of coins. For example, if there are three chests A, B, and C containing 3, 5, and 8 coins, then the first person in the queue selects C, the second selects B, the third selects C, the fourth selects A, and so forth.

After the mth person has chosen a chest, how many people have selected each chest? Express your answer in terms of the number of coins per chest. What more can be said about people's chest selections?

This problem is deliberately open ended, to encourage discussion and generalisation.

We invite readers to suggest their own problems, and to offer comments and solutions to past problems. In later issues we will publish solutions and acknowledge problem solvers. Please email submissions to `imsproblems@gmail.com`.

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