CONTENTS

EDITORIAL

This last volume of the Bulletin *that I am editing once again gives a nice overview on the mathematical activities in Ireland in the past year. Finbarr Holland's note on what he was taught in secondary school ties in very well with Bernd Kreussler's report on the 51st IMO and the lack of knowledge of the Irish participants in some areas of Mathematics. Four survey articles on Mathematical Education highlight the need for future reflection and development at both the secondary and tertiary level. Are some parts of mathematical thinking getting lost in our society or is it just a shift towards a new era? The diversity of mathematical research done on this island is illustrated by not less than 11 abstracts of PhD theses submitted in 2010.*

Leaving my job as editor of this Bulletin *now allows me to look back on 10 years of service to the Society which I enjoyed and found very rewarding. To my successor, Tony O'Farrell, I wish* ádh mór ort *and that he will get the same valuable support that I received during my time as editor. While the web page for the* Bulletin *remains the same*

`http://www.maths.tcd.ie/pub/ims/bulletin/index.php`

all future correspondence should be addressed to

`ims.bulletin@gmail.com`

My thanks go to the readers of the Bulletin *and all those who contributed to its success.*

—MM

NOTICES FROM THE SOCIETY

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Vice-President	Dr M. Mathieu	Dept. of Pure Mathematics QUB
Secretary	Dr S. O'Rourke	Dept. of Mathematics Cork Inst. Technology
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Cork	IT	Dr D. Flannery
	UCC	Prof. M. Stynes
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	DIT	Dr C. Hills
	DCU	Dr M. Clancy
	St Patrick's	Dr S. Breen
	TCD	Prof R. Timoney
	UCD	Dr R. Higgs
Dundalk	IT	Mr Seamus Bellew
Galway	UCG	Dr J. Cruickshank
Limerick	MIC	Dr G. Enright
	UL	Mr G. Lessells
Maynooth	NUI	Prof S. Buckley
Tallaght	IT	Dr C. Stack
Tralee	IT	Dr B. Guilfoyle
Waterford	IT	Dr P. Kirwan

Applying for I.M.S. Membership

1. The Irish Mathematical Society has reciprocity agreements with the American Mathematical Society, the Irish Mathematics Teachers Association, the New Zealand Mathematical Society and the Real Sociedad Matemática Española.

2. The current subscription fees are given below:

Institutional member	160 euro
Ordinary member	25 euro
Student member	12.50 euro
I.M.T.A., NZMS or RSME reciprocity member	12.50 euro
AMS reciprocity member	15 US$

   The subscription fees listed above should be paid in euro by means of a cheque drawn on a bank in the Irish Republic, a Eurocheque, or an international money-order.

3. The subscription fee for ordinary membership can also be paid in a currency other than euro using a cheque drawn on a foreign bank according to the following schedule:

   If paid in United States currency then the subscription fee is US$ 30.00.
   If paid in sterling then the subscription is £20.00.
   If paid in any other currency then the subscription fee is the amount in that currency equivalent to US$ 30.00.

   The amounts given in the table above have been set for the current year to allow for bank charges and possible changes in exchange rates.

4. Any member with a bank account in the Irish Republic may pay his or her subscription by a bank standing order using the form supplied by the Society.

5. Any ordinary member who has reached the age of 65 years and has been a fully paid up member for the previous five years may pay at the student membership rate of subscription.

6. Subscriptions normally fall due on 1 February each year.

7. Cheques should be made payable to the Irish Mathematical Society. If a Eurocheque is used then the card number should be written on the back of the cheque.

8. Any application for membership must be presented to the Committee of the I.M.S. before it can be accepted. This Committee meets twice each year.

9. Please send the completed application form with one year's subscription to:

   The Treasurer, I.M.S.
   Department of Mathematics
   St Patrick's College
   Drumcondra
   Dublin 9, Ireland

IRISH MATHEMATICAL SOCIETY

President's Report 2010

In Society news, Richard Timoney will leave the committee at the end of 2010 and I would like to express my gratitude to him for his outstanding service on the committee. Martin Mathieu leaves his post as editor of the Bulletin at the end of 2010—he leaves with the gratitude of the Society for a job well done. From January 2011 Tony O'Farrell will take up the reins at the Bulletin and the Society is fortunate to have such able and committed editors to call on. Our September meeting was held this year in DIT and was a great success. As well as our September meeting, we have continued our programme of conference support and I encourage any prospective organisers of conferences in 2011 to submit applications for funding to our treasurer as soon as possible.

The economic downturn has affected our members in many ways, with hiring freezes and pay cuts being some of the more obvious. In February 2010 we saw another proposed cutback with the announcement by IreL of wide ranging cuts to electronic journal access in seven of our universities—mathematics was particularly affected by the proposed cuts. The IMS, along with the RIA and representatives from the universities was involved in discussions with IreL which will hopefully ensure that we have continued access to these valuable resources. Incidents such as this serve to highlight the fact that the IMS has an important role to play in representing the interests of mathematicians in Ireland.

More recently, mathematics has been in the public spotlight again with the announcement of the so called "bonus points scheme" for higher level leaving certificate mathematics. A lively discussion on this issue took place at our AGM in September. While the IMS has subsequently publicly supported the introduction of a bonus points scheme, many of our members have reservations, especially concerning the details of the proposed scheme. It is clear that there is a crisis in second level mathematical education in Ireland and it is to be hoped that our politicians will provide the resources to back up the introduction of bonus points with a corresponding improvement in the level of second level mathematics teaching.

As I come to the end of my term as president, I would like to thank all the members and especially the committee members for their support and I look forward to meeting everyone in late August in Limerick.

James Cruickshank
Galway, January 19, 2011.

Minutes of the Meeting
of the Irish Mathematical Society
Annual General Meeting
3rd September 2010

The Irish Mathematical Society held its Annual General Meeting from 15:30 to 17:00 on Friday 3 September at Dublin Institute of Technology. There were 16 members present at the meeting.

The meeting began with a minute's silence as a mark of respect for Alexei Pokrovskii, Professor of Applied Mathematics in UCC, who had recently died.

1. Minutes

After correcting a minor typographical error, the minutes of the last AGM were approved and signed.

2. Matters arising

13: Daphne Gilbert informed the meeting that the overseas student whose case was highlighted at the last AGM is not being charged fees by DIT at the non-EU rate as originally feared. The SFI however has not changed its position regarding funding of non-EU students.

Colm Egan is the winner of this year's Fergus Gaines Cup. It was suggested that the cup could be presented at the same time as the Hamilton Prize.

3. Correspondence

The British Library, as a copyright library, has sought copies of the Bulletin. Gordon Lessells informed the meeting that five copies are sent to Trinity, four of which are supposed to be sent to the other copyright libraries, including the British Library.

4. New Members

Ciaran Mac an Bhaird (NUIM); Eabhnat Ni Fhloinn (DCU); Joe Gildea (Sligo IT); Edwin O'Shea (UCC); Stephen O'Sullivan (DCU); Paul Smyth (DIT); Georgi Grahovski (DIT); Fiona Murray (DIT); James Gleeson (UL); Noel White (Wicklow VEC); Rossen Ivanov (DIT); Roman Sedakov (student, UL); Carlos Argaez Garcia (student, DIT); Peter Fennell (student, TCD); James B. Gillespie (student, TCD); Garrett Greene (student, TCD).

A rule change concerning membership of the society was proposed: that members near retirement of at least 10 years' standing be given life membership. However this was generally felt to be rather generous compared to what other national mathematical societies offer, and to have a possibly detrimental effect on the Society's finances, so the proposal was withdrawn.

5. President's Report

Following conversations between the President and Cathal McCauley of IReL most of the resources that were in danger of being cancelled have been restored.

The scheduled EMS meeting in Bucharest was cancelled due to the Icelandic volcanic ash, which would have affected many attendees. R. Higgs also decided against attending the EMS Council in Sofia as he considered it to be too expensive to be worthwhile.

The President stated that we should increase our conference support and expand our activities.

6. Treasurer's Report

In the absence of S. Breen (who sent her apologies) S. Wills presented the Treasurer's report for 2009. It showed a shortfall of €580.46.

He informed the meeting that about half of the Society's members haven't updated their standing orders to reflect the recent increase in membership fees. Moreover there are relatively few new members compared to retired members, which may increasingly pose a challenge to the Society's finances. Finally many members have failed to pay anything for several years. It was accepted that members who have not paid membership fees in over three years would have their membership terminated, as provided for by the constitution.

7. The Bulletin

The Editor, M. Mathieu, informed the meeting that Volume 65 of the Bulletin was ready for distribution, and contained for the first time some colour pictures and even sheet music.

He was thanked for his invaluable long-standing contribution to the Society by editing the Bulletin over the years. Thanks were also extended to his successor, Tony O'Farrell for agreeing to take over as Editor. The President said that the Society was fortunate to have editors of such high calibre.

8. Election to Committee

The following were elected unopposed to the committee:

Committee Member	Proposer	Seconder
S. Wills (President)	G. Lessells	R. Timoney
M. Mathieu (Vice President)	J. Cruickshank	M. Mackey
M. Mackey	M. Mathieu	G. Lessells

C. Stack agreed to serve another two-year term on the Committee. As editor from January 2011, Tony O'Farrell will be invited to committee meetings. The total number of years each existing member will have been on the committee as of 31 December 2010 will be: J. Cruickshank (8), R. Timoney (6), N. Kopteva (5), B. Guilfoyle (5), S. Breen (4), S. O Rourke (4), S. Buckley (3), C. Hills (3), A. Wickstead (3), S. Wills (3), C. Stack (2), M. Mackey (1), R. Quinlan (1).

The following will then have one more year of office: S. Breen, S. O Rourke, N. Kopteva, B. Guilfoyle, S. Buckley, A. Wickstead, R. Quinlan.

9. Membership of ICIAM.

The International Council for Industrial and Applied Mathematics (ICIAM) were in contact with the Society, regarding possible associate membership of the former. The cost is currently $150 per annum. It was suggested that opinions of IMS members be canvassed, possibly via mathdep.

10. Future Meetings

The 2011 September Meeting will take place in the University of Limerick. The 2012 meeting will take place in IT Tallaght. There was some discussion about the timing of the annual meeting, and whether late August, for example, would be more convenient.

11. Issues Related to Second-Level Mathematics Education

There was an extensive discussion on two topics of current interest: Project Maths, and bonus points for higher level Leaving Certificate Mathematics.

Some points made concerning Project Maths (PM):

- A group of IMS Committee members sent a letter to the Project Maths Implementation Support Group (PMISG) with a view to having an augmented role in Project Maths. (S. Buckley)
- PM is not connected with PMISG, but with the National Council for Curriculum and Assessment (NCCA). There is already strong third-level representation on the NCCA informing PM. (S. Dineen)
- PM is intended ultimately to reform mathematics education throughout the entire second-level, not merely the senior cycle, so it is best not to judge this year's PM Leaving Certificate papers in isolation. (S. Dineen)
- PM is an attempt to reform second-level mathematics education which is currently unsatisfactory, not least in the way that students learn by rote and depend on routines to answer questions. (A. O'Farrell)
- There are few materials (such as sample papers) available on the internet about PM. (C. Stack)
- The Society should consider inviting members of the inspectorate to explain their position. (B. Goldsmith)
- The prospect of students being forced to think more under the new regime is likely to frighten students, however preferable it might be to the current system. (R. Timoney)

Some points made concerning bonus points for higher level Mathematics:

- One of the main reasons students don't do higher level mathematics is the time required, estimated to be 50% more than other higher level subjects. (S. Dineen)
- However the level of time and effort required for higher level mathematics is not expected to decrease under PM. Mathematics should receive correspondingly more teaching time, but this is a decision of school principals, and is likely to be determined by points (A. O'Farrell)

- Whether bonus points would be a good thing depends a lot on the detail of any proposal; crucially what multiplier should be applied, e.g. 1.4, 2,. . . ?
- However, bonus points are a prominent issue in the news lately, and the Society should have a clear position on it. (J. Cruickshank)
- Many students are lost from the higher level, because their teachers are not confident about teaching it, and are afraid that the students will fail. (B. Goldsmith)
- We should beware that other organisations are better at dealing with the media.

Following the discussion, the following motion was passed by the meeting, and is now policy of the IMS.

The Irish Mathematical Society supports the introduction of bonus points for higher level Leaving Certificate Mathematics.

12. Any other business

R. Timoney's term on the Committee ends at the end of 2010. Richard was thanked for his service to the Society.

Shane O Rourke
CIT.

These minutes still need to be approved by the next AGM.

PROGRAMME
23rd SEPTEMBER MEETING

Dublin Institute of Technology
2–3 September 2010

Thursday 2 September

9:30–10:45	Registration and coffee
10:40–10:45	**Michael Devereux** *Introduction and welcome*
10:45–11:00	**Sandra Collins (SFI)** *Short address*
11:00–12:00	**Alexander Mikhailov (Univ Leeds)** *Automorphic Lie algebras and corresponding integrable systems*
12:00–12:30	**Michael Melgaard (DIT)** *Resonances in Quantum Chemistry: Complex absorbing potential method for systems*
12:30–13:30	Lunch
13:30–14:30	**Richard Timoney (TCD)** *Computation versus formulae for norms of elementary operators*
14:30–15:00	**Conor Masterson (Renaissance Reinsurance)** *Portfolio optimisation in a volatile market*
15:00–15:30	Coffee
15:30–16:00	**Frederic Dias (UCD)** *Breathers, solitons and freak waves*

16:00–17:00	**Eabhnat Ní Fhloinn (DCU)** *Mathematics Support and other strategies for tackling issues in the third-level mathematics classroom*
17:30–19:00	Social Event
19:30	Conference Dinner

Friday 3 September

9:30–10:30	**Hilary Ockendon (Univ Oxford)** *A Case History in Interdisciplinary Theoretical Mechanics*
10:30–11.00	**Benjamin McKay (UCC)** *On the powers of a nilpotent algebra*
11:00–11:30	Coffee
11:30–11:50	**Sally McClean (Univ Ulster)** *Markov and Semi-Markov reward systems for patient care*
12:00–12:30	**Brendan Goldsmith (DIT)** *On Hopfian and co-Hopfian groups*
12:30–13:30	Lunch
13:30–14:30	**Andrew Fowler (Univ Limerick)** *Drumlins*
14:30–15:00	**Paolo Guasoni (DCU)** *The Incentives of hedge fund fees and high-water marks*
15:00–15:30	Coffee
15:30–16:30	AGM of the IMS

The IMS September Meeting 2010 at DIT
Abstracts of Invited Lectures

Automorphic Lie algebras and corresponding integrable systems

Alexander Mikhailov (University of Leeds)

We study a new class of infinite dimensional Lie algebras, which has important applications to the theory of integrable equations. The construction of these algebras is very similar to the one for automorphic functions and this motivates the name automorphic Lie algebras. It is also a natural generalisation of the construction used by Victor Kac in his study of graded Lie algebras. In contrast to the Kac–Moody algebras, automorphic Lie algebras are quasi-graded and, in a certain sense, are deformations of the Kac–Moody Lie algebras. We discuss the progress in the classification problem for automorphic Lie algebras corresponding to finite groups of automorphisms (finite reduction groups).

Integrable systems related to quasi-graded Lie algebras are nonhomogeneous and are deformations of well known integrable systems in the graded case. The two dimensional generalisation of the Volterra chain is an interesting example of such a system. Its continuous limit is the famous Kadomtsev–Petviashvili equation. Using the dressing method we study exact solutions of the 2-d Volterra system. Apart of soliton-like solutions we have found exact solutions corresponding to wave fronts. Classification of soliton and wave front solutions is related to the Schubert decomposition of a Grassmanian.

Resonances in Quantum Chemistry: Complex Absorbing Potential Method for Systems

Michael Melgaard (DIT)

The Complex Absorbing Potential (CAP) method is widely used to compute resonances in Quantum Chemistry, both for scalar valued and matrix valued Hamiltonians. In the semiclassical limit $\hbar \to 0$ we consider resonances near the real axis and we establish the CAP method rigorously in an abstract matrix valued setting. The proof is based on pseudodifferential operator theory and microlocal analysis.

Computation versus formulae for norms of elementary operators

RICHARD TIMONEY (TCD)

A long-standing problem was to find a formula for the norm of an elementary operator acting on a C*-algebra (or a matrix algebra). As with the best problems, the problem was quick to explain, although there was no conjectured answer. The solution which has been found involves concepts that are almost equally simple to explain, but the effective use of the formula is more subtle. It does lead to insights into the structure of the algebras. The techniques involved in justifying the formula are useful in practical and theoretical ways.

Portfolio optimisation in a volatile market

CONOR MASTERSON (RENAISSANCE REINSURANCE)

The recent financial crash has led to unprecedented volatility in the European Government bond markets, with downward pressure on interest rates, increased government borrowing to fund extensive public intervention in the banking system adding to concerns over the fiscal stability and the future of the Euro system. Core European countries including Germany have seen bond yields at record low levels resulting from the flight to quality, while peripheral countries, led by Greece have seen unprecedented high yields. This situation has posed major new challenges in the managing government bond portfolios, which are generally marked by steady returns and low volatility. This talk addresses the challenges of managing this type of portfolio and discusses quantitative methods used to manage risk and optimize returns in the current market.

Breathers, solitons and freak waves

FRÉDÉRIC DIAS (UCD)

The Peregrine soliton is a localised nonlinear structure whose existence was predicted over 25 years ago but which has not to date been experimentally observed in any physical system. It is of fundamental significance because it is localised in both time and space, and because it defines the limit of a wide class of solutions to the nonlinear Schrödinger equation (NLSE). Here, we use an analytic description of NLSE breather propagation to implement experiments in optical fibre generating femtosecond pulses with strong temporal and spatial localization, and near-ideal temporal Peregrine soliton characteristics.

In showing that Peregrine Soliton characteristics appear with initial conditions that do not correspond to the mathematical ideal, our results may impact widely on studies of hydrodynamic wave instabilities where the Peregrine soliton is considered a freak wave prototype. (This is joint work with B. Kibler, J. Fatome, C. Finot, G. Millot, G. Genty, N. khmediev, J. M. Dudley.)

Mathematics support and other strategies for tackling issues in the third-level mathematics classroom

Eabhnat Ní Fhloinn (DCU)

Increasingly diverse student populations combined with large class sizes lead to serious challenges for many mathematics educators in third-level. As a result, various forms of mathematics support have been introduced in numerous third-level institutes. However, such support systems function most effectively when integrated into the overall mathematics education of the student, and are not a complete solution in themselves to the issues encountered in the third-level mathematics classroom. This talk will focus in particular on the experience of providing mathematics support in Dublin City University, largely through the form of a Maths Learning Centre, before moving on to address possible approaches to addressing challenges in the classroom.

A case history in interdisciplinary theoretical mechanics

Hilary Ockendon (University of Oxford)

The talk will describe a project from the defence industry which started as a student project and has developed into basic research into the modelling of materials under extreme stress. It will illustrate the breadth of knowledge required to attack such problems and will touch on ideas from fluid and solid dynamics, plasticity, Greens functions, inverse problems and hyperbolic PDEs.

The Hartogs phenomenon and holomorphic geometric structures

BENJAMIN MCKAY (UCC)

One can often find an open set in a complex manifold so that every holomorphic function on the open set extends to the manifold. This is called the Hartogs phenomenon. I will describe Hartogs phenomenon for many other complex analytic geometric objects on complex manifolds.

Markov and semi-Markov reward systems for patient care

SALLY MCCLEAN (UU)

Previously, (McClean 1976, 1980) we have developed Markov and semi-Markov models for a multi-grade system with Poisson arrivals. This system consisted of a number of transient states and an absorbing state. Joint distributions for the numbers in each transient state at any time were found in each case and the limiting distributions were shown to be independent Poisson. Such models have been applied to manpower planning (Bartholomew et al., 1991) and healthcare, for movements of patients through a hospital system (Taylor et al., 1998, 2000). Semi-Markov systems with constant size and known growth have also been discussed by a number of authors e.g. Papadopolou and Vassiliou (1990), Vassiliou and Papadopolou (1992), McClean et al., (2004).

More recently, results have been obtained for a reward (cost) model for a two (McClean et al., 1998a) and three (McClean et al., 1998b) transient state system. These models were applied to a hospital in-patient system where the“reward” are costs and the total cost for a group of patients moving through the system, with no new admissions, is known as spend-down. However, it is often required to identify daily and long-term costs of a system where, in addition to movements between the transient states, and departures to the absorbing state(s), there are arrivals to the transient states. These arrivals correspond to recruitment in a manpower system and admissions of new patients for the hospital patient system. A semi-Markov approach provides more generality that can serve to describe the complex semantics of such models; for example Bartholomew et al. (1991) describe the use of semi-Markov models for manpower systems while Kao (1978) has previously used a semi-Markov model to describe patient stays in hospital. Results for a semi-Markov system, in which the total size of the system at any time are known, have been provided by Papadopolou (2001).

In McClean et al. (2004), we have thus developed a reward model for a discrete time homogeneous semi-Markov system with Poisson arrivals and, also, for the case where the system has grown by a known amount at each discrete time point. Results were obtained for the distribution, mean and variance of daily costs of such a system at any time, and in steady state.

For the healthcare application these models can also help us to assess the complex relationship between hospital and community care where there may be possible trade-offs between hospital treatment and community care costs (McClean and Millard, 2006). Stroke disease is particularly suited to our approach as patients that do not receive appropriate therapy or rehabilitation in a timely manner may subsequently buildup huge costs over time. Semi-Markov models can assess where and how patients should be treated. We have thus developed Markov and semi-Markov models to describe the whole integrated system of stroke patient care and facilitate planning of services. Based on data on stroke patient data from the Belfast City Hospital, various scenarios are being explored, such as the potential efficiency gains if length of stay in hospital, prior to discharge to a Private Nursing Home, can be reduced.

On Hopfian and co-Hopfian groups

Brendan Goldsmith (DIT)

The notions of Hopfian and co-Hopfian groups have been studied for a long time. The terminology "Hopfian" seems to have arisen from the fact that the topologist H. Hopf showed that the fundamental groups of certain closed two-dimensional manifolds have the defining property. In modern terminology we say that a group G is Hopfian if every surjection: $G \to G$ is an automorphism; it is said to be co-Hopfian if every injection: $G \hookrightarrow G$ is an automorphism. Finite groups are, of course, the prototypes for both Hopfian and co-Hopfian groups. Hopfian and co-Hopfian groups have arisen recently in the study of algebraic entropy and dual entropy. Despite the seeming simplicity of their definitions, Hopfian and co-Hopfian groups are notoriously difficult to handle and easily stated problems have remained open for a long time: if G is Hopfian, is the direct product $G \times Z$ Hopfian; this is still an open question. Our interest in this paper shall be principally focussed on Abelian Hopfian and co-Hopfian groups. The assumption of commutativity does, of course, make the situation somewhat more tractable—for example, the problem above has a positive answer—but it by no means removes all the difficulties; we still appear to be a long way away from any satisfactory description of these classes of groups. In the case of Abelian p-groups, it is rather easy to bound the size of a group from either class and torsion-free co-Hopfian groups are countable and classifiable by a single finite cardinal invariant. It is easy to see that an Abelian group with endomorphism ring isomorphic to the ring of integers, is necessarily Hopfian and consequently, it follows from modern realization theorems that arbitrarily large torsion-free Hopfian groups exist. But beyond that, little by way of classification is known. It is still not known whether the direct sum of two co-Hopfian groups which are not torsion-free, is co-Hopfian. The talk, which shall be accessible to non-experts, will give an overview of recent developments based on the work of the author and his student, Ketao Gong.

Drumlins

Andrew Fowler (University of Limerick)

Drumlins pervade the Irish landscape, and also many other parts of the world where ice sheets were formerly present. They present an intriguing problem in pattern formation, and although they have been avidly studied for well over one hundred years, it is only recently that anything approaching a mathematical theory for their formation has been put forward. This talk will attempt to describe what drumlins are, what processes are involved in their formation, and how mathematical models are being developed to describe their origin.

The incentives of hedge fund fees and high-water marks

Paolo Guasoni (DCU)

Hedge fund managers receive as performance fees a large fraction of their funds' profits, in addition to regular fees proportional to funds' assets. Performance fees are paid only when a fund exceeds its previous maximum—the high-water mark. The most common scheme, dubbed Two and Twenty, entails performance fees of 20 percent of profits plus regular fees of 2 percent of assets.

We study the risk-shifting incentives created by such fees, solving the portfolio choice problem of a manager with constant relative risk aversion, constant investment opportunities, and a long horizon. The portfolio that maximizes expected utility from future fees is constant, and coincides with a Merton portfolio with effective risk aversion equal to the weighted average of the manager's true risk aversion and the myopic value of one, with the performance fee as the myopic weight. Moreover, the optimal portfolio coincides with that of an investor facing the constraint of a maximum drawdown less than one minus the performance fee, as a fraction of the last recorded maximum.

Since performance fees modify a manager's risk aversion, we investigate their potential as agency tools, solving a Stackelberg equilibrium between an investor and a manager. We find that an equilibrium exists only if both the manager and the investor have very low risk aversion. In all other cases, no equilibrium is consistent with positive performance fees. Joint work with Jan Obloj (Oxford).

Thanks from the Organisers

The organising committee would like to warmly thank all delegates for participating in a hugely successful meeting and to acknowledge the generous support of the Irish Mathematical Society, the Mathematics Applications Consortium for Science and Industry (MACSI) and the College of Sciences and Health, DIT. In particular, the organisers are grateful to all the invited speakers whose contributions ensured an interesting, dynamic and enjoyable conference.

ANNOUNCEMENTS OF MEETINGS AND CONFERENCES

This section contains the announcement of the annual meeting of the IMS and closely related conferences (satellites) as supplied by organisers. The Editor does not take any responsibility for the accuracy of the information provided.

24th Annual Meeting of the IMS

University of Limerick
August 29–30, 2011

The 2011 IMS Meeting is scheduled for Monday 29th and Tuesday 30th August in the University of Limerick. Accommodation will be available in the Castletroy Park Hotel, Kilmurry Lodge Hotel and Travelodge Hotel all within walking distance of the Campus. Further information from eugene.gath@ul.ie, gordon.lessells@ul.ie or natalia.kopteva@ul.ie. A list of intended speakers will shortly be available at the conference website.

Abstracts of PhD Theses at Irish Universities 2010

Supersymmetric Quantum Stochastic Analysis

Clodagh Carroll
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This is an abstract of the PhD thesis *Supersymmetric Quantum Stochastic Analysis* written by Clodagh Carroll under the supervision of Dr. Stephen Wills at the School of Mathematical Sciences, UCC and submitted in June 2010.

$\mathbb{Z}_2$-graded quantum stochastic calculus is reformulated into a basis independent, infinite-dimensional calculus, expanding on the finite dimensional calculus developed by T.M.W. Eyre and R.L. Hudson. We follow J.M. Lindsay's synthesis, in the sense of Hudson and Parthasarathy, making use of the Hitsuda–Skorohod integral and Operator Space Theory to develop this calculus. $\mathbb{Z}_2$-graded integrals are expressed in terms of Bosonic integrals and the fundamental formulae and estimate of $\mathbb{Z}_2$-graded quantum stochastic calculus are derived, followed by their higher order analogues.

Two types of quantum stochastic differential equation are analysed; the first is the Hudson–Parthasarathy equation

$$dX_t = F_t \widehat{X}_t \, d\Xi(t), \qquad X_0 = \Psi \otimes I_{\mathcal{F}}, \tag{1}$$

$$dY_t = \widehat{Y}_t (L \underline{\otimes} I_{\mathcal{E}(\mathcal{S})}) \, d\Xi(t), \qquad Y_0 = \Phi \otimes I_{\mathcal{F}}, \tag{2}$$

with bounded generators F_t, L and bounded operators Ψ, Φ. Existence and uniqueness results for (1) and (2) are proved, with suitable local uniform boundedness and regularity conditions imposed. Necessary and sufficient conditions for isometric, co-isometric and unitary solutions are derived for each differential equation and the relationship between solutions of (1) and (2) is examined.

Existence and uniqueness of solutions of the quantum stochastic differential equation

$$dk_t = k_t \circ \phi \, d\Xi(t), \qquad k_0 = \kappa \otimes I_{B(\mathcal{F})} \tag{3}$$

is likewise demonstrated, once more, under appropriate regularity conditions. In addition, the question of which generators ϕ yield *-homomorphic solutions is addressed, where ϕ and κ are completely bounded maps. We demonstrate how a solution of (3) can be used to realise solutions of either of the differential equations (1) or (2) and vice versa.

For both types of quantum stochastic differential equation, we determine the necessary and sufficient criteria for homogeneous solutions and finally combine our results.

A Filtering Laplace Transform Integration Scheme For Numerical Weather Prediction

COLM CLANCY
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This is an abstract of the PhD thesis *A Filtering Laplace Transform Integration Scheme For Numerical Weather Prediction* written by Colm Clancy under the supervision of Prof. Peter Lynch at the School of Mathematical Sciences, University College Dublin and submitted in November 2010.

A filtering time integration scheme is developed and tested for use in atmospheric models. The method uses a modified inversion of the Laplace transform (LT) and is designed to eliminate spurious high frequency components while faithfully simulating low frequency modes. The method is examined both analytically and numerically.

For the numerical studies, two atmospheric models are developed, based on the shallow water equations. The first uses an Eulerian form of the governing equations and is based on the reference Spectral Transform Shallow Water Model (STSWM). The second uses a semi-Lagrangian trajectory approach. The LT method is implemented in both models. The models are tested against reference semi-implicit models using standard test cases and perform competitively in terms of accuracy and efficiency. Like semi-implicit schemes, the LT method has attractive stability properties. In particular, the semi-Lagrangian LT discretisation permits simulations with high timesteps, exceeding the CFL cutoff of Eulerian models.

There are a number of additional benefits. The LT scheme is proven to simulate accurately the phase speed of gravity waves. This is in contrast to semi-implicit methods, which maintain stability by slowing down fast-moving waves. This improved representation is shown both analytically and numerically in the case of dynamically significant Kelvin waves.

In addition, the semi-Lagrangian LT method has advantages for the treatment of orography. Semi-Lagrangian semi-implicit discretisations have been shown to generate a spurious resonance where there is flow over a mountain at high Courant number. It is demonstrated, with both a linear analysis and numerical simulations with the fully nonlinear shallow water equations, that the LT discretisation does not suffer from this problem.

Large Fluctuations of Stochastic Differential Equations with Regime Switching: Applications to Simulation and Finance

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This is an abstract of the PhD thesis *Large Fluctuations of Stochastic Differential Equations with Regime Switching: Applications to Simulation and Finance* written by Terry Lynch under the supervision of John Appleby at the School of Mathematical Sciences in Dublin City University and submitted in September 2010.

This thesis [1] deals with the asymptotic behaviour of various classes of stochastic differential equations (SDEs) and their discretisations. More specifically, it concerns the largest fluctuations of such equations by considering the rate of growth of the almost sure running maxima of the solutions. The first chapter gives a brief overview of the main ideas and motivations for this thesis. Chapter 2 examines a class of nonlinear finite–dimensional SDEs which have mean–reverting drift terms and bounded noise intensity or, by extension, unbounded noise intensity. Equations subject to Markovian switching are also studied, allowing the drift and diffusion coefficients to switch randomly according to a Markov jump process. The assumptions are motivated by the large fluctuations experienced by financial markets which are subjected to random regime shifts. We determine sharp upper and lower bounds on the rate of growth of the large fluctuations of the process by means of stochastic comparison methods and time change techniques. Chapter 3 applies similar techniques to a variant of the classical Geometric Brownian Motion (GBM) market model which is subject to random regime shifts. We prove that the model exhibits the same longrun growth properties and deviations from the trend rate of growth as conventional GBM. The fourth chapter examines the consistency of the asymptotic behaviour of a discretisation of the model detailed in Chapter 3. More specifically, it is shown that the discrete approximation to the stock price grows exponentially and that the large fluctuations from this exponential growth trend are governed by a Law of the Iterated Logarithm. The results about the asymptotic behaviour of discretised SDEs found in Chapter 4, rely on the use of an exponential martingale inequality (EMI). Chapter 5 considers a discrete version of the EMI driven by independent Gaussian sequences. Some extensions, applications and ramifications of the results are detailed. The final chapter uses the EMI developed in Chapter 5 to analyse the asymptotic behaviour of discretised SDEs. Two different methods of discretisation are considered: a standard Euler–Maruyama method and an implicit split-step variant of Euler–Maruyama.

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p-adic Hypergeometric Series and Supercongruences

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This is an abstract of the PhD thesis *p-adic Hypergeometric Series and Supercongruences* written by Dermot McCarthy under the supervision of Dr. Robert Osburn at the School of Mathematical Sciences, University College Dublin and submitted in June 2010.

In examining the relationship between the number of points over $\mathbb{F}_p$ on certain Calabi-Yau manifolds and hypergeometric series which correspond to a particular period of the manifold, Rodriguez-Villegas identified 22 possible supercongruences. In this thesis, we extend Greene's hypergeometric series over finite fields in the p-adic setting. We prove various congruences between this new p-adic hypergeometric series and truncated generalised hypergeometric series. These congruences provide a framework for proving all 22 supercongruence conjectures of Rodriguez-Villegas. Using this framework we prove one of the outstanding supercongruence conjectures between a special value of a truncated generalised hypergeometric series and the p-th Fourier coefficient of a modular form. In the course of this work we also establish a relationship between this new series and the p-th Fourier coefficient of the modular form in question, and two new binomial coefficient–harmonic sum identities.

K-Theory of Azumaya algebras

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This is an abstract of the PhD thesis *K-Theory of Azumaya algebras* written by Judith Millar under the supervision of Dr. Roozbeh Hazrat at the Department of Pure Mathematics, Queen's University Belfast and submitted in September 2010.

Green et al. [1] proved that for a division algebra finite dimensional over its centre, its K-theory is "essentially the same" as the K-theory of its centre; that is, for a division algebra D over its centre F of index n,

$$K_i(D) \otimes \mathbb{Z}[1/n] \cong K_i(F) \otimes \mathbb{Z}[1/n].$$

By the Artin-Wedderburn Theorem, a central simple algebra over a field is isomorphic to a matrix over a division algebra. Central simple algebras over fields are generalised by Azumaya algebras over commutative rings. An Azumaya algebra A over a commutative ring R can be defined as an R-algebra A such that A is finitely generated as an R-module and $A/\mathrm{m}A$ is a central simple R/m-algebra for all $\mathrm{m} \in \mathrm{Max}(R)$.

For an Azumaya algebra A which is free over its centre R of rank n, we prove that the K-theory of A is isomorphic to the K-theory of R up to its rank torsion; that is,

$$K_i(A) \otimes \mathbb{Z}[1/n] \cong K_i(R) \otimes \mathbb{Z}[1/n]$$

for any $i \geq 0$. This result appeared in [2, Thm. 6]. We observe that a graded central simple algebra, graded by an abelian group, is a graded Azumaya algebra and it is free over its centre. So the above result, from the non-graded setting, covers graded central simple algebras. For a graded central simple algebra A, we can also consider graded projective modules. Let $\mathcal{P}\mathrm{gr}(R)$ be the category of graded finitely generated projective R-modules and K_i, $i \geq 0$, be the Quillen K-groups. Then $K_i^{\mathrm{gr}}(R)$ is defined to be $K_i(\mathcal{P}\mathrm{gr}(R))$. We give some examples to show that the graded K-theory of A does not necessarily coincide with its usual K-theory. For a graded Azumaya algebra A, free over its centre R and subject to some conditions, we show that $K_i^{\mathrm{gr}}(A)$ is "very close" to $K_i^{\mathrm{gr}}(R)$. These results were published in [3].

We consider additive commutators in the setting of graded division algebras. For a graded division algebra D with a totally ordered abelian grade group, we show how the submodule generated by the additive commutators in QD relates to that of D, where QD is the quotient division ring.

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The Information-carrying Capacity of Certain Quantum Channels

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This is an abstract of the PhD thesis *The information-carrying capacity of certain quantum channels* written by Ciara Morgan under the supervision of Prof. Tony Dorlas at the School of Theoretical Physics, Dublin Institute for Advanced Studies and Prof. Joe Pulé at the School of Mathematical Sciences, University College Dublin and submitted in January 2010.

In the thesis we consider the classical capacity of certain quantum channels, that is, the maximum rate at which classical information, encoded as quantum states, can be transmitted reliably over a quantum channel.

We first concentrate on the *product-state* capacity of a particular quantum channel, that is, the capacity which is achieved by encoding the output states from a source into codewords comprising of states taken from ensembles of non-entangled (i.e. separable) states and sending them over copies of the quantum channel. Using the “single-letter” formula proved by Holevo [1] and Schumacher and Westmoreland [2] we obtain the product-state capacity of the qubit quantum amplitude-damping channel, which is determined by a transcendental equation in a single real variable and can be solved numerically. We demonstrate that the product-state capacity of this channel can be achieved using a minimal ensemble of *non-orthogonal* pure states. We also consider the *generalised* amplitude-damping channel and show that the technique used to calculate the product-state capacity for the “traditional” amplitude damping channel also holds for this channel.

Next we consider the *classical capacity* of two quantum channels with memory namely, a periodic channel with quantum depolarising channel branches and a convex combination of quantum channels. The classical capacity is defined as the limit of the capacity of a channel, using a block of states which are permitted to be entangled over n channel uses and divided by n, as n tends to infinity.

We prove that the classical capacity for each of the classical memory channels mentioned above is, in fact, equal to the respective product-state capacities. For those channels this means that the classical capacity is achieved without the use of entangled input-states. We also demonstrate that the method used in the proof of the classical capacity of a periodic channel with depolarising channels does not hold for a periodic channel with *amplitude-damping* channel branches. This is due to the fact that, unlike the depolarising channel, the maximising ensemble for a qubit amplitude-damping channel is not the same for all amplitude-damping channels.

We also investigate the product-state capacity of a convex combination of two memoryless channels, which was shown in [3] to be given by the supremum of the minimum of the corresponding Holevo quantities, and we show in particular that the product-state capacity of a convex combination of a depolarising and an amplitude-damping channel, is not equal to the minimum of their product-state capacities.

Next we introduce the channel coding theorem for memoryless quantum channels, providing a known proof [4] for the strong converse of the theorem. We then consider the strong converse to the channel coding theorem for a periodic quantum channel.

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Yang-Mills Instantons on the Taub-NUT Space and Supersymmetric $N = 2$ Gauge Theories with Impurities

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This is an abstract of the PhD thesis *Yang-Mills Instantons on the Taub-NUT Space and Supersymmetric* $N = 2$ *Gauge Theories with Impurities* written by Clare O'Hara under the supervision of Dr. Sergey Cherkis at the School of Mathematics, Trinity College Dublin and submitted in August 2010.

We write a formula for arbitrary charge calorons, instantons on $\mathbb{R}^3 \times S^1$, in terms of the Green's function of the Laplacian defined for the Nahm Transform, thus generalising the formula for the charge one caloron derived by Kraan and van Baal in [1]. The Laplacian is constructed from Nahm data. The usual approach to the Nahm Transform involves an integration over the interval on which the Nahm data are defined. By using Green's functions we avoid this integration and our formula is straightforward to use.

Using the same approach, we derive a formula for an $SU(2)$ instanton on the Taub-NUT space. Here, the Laplacian is constructed from Bow data that solve the Nahm Equations in the interior of the interval. The Bow data includes bifundamental data at the end-points of the interval.

We write the Lagrangian for the low-energy effective field theory on the D3-brane in a Chalmers-Hanany-Witten configuration of intersecting D3-, D5- and NS5-branes [2] [3], by adding bifundamental fields to the Lagrangian written in [4]. The low-energy theory on the D3-branes is described by $N = 2$ Super-Yang-Mills gauge theory with codimension one defects. The supersymmetric vacuum conditions for the gauge theory give the Bow data for an instanton on the Taub-NUT space.

We write an explicit expression for a charge one $SU(2)$ instanton on the Taub-NUT space, in terms of the Green's function values at jumping points and end-points of the Nahm interval. For the charge one instanton we find the Green's function explicitly.

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Convolutional Codes from Group Rings

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This is an abstract of the PhD thesis *Convolutional Codes from Group Rings* written by Jessica OShaughnessy under the supervision of Prof. Ted Hurley at the School of Mathematics, Statistics, and Applied Mathematics, National University of Ireland, Galway and submitted in April 2010.

Convolutional codes have been widely researched and have many practical applications. Convolutional codes from group rings have also been considered.

First, this thesis extends known group ring constructions and applies them to an interesting class of convolutional codes known as 'Quick Look In' (QLI) convolutional codes. QLI convolutional codes have previously been extended to turbo codes. QLI convolutional codes were proposed as an alternative to systematic convolutional codes. It is shown that all QLI convolutional codes can be constructed using units in the group ring $\mathbb{Z}_2C_2C_\infty$.

Some new QLI convolutional codes are also given. Simulations are provided comparing the new codes alongside existing QLI convolutional codes and existing optimal (2.1) convolutional codes. Next, group ring constructions are extended to a new group ring construction. This construction uses units in the group ring $\mathbb{Z}_2GC_\infty$ and the group ring matrices corresponding to these units. For this construction, the order of the group G must be even. Properties of the new construction allow for calculations of lower bounds on the free distances of the convolutional codes for some types of generators. The new construction can also be used in the construction of some optimal $(2, 1)$ convolutional codes up to degree 10. It is shown that all $(2, 1)$ systematic convolutional codes can be constructed using this new construction. Finally, LDPC convolutional codes using the proposed construction are considered.

Harmonic Functions on Quadrature Domains and Denjoy-type Domains

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This is an abstract of the PhD thesis *Harmonic functions on quadrature domains and Denjoy-type domains* written by Joanna Pres under the supervision of Prof. Stephen Gardiner at the School of Mathematical Sciences, University College Dublin, and submitted in February 2010.

The main purpose of this thesis is to investigate how the geometry of certain domains affects the boundary behaviour of positive harmonic functions. It also deals with related problems of integrability and approximation of positive harmonic functions.

Chapter 1 concerns problems related to quadrature domains. Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ and let μ be a signed measure with compact support in Ω. We say that Ω is *a quadrature domain for harmonic functions with respect to* μ, if

$$\int_\Omega h(x)dx = \int h d\mu \quad \text{for all integrable harmonic functions } h \text{ on } \Omega.$$

It was shown by Gustafsson, Sakai and Shapiro $(N = 2)$, and Gardiner and Sjödin $(N \geq 2)$, that all positive harmonic functions are integrable on a quadrature domain. This leads to the significant conclusion that quadrature domains with respect to signed measures are, in fact, quadrature domains with respect to positive measures. In Chapter 1 we establish a corresponding result in the case of quadrature domains with weight. Counterexamples to a certain density question that arose in the above work of Gustafsson et al., are constructed in the second chapter of the thesis, where domains of Denjoy-type are investigated.

A domain Ω in $\mathbb{R}^N$ whose complement $\mathbb{R}^N \setminus \Omega$ is contained in the hyperplane $\mathbb{R}^{N-1} \times \{0\}$ is called *a Denjoy domain.* Benedicks established a harmonic measure condition that describes when Ω inherits the potential theoretic character of the half-space $\mathbb{R}^{N-1} \times (0, +\infty)$, in the sense that there is a minimal harmonic function u on Ω such that $u(x) \geq x_N$. (A positive harmonic function u on Ω is called *minimal* if any harmonic function v on Ω satisfying $0 \leq v \leq u$ is a constant multiple of u.) In Chapter 2 we investigate positive harmonic functions on a domain Ω whose complement is contained in the boundary of the infinite cylinder $U = B' \times \mathbb{R}$, where B' is the unit ball in $\mathbb{R}^{N-1}$. Let α denote the square root of the first eigenvalue of $-\Delta = \sum_{k=1}^{N-1} \partial^2/\partial x_k^2$ on B' and let ϕ be the corresponding eigenfunction normalized by $\phi(0) = 1$. We establish explicit criteria of Benedicks and Wiener-type that describe when Ω inherits the potential theoretic character of the cylinder U, in the sense that there exists a minimal harmonic function u on Ω such that $u(x', x_N) \geq e^{\alpha x_N}\phi(x')$ on U. We also provide illustrative examples and two applications. The first application is a quantitative version of a recent construction of Gardiner and Hansen concerning minimal harmonic functions associated with an irregular boundary point. The second application gives an answer to the approximation question that arose in the study of quadrature domains, by showing that there are bounded domains in $\mathbb{R}^N$ $(N \geq 2)$ for which the positive L^p-integrable harmonic functions are not dense among all positive harmonic functions for any $p \in (0,1]$. The results of this chapter were published in *Positive harmonic functions that vanish on a subset of a cylindrical surface* [written jointly with Marius Ghergu, Potential Analysis 31, 147–181 (2009)].

In Chapter 3 we further investigate the boundary behaviour of positive harmonic functions in relation to the classical angular derivative problem. Suppose (a_n) is a strictly increasing sequence of non-negative numbers such that $a_n \to +\infty$ and $a_{n+1} - a_n \to 0$ as $n \to \infty$. We consider a domain Ω in $\mathbb{R}^N$ such that

$$\mathbb{R}^N \setminus \Omega = \bigcup_{n \in \mathbb{N}} (\mathbb{R}^{N-1} \setminus B') \times \{a_n\}.$$

In two dimensions domains of such a form are known as *comb domains*. We reformulate the angular derivative problem for comb domains in terms of harmonic functions. Although the concept of an angular derivative cannot be defined in higher dimensions in the classical sense, we show that it is possible to investigate its counterpart for harmonic functions. We thus characterize comb-like domains of the above form that admit a minimal harmonic function u such that $u(x', x_N) \geq e^{\alpha x_N} \phi(x')$ on U, in terms of the -spacing between the hyperplanes $\mathbb{R}^{N-1} \times \{a_n\}$.

High Frequency Elastic Wave Inversion

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This is an abstract of the PhD thesis *High Frequency Elastic Wave Inversion* written by Niall Ryan under the supervision of Dr. Clifford Nolan at the Department of Mathematics and Statistics, University of Limerick and submitted in November 2010.

High frequency elastic wave inversion is the problem of determining sharp, localised changes in the properties of materials beneath the surface of the earth using only measurements of reflected seismic waves taken at or near the surface.

The central objective of this thesis is to construct multiparameter inversion operators which map data from surface wave measurements into accurate estimates for high frequency perturbations in the elastic parameters of underground anisotropic inclusions; namely the density, ρ, and the 21 independent Hooke's tensor components,

c_{ijkl}. Using results from the field of microlocal analysis of Fourier Integral Operators, it is shown that asymptotically valid inversion operators exist which can invert all 22 independent elastic parameter perturbations directly, without relying on statistical estimates.

Building on work by Burridge and others in [1], and by Nolan in [2] and [3], the technique of using ensembles of linked seismic experiments is introduced and analysed in the context of a standard linearised single scattering model for elastic waves based on the Born approximation. This forward model gives an asymptotically valid representation for the data, $\mathbf{d}$, gathered by the ensembles, as a function of the 22 the elastic parameter perturbations, $\mathbf{c}^1$, being sought.

It is shown that the data gathered in ensembles of experiments is given asymptotically by the equations

$$\mathbf{d} = AW^T\mathbf{c}^1, \quad \text{and} \quad \mathbf{d} = \int e^{i\omega\Phi} AW^T\mathbf{c}^1 d\mathbf{x} đ\omega$$

in the cases of point and volume inclusions respectively. To show that the data can be inverted in principle, a theoretical framework is introduced which shows under what circumstances multiparameter inversion can be achieved; specifically showing which types of seismic ensemble setups and elastic wave modes permit non-singular W. These results will also show under which circumstances multiparameter inversion is not possible, in particular for the case of volume inclusions. To complement these theoretical results, an application of evolutionary algorithms is presented which is used to find practical seismic ensembles that allow inversion to be carried out feasibly.

Following this analysis, asymptotic inversion operators are constructed for invertible ensembles, and are shown to take the forms

$$\mathbf{c}^1 \cong B^+\mathbf{d}, \quad \text{and} \quad \mathbf{c}^1 \cong \int B^+ e^{-i\omega\Phi}\mathbf{d}\ d\mathbf{r}_0 đ\omega dt$$

respectively in the cases of point and volume anisotropic inclusions which are embedded in isotropic elastic backgrounds. It is also shown that these results can be extended to inversion in multiple types of elastic background materials.

The thesis also presents an introductory overview of the techniques of Fourier Integral Operators [4] and microlocal analysis, used to construct later inversion operators. Other more elaborate mathematical techniques used in the thesis are also introduced or expanded on in the appendices for the benefit of the general reader.

Finally, supporting lemmas in the appendices introduce a novel method for determining the dependencies of Hooke's tensor components in linear elastic materials with symmetries.

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Numerical Modelling of Industrial Processes Exhibiting Layer Phenomena

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This is an abstract of the PhD thesis *Numerical Modelling of Industrial Processes Exhibiting Layer Phenomena* written by Martin Viscor under the supervision of Prof. Martin Stynes at the School of Mathematical Sciences, University College Cork and submitted in September 2010.

Part I of this thesis examines the singularly perturbed degenerate parabolic problem

$$Lu(x,t) := \varepsilon u_{xx}(x,t) - x^{\alpha} u_t(x,t) = f(x,t,\varepsilon), \tag{1}$$

for $(x,t) \in (0,1) \times (0,T]$, with Dirichlet initial and boundary conditions. Here $\varepsilon \in (0,1]$ is a small parameter and $\alpha > 0$ is a positive constant. The parabolic differential operator L degenerates at the boundary $x = 0$ of the domain and consequently its properties are not encompassed by the standard theory of parabolic partial differential equations.

Existence of the solution u of (1) is established for $0 < \alpha < 4$ and bounds on the solution and its derivatives are derived. These clarify the interaction between the singularly perturbed nature of the problem ($0 < \varepsilon \ll 1$) and its degenerate character (vanishing of the coefficient x^α along the boundary $x = 0$). Furthermore, a finite difference method to approximate u is developed. Using the bounds on the derivatives of u, convergence results are proved for this numerical method; these depend on the parameter α but not on ε. Finally, several numerical examples are provided to support the theory.

Part II of the thesis is devoted to the spray drying of a single spherical particle. A model is developed for this process, then attention is focussed on tracking the moving interface that separates the wet core of the particle from its dry crust during the drying process. The new aspect of our model, compared with existing models in the research literature, is that both temperature and moisture content are allowed to vary inside the particle. This results in a Stefan-type problem describing the location of the moving interface, but this problem is of a non-standard mathematical type that is difficult to solve numerically. A new numerical method is developed for its solution and numerical examples are provided to illustrate its accuracy.

Mathematical Thinking and Task Design

SINÉAD BREEN AND ANN O'SHEA

Abstract. Mathematical thinking is difficult to define precisely but most authors agree that the following are important aspects of it: conjecturing, reasoning and proving, abstraction, generalization and specialization. However, recent studies have shown that many sets of mathematical tasks produced emphasize lower level skills, such as memorization and the routine application of algorithms or procedures. In this paper we survey the literature on the design and use of tasks that aim to encourage higher level aspects of mathematical thinking in learners of mathematics at all levels. The frameworks presented here aim to guide task designers when writing a set of exercises.

1. Introduction

There are many different definitions and interpretations of the term *mathematical thinking*. We all have an intuitive feel for what this term means and whether we have formulated a clear definition of it or not, we probably aim to promote mathematical thinking in our students. In this article, we will look at different definitions of the term and at some suggestions from the literature as to how this type of thinking might be fostered in students, through the use of mathematical tasks. In this paper, the term *task* will refer to both homework problems and classroom activities where the student is asked to work on an exercise on their own or in a group. To this end, we will discuss features that have been suggested as being desirable in mathematical tasks and survey various frameworks proposed to aid the design of tasks, reporting briefly on some studies where these frameworks have been used. We will also consider the findings of a number of authors on the types of tasks that have been assigned to students taking mathematics courses at upper second-level and early undergraduate level. Finally, we pass on some advice garnered from the literature in relation to factors that should be attended to when implementing mathematical tasks in order to protect their integrity.

2. Mathematical Thinking

Let us start with the notion of *mathematical proficiency* as used by the Mathematics Learning Study Committee of the US National Research Council [6]. For them, mathematical proficiency is what allows people to learn mathematics successfully and they believe that this has five strands: conceptual understanding; procedural fluency; strategic competence (the ability to formulate and solve mathematical problems); adaptive reasoning (capacity for logical thought, reflections and justification); productive disposition (seeing mathematics as worthwhile and being confident in one's own abilities), (p.116). The authors claim that these five strands are interwoven and that all five should be encouraged and developed together. The focus in [6] is on school mathematics. At third level, the notion of *advanced mathematical thinking* is often considered. There is some debate as to whether this term means thinking about advanced mathematics or thinking about any mathematics in an advanced way. David Tall [19] claims that the distinguishing features of advanced mathematical thinking are abstraction, and the insistence on proof rather than justification.

Many authors agree that the mathematical practices and thinking to be encouraged in learners of mathematics should mirror the practices of professional mathematicians. For instance, Hyman Bass [1] speaks about the mathematical practices or habits of mind of research mathematicians and argues that these practices such as experimentation, reasoning, generalization, the use of definitions and the use of mathematical language can be fostered at any stage in the education system. Mason and Johnston-Wilder [9] provide a detailed list of words they believe denote processes and actions that mathematicians employ when they pose and tackle mathematical problems: "exemplifying, specializing, completing, deleting, correcting, comparing, sorting, organizing, changing, varying, reversing, altering, generalizing, conjecturing, explaining, justifying, verifying, convincing, refuting" (p.109). They propose that questions posed to students should draw on these words to enable students to experience aspects of mathematical thinking. Acquiring a mathematical disposition is how Henningsen and Stein [5] speak of students' learning of mathematics—such a disposition being characterized by activities such as "exploring patterns to understand mathematical structures and underlying relationships; using available resources effectively and appropriately to formulate and solve problems; making sense of mathematical ideas, thinking and reasoning in flexible ways" (p.525), in addition to those mentioned above.

However, in popular culture, it seems that mathematics is often associated with certainty—Lampert [8] describes how school experience can shape the cultural belief that "*doing* mathematics means following the rules laid down by the teacher; *knowing* mathematics means remembering and applying the correct rule when the teacher asks a question; and mathematical *truth is determined* when the answer is ratified by the teacher" (p.32). She points out that the activity of doing mathematics is different to what is recorded once it is done and she has worked towards bringing the practice of doing mathematics in school closer to what it means to do mathematics within the discipline itself. This, she believes, involves students at all levels of schooling engaging in activities similar to those in which mathematicians engage (in agreement with [1], [5], [9]).

Bell [2] asserts that most uses of mathematics involve a cycle of mathematization (that is, recognizing the relevance of some mathematical relationship in a given situation and expressing it symbolically), manipulation and interpretation. He laments the fact that traditional mathematics instruction has assumed that the part of this process that needs most teaching is the phase of manipulation and so traditional school lessons have consisted of demonstration of a single technique followed by practice with a variety of numbers. He too believes that the main lesson experience should instead be one of genuine and substantial mathematical activities bringing into play the sort of strategies mentioned above (abstracting, formulating questions etc.).

3. Types of Tasks Observed

In [14] Sangwin takes a different approach. He maintains that assessment drives what and how mathematics is learned and so "any attempt to elaborate on what is meant by mathematical skills must be based on an analysis of what *in reality* we ask students to do" (p.814). In collaboration with Pointon, he developed a mathematical question taxonomy in order to undertake a classification of course-work questions. Table 1 illustrates this taxonomy.

1. Factual recall
2. Carry out a routine calculation or algorithm
3. Classify some mathematical object
4. Interpret situation or answer
5. Prove, show, justify - (general argument)
6. Extend a concept
7. Construct an instance
8. Criticize a fallacy

TABLE 1. Mathematical question taxonomy of Pointon and Sangwin [11, 14]

Successful completion of tasks following 1-4 of Table 1 are characteristic of 'adoptive learning' in which students engage in an essentially reproductive process requiring the application of well-understood knowledge in bounded situations [14] — that is, the students behave as 'competent practitioners'. While questions in classes 4-8 of Table 2 typically require higher cognitive processes such as creativity, reflection, criticism, and would be characterized as 'adaptive learning', requiring students to behave as 'experts'.

This taxonomy was used to classify a total of 486 course-work and examination questions used on two first year undergraduate mathematics courses, leading to the finding that "(i) the vast majority of current work may be successfully completed by routine procedures or minor adaption of results learned verbatim and (ii) the vast majority of questions asked may be successfully completed without the use of higher skills" ([14] p.8). In fact, further details given in [11] show that 61.4% of all questions inspected related to class 2 of Table 1 while only 3.4% of questions related to classes 6-8.

Others have also undertaken work on investigating the types of tasks that are assigned to students as homework or that appear on examinations. In [4], Boesen, Lithner and Palm considered tasks from Swedish national second level high stakes examinations and classified them according to how familiar they were to students. They used textbooks to decide if the tasks are familiar or not. They also characterized the types of reasoning that students might use to solve problems: imitative reasoning (using memorization or well-rehearsed procedures); creative mathematically founded reasoning (novel reasoning with arguments to back it up and anchored in appropriate mathematical foundations). Not surprisingly, they found that often no conceptual understanding was needed to solve familiar tasks. Boesen et al claim that exposure to familiar tasks alone affects students' ability to reason and so influences student learning. Bergqvist [3] analyzed 16 examinations from introductory courses in Calculus in four Swedish universities. She found that 70% of the exam questions could be solved using imitative reasoning alone and that 15 of the 16 examinations could be passed without using creative reasoning.

4. Purposes of Tasks and Frameworks for Task Design

Polya, in his preface to *How to Solve It* [12], states

'Thus, a teacher of mathematics has a great opportunity. If he fills his allotted time with drilling his students in routine operations he kills their interest, hampers their intellectual development, and misuses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate to their knowledge, and helps them to solve their problems with stimulating questions, he may give them a taste for, and some means of, independent thinking.' (p.v).

How might mathematical tasks be designed in order to make best use of the opportunity Polya describes? Mason and Johnston-Wilder [9] define the purpose of a mathematical task as being "to initiate mathematically fruitful activity" (p.25) which entails harnessing learners' innate abilities to stress and ignore, specialize and generalize, distinguish and connect, imagine and express, conjecture and convince, organize and characterize. In their opinion, tasks should involve a range of possibilities and offer students opportunities to discuss ideas and to make choices, in order for students to view mathematics as a constructive enterprise. In addition, tasks should be chosen to enable learners to encounter significant mathematical ideas and themes, and should be appropriately challenging without being over-taxing. An individual task may have one of a number of intentions — according to Mason and Johnston-Wilder these are providing a context for practising ideas met previously, providing a context to encounter new ideas, acting as revision or consolidation, prompting reflection, prompting the connection and integration of various ideas.

Stein et al [16] also emphasize the importance of the type of mathematical tasks presented to students as these highly influence the kinds of thinking processes in which they engage, their level of engagement, and, thus, the learning outcomes achieved. Tasks with which students engage determine not only what substance they learn but, more importantly, how they come to think about, develop, use and make sense of mathematics. As a result, they should be exposed to meaningful and worthwhile mathematical tasks. Truly problematic tasks should require students to impose meaning and structure, to make decisions about what to do and how to do it, and to interpret the reasonableness of their actions and solutions. They contend that such tasks are characterized by features such as: having more than one solution strategy; being capable of being represented in multiple ways; demanding that students communicate and justify their processes and understandings in written and/or oral form.

Writing in 1993, Krainer [7] described the move away from traditional methods of teaching to one in which students are more active and are given the opportunity to create their own knowledge. He says '...learners should be seen not only as consumers but also producers of knowledge. The teacher's task is to organize an active confrontation of the pupils with mathematics. Powerful tasks are important points of contact between the actions of the teacher and those of the learner' (p.68). He describes some properties of powerful tasks: tasks should have connections with other tasks and areas of mathematics; tasks should generate other interesting questions; tasks should involve actions that promote concept formation; tasks should be structured so that acting and reflecting are closely linked. In the remainder of his paper Krainer [7] reports on a set of 69 tasks written by him to encourage students to think about the concept of angle. He describes 5 of the tasks in detail, giving learning objectives for each one and explaining the connections with other tasks.

Bell [2] also describes some desirable properties or features of mathematical tasks. Tasks should be connected so that knowledge is more easily retained; an element of feedback should be incorporated (e.g. self-verification) for immediate detection of misconceptions; a reflection and review phase should be built-in to provide for the expression and sharing of different understandings and help place new knowledge within the broader field of mathematics.

More recently, Swan [18] has created a framework of 5 task types that he believes foster conceptual understanding at second level. They are classifying mathematical objects (asking students to devise or apply a classification); interpreting multiple representations (drawing links and developing mental images for concepts); evaluating mathematical statements (asking students whether statements are always, sometimes or never true, and developing proofs); creating problems (asking students to create problems for the class); analyzing reasoning and solutions (diagnosing errors and comparing solutions). The tasks described are designed as classroom activities. He asserts that teaching is more effective when rich tasks are used. He says 'The tasks we use should be accessible, extendable, encourage decision-making, promote discussion, encourage creativity, encourage 'what if' and 'what if not?' questions' (p.8). Along with the use of tasks, he contends that teaching for conceptual understanding is more effective when it builds on the students' previous knowledge, confronts difficulties rather than avoiding them and exposes common misconceptions. He also believes that students should be encouraged to talk and write about mathematical ideas, that teachers should emphasize reasoning and not 'answer-getting'. This paper also describes the design of one task in detail.

Schoenfeld [15] created a framework for balanced assessment in an NSF-funded project. His framework introduces seven dimensions under which tasks could be measured: content (including procedure and technique, representations and connections); thinking processes; student products; mathematical point of view; diversity; circumstances of performance; pedagogics-aesthetics. Each dimension is further refined, for example, the student products dimension considers the type of work the students will produce as a result of the task such as a model, an investigation, an explanation, a decision and justification, a problem solution, and/or the exhibition of a technique.

Schoenfeld's emphasis is on balance [15] and he recognizes that any one task could not foster all types of thinking, for example, but that when a set of tasks is being designed (whether they be homework assignments, examination questions or classroom activities) one should aim to cover as many different dimensions as possible. Mason and Johnston-Wilder [9] also advocate a "mixed economy" (p.6) in which learners are given a variety of types of tasks to develop mathematical thinking; Sangwin [13] and Bell [2] reinforce this viewpoint.

In general, the types of tasks considered by the authors mentioned in this section concern asking students to generalize and specialize, generate examples, make conjectures, reason, make decisions, explore, make connections, and reflect. Maria Meehan [10] has already written in this journal about example generation. Other ways of creating tasks on these topics can be found in Watson and Mason's [20] book *Questions and Prompts for Mathematical Thinking.* For example, to write tasks on generalization they suggest that questions like 'What happens in general?', 'Of what is this a special case?', 'What can change and what has to stay the same so that ... is still true?' could be used. Tasks like the ones mentioned here can be adapted and assigned to students at all levels.

5. Implementation of Tasks

Swan [17] describes using some of his tasks in a professional development course for mathematics teachers which ran for 6 months and had participants from 44 different schools in the UK. He gives examples of the 5 task types described in the previous section. The teachers used these tasks in their own classrooms and reported changes in their beliefs and practices. The tasks encouraged the teachers to use more challenging examples than normal, to confront students with conceptual obstacles and to encourage collaboration in their classes.

In reaction to the analysis of tasks Sangwin undertook ([14]), which showed how rarely students were asked to think in creative ways, he developed a number of questions in which students were asked to 'create an instance' — that is, generate an example or provide an object satisfying certain mathematical properties. Although typically there will be many correct answers to such a question and no general method for constructing an instance, Sangwin describes how such questions may be assessed in practice without the imposition on staff of an onerous marking load [14]. He also reports that feedback from a small number of students, with whom the questions were trialled, showed they had a mature understanding of the purposes of these questions.

Bell [2] states that a demonstration-plus-exercises method of teaching risks failing to make contact with students' actual knowledge and thus makes it more difficult for the new knowledge to be embedded in the students' existing cognitive structure. Instead he recommends that tasks should be attempted by students initially and only when their responses have been given should the teacher intervene to offer hints or help towards a solution. He also notes that effective methods of teaching involve crucial management of the learning situation by the teacher, for instance, by adjusting the challenge of a task presented to keep it at an appropriate level for all learners.

In working to challenge conventional ideas about 'doing mathematics' at school, Lampert [8] based her approach on the assumption that students would not learn a different way of thinking about what it means to do mathematics simply by being told what to do or having mathematical problems explained to them. Instead she modelled and demonstrated mathematical thinking in a public manner in her classroom, engaging in mathematical arguments with her students and allowing such arguments to "wander around in various mathematical terrain" (p.41), encouraging the students to make conjectures and to muster appropriate evidence to support or challenge each others' assertions.

Henningsen and Stein [5] warn that it is not enough for a teacher to select and appropriately set-up worthwhile mathematical tasks: he/she must also proactively and consistently support students' cognitive activity in order to ensure that the complexity and cognitive demands of the tasks are not reduced during implementation. Furthermore, Stein et al [16] found that those mathematical tasks which were most cognitively demanding at the design phase (e.g. involving conjecturing, justifying, interpreting) were most likely to decline into somewhat less-demanding activities in implementation. Their research found that tasks that are most likely to maintain high-level cognitive engagement are those that are built on students' prior knowledge. Mason and Johnston-Wilder [9] also warn that there may be a mismatch between the intentions of the author or designer of a task and the motives of a learner. Tasks may be altered, often unconsciously, so that the intentions of a task are not effectively translated in its implementation and therefore the proposed learning outcomes are not attained. For instance, learners may reconstruct a task for themselves so that it becomes something they can do, or by issuing instructions in order to make a task as accessible as possible for his/her students, a teacher might undermine the very purpose of a task.

6. Conclusion

In order to develop mathematically, it is necessary for learners of mathematics not only to master new mathematical content but also to develop a wide range of thinking skills. Most courses emphasize content but students are often expected to pick up mathematical habits of mind on their own. This is sufficient for only a limited proportion of students. One way that the mathematical community could aid students would be to assign a wider range of tasks which would develop their mathematical thinking skills. In this paper we have surveyed some of the mathematics education literature on this topic and for those who wish to learn more and actively incorporate these ideas into their teaching, we hope that the references in this paper will be a good starting point.

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The Origins, Development and Evaluation of Mathematics Support Services

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Abstract. This article is an introductory overview of the recent expansion in the development and provision of mathematics support services at third level. In the last ten years the establishment of Mathematics Support and Learning Centres has increased significantly in Ireland and the UK. Most third level institutes in Ireland now supply some level of mathematics support. We will discuss the development of these supports as well as the reasons why third level institutes have decided to introduce them. We also give an overview of how these services are evaluated and the impact that they appear to have on student retention and performance.

1. Introduction

There is widespread concern about the number of students entering who have basic mathematical problems. Recent reports, (Hourigan & ODonoghue, 2007; Lynch et al., 2003), contain detailed analysis of these issues in the teaching and learning of mathematics at second level in Ireland. Some of the main factors listed include: bad publicity for mathematics, negative attitudes towards the subject, little understanding of the context or background of mathematics and rote learning. The National Council for Curriculum and Assessment review (NCCA, 2005) and Cosgrove et al. (2004) have highlighted possible impacts of these problems. The Organisation for Economic Co-operation and Development report (1999) discusses the impact that this has when these students enter third level. Low attainment in mathematics is cited as a contributing factor in low enrolment and retention rates in science and technology courses (OECD, 1999; IDES 2002). There is also significant international research on these issues (Picker & Berry, 2001).

The provision of mathematics support is one response from third level institutions to try and address these issues. The main aim is to assist students in overcoming their mathematical difficulties. Mathematics support is also active in helping students with different challenges and backgrounds such as Access, Disability and Mature students. It aims to provide a better mathematics learning experience for students from all these backgrounds. Organisations such as NCE-MSTL (The National Centre for Excellence in Mathematics and Science Teaching and Learning) based in the University of Limerick and **sigma** (The Centre of Excellence in Mathematics and Statistics Support) based in the UK have been established to promote and expand, amongst other things, the area of mathematics support. **sigma**, for example, has helped establish mathematics support networks (hubs) in various parts of the UK. The Irish Mathematics Support Network (IMSN) was also established in 2009 to promote mathematics support on a national basis.

In this paper we will present an overview of the creation and subsequent development of mathematics support and learning services. We will review the reasons why such supports have been established, their rapid development over the past decade and the evaluations that are carried out on these services. To some extent, this paper is an introductory literature review of the research carried out in these areas.

2. Why are additional Maths Support Services being provided?

The poor core mathematical skills of a large number of students entering third-level education has been a growing cause for concern for mathematics educators for many years now. This concern has been expressed in numerous journal articles and conference proceedings, and inquiries have been undertaken to ascertain the mathematical accomplishment of these students.

In Ireland, studies were being undertaken as early as 1985. A paper from Cork Regional Technical College (now Cork Institute of Technology (CIT)) in 1985 concluded that their incoming undergraduates were deficient in basic mathematics. Similar findings were reported in University College Cork (Hurley & Stynes, 1986), and a number of other universities and institutes soon followed suit (Brennan, 1997; O'Donoghue,1999). In Hourigan & O'Donoghue (2007), the authors discuss some of the details of the problems that are apparent at third-level. They state (p.461) *"the inability of students to successfully make the transition to tertiary level mathematics education lies in the substantial mismatch between the nature of entrants pre-tertiary mathematical experiences and subsequent tertiary level mathematics-intensive courses."* Indeed, the *"mathematics problem"* at all levels in Ireland has received considerable focus of late, with a comprehensive review of post-primary mathematics education in Ireland (NCCA, 2005) and internationally (Conway & Sloane, 2005) undertaken by the National Council for Curriculum and Assessment (NCCA) leading to the introduction of the new Project Maths syllabus this year.

In the UK, in 1995, the London Mathematical Society (LMS), the Institute of Mathematics and its Applications (IMA) and the Royal Statistical Society (RSS) produced their report *Tackling the Mathematics Problem*. They investigated concerns amongst mathematicians, scientists and engineers in Higher Education about the mathematical preparedness of new undergraduates. In the same year, the Engineering Council also commissioned a report to investigate anecdotal evidence and growing speculation that the mathematical background of undergraduate engineers had changed. The findings showed that, amongst this body of students, mathematical knowledge was weaker than it had been ten years previously (Sutherland & Pozzi, 1995). On a wider scale, the Gatsby Charitable Foundation sponsored a seminar to investigate the same issue in departments of Mathematics, Physics and Engineering. Again, the findings showed strong evidence of a steady decline in basic mathematical skills and increasing inhomogeneity in mathematical attainment and knowledge (Savage et al., 2000). There has, subsequently, been a number of government-funded inquiries, for example, *Inquiry Into A Level Standards* (Tomlinson, 2002), *SET for success* (Roberts, 2002) and *Making Mathematics Count* (Smith, 2004) investigating the standards, suitability and uptake of pre-19 mathematics qualifications.

These issues are not exclusive to Ireland and the United Kingdom; reports of this kind have been produced worldwide. Eight years ago, when the Mathematics Working Group for the European Society for Engineering Education (SEFI-MWG) produced the first revision to their report on a mathematics curriculum for engineers, they noted that *" In increasingly more countries, there is concern over the deterioration in the mathematical ability of new entrants to engineering* *degree programmes."* (SEFI, 2002). In Australia, a government report on teaching produced by McInnes & James (1995) focused on the experiences of first-year undergraduate students. It *"identified weaknesses in mathematical skills and confidence as a barrier for success for many students."* (MacGillivray, 2008).

These reports give credibility to the fact that real difficulties are being experienced by both students and staff in numerate disciplines. Added to this perceived decline in knowledge, a shift in effort and student attitude was reported by the SEC (2001), with evidence of a distinct lack of perseverance in the answering of questions in the Leaving Certificate examination. Even higher level students were a cause of unease, which is similar to findings reported in the UK (LMS, 1995). A lack of competence in mathematics is problematic as it can lead to: *" Stunted advancement in other areas of the degree programmes... Compromised standard of degrees... Amplified failure rates and deflated self esteem.."* (O'Donoghue, 1999). These mathematical deficiencies need to be addressed as early as possible in students' university lives.

3. The Growth and Expansion of Mathematics Support Services

To determine the extent of the "mathematics problem", diagnostic testing has been carried out across Irish third level institutions since the 1980s. In a study carried out in the University of Limerick (UL) in 1997 (O'Donoghue, 1999), it was revealed that 30% of first year service mathematics undergraduates were mathematically underprepared for the demands of their service mathematics modules. While the support tutorials which were initially provided indicated improvements in student performance in general, the need for a one-to-one approach was becoming more critical.

In October 2001, Ireland's first Mathematics Learning Centre opened in UL. The Centre, which was based on the model of the Loughborough University support centre

`http://www.mathcentre.ac.uk/`

continues to carry out diagnostic testing and uses the results to diagnose and support students, as well as for research purposes. The need for similar type centres was recognised across all higher education institutions and most have followed suit.

The Learner Support Unit at Mary Immaculate College of Education was established in 1997 to deliver academic support to degree level students. Mathematics support provision was first provided by the unit in 2002/03. Also, in 2002, Dublin Institute of Technology opened their Students' Maths Learning Centre for all students participating in service mathematics modules. In 2003, the Department of Computing and Networking in IT Carlow received funding from the "Information Technology Investment Fund" to set up a "drop-in" centre to help first year students in programming and mathematics, the two most problematic subjects. While mathematics support was on-going in Tallaght Institute of Technology in the 1990s, in 2003 the Engineering Learning Support Unit was formally established. This centre was also based on a model inspired by mathematics learning support in Loughborough University. One of its aims was to provide mathematics help to engineering students. In January 2005, this facility was renamed the Learning Support Unit. Also in 2003, the Department of Mathematics in University College Dublin (UCD) received Higher Education Authority funding to set up the UCD Mathematics Support Centre which formally opened on 16th February 2004.

The Maths Learning Centre in Dublin City University (DCU) originally took the form of Maths Clinics, scheduled two evenings a week for first year students between 2001 and 2003. The Maths Learning Centre then opened in February 2004 and was established as a permanent entity (funded by the Faculty of Science and Health) in September 2007. Limerick Institute of Technology also launched its Learning Support Unit in March 2004. The Mathematics Support Centre (MSC) in the National University of Ireland Maynooth (NUIM) started on a small scale initially in 2004/2005. It opened on a larger basis in September 2007 and has grown significantly since due to student demand. CIT followed closely behind, opening their Learning Support Centre in September 2005 with a mission to support students predominantly in Mathematics, Physics, Programming and Electronics. The Mathematics Learning Centre in Letterkenny Institute of Technology was formally established in 2006.

In 2001, a total of 95 Higher Education Institutions in the U.K. were surveyed to investigate the extent to which mathematics support was being offered. 46 stated that they offered support over and above traditional lectures and tutorials (Lawson et al., 2001). In 2004, a survey uncovered that 62.3% of 106 U.K. universities stated that they were providing some form of mathematics support (Perkin & Croft, 2004). This represents a 14% increase in supports being offered over a 3 year period.

Gill et al. (2008) carried out an audit of Mathematics Learning Support in 13 Irish third level institutions in 2008, many of the findings are detailed above. Institutes who at the time of writing had no formal physical presence in terms of a drop-in centre were on the path to doing so and were providing mathematics support in the form of support tutorials and online support. Therefore, it is reasonable to assume that not only are more Higher Education institutions in Ireland offering support than those detailed above, but there has been an increase in the number of facilities being provided. Participants at the 4th Irish Workshop on Maths Learning and Support Centres at DCU reported on many of these supports, especially from a technological point of view. Ní Fhloinn (2010a) contains an overview of this conference.

4. The Evaluation of Maths Support Services.

Continuous and thorough evaluations of mathematics support services is of critical importance to the establishment of best practice and the maintenance of these services for the students who need them. In the previous sections, many of the reasons behind the widespread establishment and development of support services has been documented. In this section we look at how these services are evaluated nationally and internationally and how this research impacts on the services provided and the establishment of new initiatives. Lawson et al. (2001 and 2002) give excellent introductory articles on what the first steps of evaluation should be.

The most basic level of evaluation are the levels of engagement or attendance figures at the various supports. Most of the services in Ireland do not have permanent funding, so the maintenance and analysis of these records is critical. It is both beneficial for the institution to see the overall levels of engagement, and also for students to observe that their peers are engaging with these services. The figures are also vital for the service administrators as they can indicate the need for additional staff and supports to deal effectively with students. This data is recorded in most third level institutes and copies of the annual audits are available upon request. The importance of maintaining these records is evident from the case of a mathematics support service in Wales. Until recently, they did not maintain concise and accurate records. They wanted to keep the support as informal as possible to encourage more students to attend. However, when their institution conducted an annual review of the service, it was almost shut down as there was very little evidence (other than anecdotal) of the numbers attending or the impact of their service. This centre now maintains a complete and thorough record of usage.

Clearly the number of users of mathematics support services does not necessarily measure the impact of the supports offered. An initial step is to consider the impact that services appear to have on students' grades and retention rates. It should also be noted at this stage that it is difficult to evaluate the performance of the services using students' mathematics grades alone because so many factors affect performance on examinations. It is impossible to measure how much time students spend studying the subject. It is possible that students who attend the services do better in their exams than those who do not simply because these students worked harder. With this in mind, many studies also consider performance on past examinations when comparing the grades of the students who attend the services with those who do not attend.

A number of studies have been carried out at Irish Universities. Gill & O'Donoghue (2007) look at various ways of measuring the success of the support service provided by a mathematics learning centre. Dowling & Nolan (2006) looked at the pass rates of at-risk students at DCU, and concluded that their Mathematics Learning Centre (MLC) made a positive contribution to student retention. Ní Fhloinn (2010b) looked at the role of student feedback in evaluation the effectivenss of DCU MLC, merging qualitative and quantitative data. Mac an Bhaird et al. (2009) discussed the impact of the Maths Support Centre (MSC) on the grades of first year students at NUIM. It appeared to have an impact on the majority of students who attended regularly, especially the most at-risk students. The importance of effectively evaluating such services in Ireland was highlighted by the 3rd Irish Workshop on Maths Learning and Support Centres held at NUIM in December 2008. The conference theme was *Is Mathematics Support worthwhile?* The speakers presented on the broad range of the various services provided in Ireland and the evaluations that are carried out. An overview is available in Mac an Bhaird & O'Shea (2009).

In the UK, where mathematics support has a slightly longer history, more detailed evaluations have been carried out. Pell & Croft (2008) consider first year Engineering students at Loughborough University. They estimated that attending the Mathematics Support Centre improved the pass rate of students by about 3%. Patel & Little (2006) had similar findings. Lee et al. (2008) used regression models and found that the results of diagnostic tests and attendance at Mathematics Support Centres were significant predictors of end of year results. Parsons (2005) describes how introducing mathematics support and implementing other changes has had a positive effect on engineering maths students. There are a number of studies in other countries also. Terlouw et al. (2008) report on the success of a special mathematics course designed to help the transition from second to third level in the Netherlands. MacGillivray & Cuthbert (2007) report on the initiatives introduced to help engineering students in Australia and the found that students who avail of supports are nearly twice as likely to complete the course and half as likely to discontinue with engineering.

One of the main developments from these initial studies are questions regarding the type of student who avails of mathematics support; whether or not support is reaching its target group; and why certain students do not avail of support. The benefits of mathematics support to students with weak mathematical backgrounds is well documented in Patel & Little (2006), Dowling & Nolan (2006), Lee et al. (2008) and Mac an Bhaird et al. (2009) for example. Patel (2004) reports on a study which identifies the effectiveness of combining appropriate mathematical diagnosis with study support. Croft & Grove (2006) look at the type of students who avail of support. In addition, some authors have been able to report on the use of support services by students with strong mathematical backgrounds. Pell & Croft (2008) consider the number of times first-year Engineering students attend the Mathematics Support Centre in Loughborough University and the grade they receive on their mathematics modules. They found that students who received the top grades were more likely to attend than those who failed or who just passed the module. They comment that the provision of mathematics support has moved from a remedial measure to one of enhancement for the whole student cohort. Similar results have been reported by MacGillivray (2009). She considered the attendance patterns of students, and found that engineering students at the Queensland University of Technology across all abilities make good use of the support services on offer there.

While the majority of research provides evidence that these supports are successful, there is also evidence that not all students are engaging; in particular it should be noted that a significant minority of at-risk students are neither availing of support nor engaging with mathematics. Some authors have found that the fear of showing a lack of knowledge or ability negatively impacts on students' willingness to ask questions (Ryan et al., 2001). Grehan et al. (2010) focuses on the fears that students expressed and how these fears prevented them from engaging with mathematics during their first year at university. This fear manifested itself in four different ways: fear of failure; fear of showing a lack of knowledge or ability; fear of being singled out; and fear of the unknown. Students also displayed a lack of awareness of services or structures within mathematics. Many of these factors were also identified in a study of students at Loughborough University (Symonds et al., 2008).

The process of evaluating mathematics support initiatives is clearly an important and very complex issue. MacGillivray & Croft (2010) contains a thorough and conclusive overview and analysis of the issues at hand. The Irish Mathematics Support Network will issue a national questionnaire for support service users in the coming months, so the impact of services on a national scale can be collated and analysed. The evaluation of services and the impact on students, why they attend or do not attend clearly has very important implications. It allows the centres to develop best practice and also allows us to investigate what students really want from the service we provide.

5. Conclusions.

The area of mathematics support services is clearly crucial to mathematics education at third level, especially considering the large numbers of students who are taking service mathematics as part of their degree. We have given some background and evidence of the *"mathematics problem"* and the issue it raises at third level. We have talked about the development of mathematics learning and support services as a response to these issues and we have presented research which suggests that students who avail of these supports tend to perform better. However, research also indicates that there are more complex issues involved and further investigations are required.

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Mathematics in Society—a Power for Change

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Introduction

> ... *I also have this sense in my bones that service is going to reemerge with greater vitality than we have seen it in the last 100 years, simply because the university must be engaged if it hopes to survive. The social imperative for service has become so urgent that the university cannot afford to ignore it* [*Boyer, 1995*].

A movement, known variously as civic engagement, community based learning and service learning among others, is quietly and effectively taking root in Ireland (`www.campusengage.ie`). This note derives from my initial uncertain understanding of, and belief in, it as a concept, all the while accompanied by a nagging sense of its potential value and relevance to the mathematical community as a whole. Accordingly, I offer an outline of the origin and spirit of the movement, and some preliminary thoughts on how and where mathematics might fit within it. Notably, this exploration has assured me that mathematics at third level has a growing participation in civic engagement, but it is as a discrete, often uncelebrated, enterprise, on the level of individuals rather than a stronger, vibrant national drive. I believe that this movement provides a structure in which to harness and celebrate individual effort and excellence into collective empowerment and impact.

Background

The origins of civic engagement arose naturally in the U.S. where institutions of higher education historically were founded with clear commitment to associated public purposes and to the common good.

Such purposes included the training of the next generation of the nation's leaders by the creation of colonial colleges, serving the nation's agricultural and technical needs by the establishment of public land grant colleges in the late 1800s, and more recently, in the 1960s, addressing the need for a more diverse work force with the establishment of two-year community and technical colleges [Hollander, 2005]. With an unequivocal mission to inspire active citizenship in its students, it was inevitable that a decline in delivering on such public aspiration would first become highlighted in the U.S. in an authoritative manner.

Campus Compact (`www.compact.org`) was the U.S. response to an increasing concern about the demonstrably diminishing impact of higher education on general social capital, and to awareness of the support institutions could potentially offer local communities in a meaningful way. Established in 1985, it has grown to a coalition of more than 1,100 institutions of higher education across the U.S. and elsewhere, and includes D.I.T. and NUI Galway as members. Its initial focus was to promote more active involvement in and partnership with communities but within a short time it became apparent that a broader scope and approach was called for. Thus Campus Compact *works to build civic engagement into campus and academic life by influencing education policy, providing training and technical assistance, developing extensive print and online resources, creating opportunities for ongoing discourse, and making available grant opportunities* (`www.compact.org`).

Ernest Boyer (1928–1995) is lauded as the individual most responsible for challenging academia to rethink *its relationship to the public good*. He took the bold step of suggesting that it was time to reassess the academy's priorities and in particular he highlighted the need for *a deeper appreciation of the value of teaching and service as scholarship*. Indeed, his remarks in a 1995 address (above) were prophetic. He further expressed his concern that the university was increasingly viewed as *a private benefit, not a public good*. Certainly in the Irish context, the great diversification that has happened in higher education over the past twenty years mitigates against such a charge. And yet, with universities warping and yielding to the increasing demands of government in the name of public good (for which, read variously *knowledge/smart economy*, *globalisation*, *skills*, *research*, *world class standard*, *entrepreneurialism*, *enterprise*), there is a certain irony and a certain inevitability in where we are today in the context of Irish higher education. Pursuing objectives ostensibly in the name of public good has led us on a charge for immediacy, relevance, focus and of course funding—at some cost to us. Significantly, I believe this cost includes that of the roundedness of education, which brings with it a holistic development of person, and also the marginalisation of so-called basic research. I believe also that this cost has paradoxically brought about the will to embrace the concept of civic engagement—and an opportunity to address the nation's need for mathematical strength. Boyer's comment resonates:

> *Unless we recast the university as a publicly engaged institution, I think our future is at stake.*

Some 15 years on, one realizes just how far-sighted his vision was.

On foot of it, a call was made in early 1995 for the development of a series of volumes on service learning in individual academic areas. The American Association for Higher Education was tasked with this, and the project reached completion in 2000. The *series explored both theoretical/contextual and practical issues involved in linking academically rigorous course work with projects involving the public good* [13]. Significantly, the Mathematical Association of America stewarded the volume for the mathematical sciences.

So what **is** civic engagement as it applies to academia, and particularly as it might or should apply to mathematics?

The mathematician's stock-in-trade: definitions and literature

As is imperative with a notion that seeks agency and authority, civic engagement is supported by an ample body of academic literature that explores a broad spectrum of critical issues. At the core of these lie the essence, the appropriateness and the viability of such engagement. But what exactly is civic engagement, particularly as it applies to the academy? Boland and McIlrath [2] capture succinctly the growing uncertainty one gleans from literature review as terminology such as service learning, community based learning, civic engagement is used interchangeably. Thus in [6], Hadlock describes civic engagement *as a set of activities that have two characteristics:*

(1) *they enhance either the delivery or the impact of curricular material, usually, but not always, within the context of a specific course, and*

(2) *they take place within a service framework where additional experience with civic engagement or social contribution will be obtained.*

Boland and McIrath remark that the term *service learning* originates in the U.S. where it has positive connotations and that it has been adopted in Ireland to describe a variety of diverse practices. Further, they note the level of discomfort felt by Irish academics concerning the use of the word *service* in this context. Such is the scale of deterrent wrought by the existence of a broad variety of terms for similar activities that these authors *advocate the suspension of any attempt at definition or labelling* and propose the use of the term *pedagogies for civic engagement* (PfCE) as a working label. Indeed terms such as *community*, *citizenship* and *engagement* are contested in both academic literature and in public debate so the need for some degree of clarity or at least agreement on meaning is surely desirable.

The scholarship of engagement

Engagement is a concept much debated in contemporary higher education. On the one hand, it refers to the level of personal commitment given by a student to his/her study and it is the engine that drives an active and enjoyable learning experience. Not surprisingly, it is an issue of great concern to mathematics educators in higher education. In the context of civic engagement, it refers to *a new partnership between the academy and civil society* [13]. Zlotowski argues convincingly that engagement *must not only characterise both student effort and institutional posture, it must also—in both its forms—play a constituent part in the way in which students learn.* In this regard, I believe that in and of itself, the learning of mathematics is a pressing civic issue. What other subject ignites such a range of emotion in the general public? There is no doubt that the learning of mathematics at all levels is and should be under the microscope and that the notion of civic engagement elevates efforts to achieve that end, namely effective teaching, to that of scholarship in its own right. Indeed, Zlotkowski claims that *it was Boyer whose unpacking of the concept of scholarship promoted the idea that teaching itself could be a form of scholarship.* Some of those who champion civic engagement argue along the following lines:

> *The method people naturally employ to acquire knowledge is largely unsupported by traditional classroom*

> *practice. The human mind is better equipped to gather information about the world by operating within it than by reading about it, hearing lectures on it, or studying abstract models of it* ([1]).

Where mathematics is concerned, this is a dangerous oversimplification. The human mind will not develop advanced mathematical thinking, for example, by being at large in the world. Such thinking is nurtured in a mathematically rich environment, indeed in a mathematical world. Such a view undermines the complexity that underpins the high-level cognitive growth characteristic of advanced mathematics. Further, this point ushers in the oft-cited value of the 'real world', particularly as it applies to mathematics. As mathematicians, we are abundantly aware that mathematics lives in the real world from elementary to, literally, rocket science and examples abound in text books to illustrate this. Yet a common and limiting interpretation of service learning (or civic engagement) for mathematics is that it is about learning mathematics 'for the real world'—which all the while suggests subliminally that practical everyday mathematics takes precedence over curiosity, exploration and discovery. These last three are vital ingredients for the welfare of mathematical thinking and they demand independent thinking, or *reflection.* Reflection is the mainstay of civic engagement programmes and it is essential for learning mathematics. This common element unites two, perhaps unlikely, partners. It is clear that I have made no argument concerning the civic aspect of learning mathematics, other than asserting that such learning impacts for better or worse every citizen of this country. Rather, my argument is that if we, as mathematics educators at all levels, can improve on the learning of mathematics by, amongst other things, adopting classroom practices that better engage and promote curiosity and creativity, then we are surely contributing to society already. Abbot and Ryan rightly allude to such a need above.

Challenge and Opportunity

Certainly, many individuals in Ireland have long since subscribed to this and contribute significantly but I believe that a fusion of energy, of ideas and of talent across higher, post-primary and primary education could create a formidable force. Zlotowski describes the MAA-published text [6] as an important resource in encouraging and facilitating a greater awareness of possibilities and opportunities that may exist so that *academically rigorous, discipline-specific civic engagement becomes accepted as an important part of what mathematicians do and what mathematics departments offer.* He continues in his quest to persuade mathematicians of the value of civic engagement for their subject by describing the degrees of reflection that can ensue. The first level, and the one for which mathematicians yearn, has already been described above. He proceeds:

> *On a second level, service-learning reflection asks the learner to become more aware of what he/she brings to the learning process: values, assumptions, biases - many of which are unexamined and potentially problematic.*

He refers to differences that students may experience in their interactions with others in terms of *educational level, mathematical attitudes and aptitudes, ethnic/racial traditions, or socio-economic circumstances* .

> *To leave these aspects unexplored would be to miss a vital educational opportunity, for they invariably stir up thoughts and feelings highly deserving of reflection and discussion.*

This puts us squarely in the realm of Mathematics Education and certainly allows for a further source of valuable data for that field. He argues that a third level of reflection, beyond mathematical competence and heightened awareness, occurs with experiencing the nature of the 'real world' from both an adult and a mathematical perspective. Undoubtedly we have moved far from the remit of an academic mathematician with such reflection—and yet there is surely recognition that Irish society is in urgent need of a mathematics 'fix'. Students involved in civic engagement projects in mathematics may develop a keener sense of how and where problems pertaining to mathematics at least derive—and may be able to contribute effectively to improvement by their civic experience. Informed opinions offer a much better steer towards improvement than rash judgement, and in public debate on pressing issues their voices may resonate. Possible associated civic engagement projects fall naturally and unsurprisingly into three categories: education, mathematical modelling and statistics—and Hadlock ventures that there may be other *untapped possibilities* [7]. The off-the-shelf Undergraduate Ambassadors Scheme (UAS, `www.uas.co.uk`) is but one highly successful civic engagement venture founded by Simon Singh and Hugh Mason (both with illustrious backgrounds in science promotion) and adopted as credit-bearing modules by some Irish universities in mathematics in the past few years. One thing is clear: the resources required to implement civic engagement projects in the fullest sense particularly for mathematics are great. The risk of 'loose cannons' at large with community partners ostensibly to contribute mathematical know-how is also very real. Yet the potential benefits to the student alone are considerable. Zlotkowski and other proponents attest to an enhanced learning experience due to

(1) motivation from seeing curricular material in action
(2) higher student energy level due to commitment and promise of delivery to community partner
(3) motivation due to higher stakes exposure
(4) greater potential for meaningful reflection
(5) greater desire and need for mastery of one's subject when perceived as a service provider.

I am convinced of an opportunity under the banner of civic engagement to unite individual effort and achievement towards greater mathematical good. Perhaps the last word is best left with Boyer's rallying call:

> *I am convinced that... the academy must become a more vigorous partner in the search for answers to our most pressing social, civic, economic and moral problems... At one level, the scholarship of engagement means connecting the rich resources of the university to our children, to our schools, to our teachers and to our cities... But at a deeper level, I have this growing conviction that what's needed is not just more programmes, but a larger purpose, a larger sense of mission, a larger clarity of direction... Increasingly, I'm convinced that ultimately, the scholarship of engagement also means creating a special climate in which the academic and civic cultures communicate more continuously and more creatively within each other... enriching the quality of life for all of us.*

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"Guessing at our Mental Models" or "Committed Knowing Acts"—What are our Students Doing?

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Abstract. This article proposes a discussion of the goals, effectiveness, and shortcomings of Mathematics curricula and assessment systems in higher education. Selected writings by mathematicians, mathematics educationalists and higher education theorists on the broad themes of curriculum and assessment are cited as points of reference. The relationship (or disaffection) between the academic communities of practitioners and theorists in these areas is briefly considered.

1. Introduction

Extensive analysis, commentary and theory (and occasional counsel) on the subject of the teaching and learning of mathematics at third level can be found in the Mathematics Education research literature. Casual perusal of the tables of contents of such journals as *Journal of Mathematical Behaviour*, *Educational Studies in Mathematics* and *Research in Mathematics Education* (to name just three examples) confirms that both the teaching practices of mathematicians and the learning practices of mathematics students command the attention of a busy and populous research community that is geographically widespread and ideologically diverse. This community, while not entirely disjoint from the academic research community in mathematics, stands mostly apart from it.

Further theory and commentary about such matters as instruction at third level and the goals and purposes of tertiary education can be found in the academic literature on Higher Education. Some of this may be of interest and value to lecturers trying to instigate and support a meaningful and stimulating intellectual experience for students while operating in an environment that is ever more constrained, for example by inflexible institutional assessment regulations and by severe resource limitations.

The meaning of the term “curriculum” does not appear to be consistent throughout educational discourse. In this article, the term is intended to include every aspect of the student’s encounter with the subject that is part of the programme of study. It includes the syllabus, which typically consists of a list of topics to be studied. It includes all lectures, tutorials, workshops, laboratory sessions and other scheduled classroom events. It includes all the resources that the students are invited to utilize, such as the library and interaction with the academic staff. It includes the tasks that are assigned for students to do, including summative assessment, and it includes the styles of thinking and investigation that have to be employed in order to complete these tasks. It includes all the activities that the students are advised, instructed or otherwise prompted to engage in, which over the course of a mathematics degree might involve such examples as

- practising the implementation of procedures (for example Gaussian elimination);
- writing a computer programme to automate such implementation (for example to apply the Euclidean algorithm);
- studying, and writing on, selected topics from the modern mathematical canon;
- participating as a team member in a mathematical modelling project;
- memorizing tracts of lecture notes for reproduction in an exam;
- assessing the validity and significance of a proof;
- investigating a phenomenon, proposing a conjecture, writing a proof.

In their book “Engaging the Curriculum in Higher Education” [2], Ronald Barnett and Kelly Coate comment on the relative responsibilities of the student and instructor in realizing the curriculum. They argue that if a curriculum is considered to be more than “a set of educational processes that is simply presented to a student”, then the responsibility for realizing this curriculum rests with the student as well as with the instructors and institution. They point out however that the curriculum experienced by the student, while its character depends in great measure on the student’s engagement, disposition and willingness to assume responsibility, is nevertheless delineated by the curriculum that is presented by lecturers.

> The curriculum as presented opens up educational possibilities on the one hand and limits educational possibilities on the other. It contains choices, whether explicit or tacit, that constrain the educational experience available to the student.

A central theme of the book by Barnett and Coate is their conceptualization of curriculum in terms of the student's experience in three distinct but intertwined domains that they refer to as *knowing*, *acting*, and *being*. Their use of these verbal forms is deliberate, intended to emphasize the critical role of the student's conscious and purposeful agency. Knowledge, they suggest, "has come to be conceived as consisting of a corpus – of ideas, proposition, theories, concepts – that stand outside students". The image of a knowledge corpus to be understood and assimilated is contrasted with the idea of the student engaging in "committed knowing acts":

> an act of knowing is just that: an act. It calls for will, an act of identity and a claim to ownership . . . individuals mark themselves out, project themselves, and claim themselves to be here rather than there. An act of knowing is a positional and personal act.

2. What do we assert that our curricula are for?

Answers to this question may obviously vary across institutions, programmes and individuals. Nevertheless, a casual search for statements by unversity mathematics units of objectives and learning outcomes of their degree programmes returns many recurrent themes. One place to which we might turn in search of a general or representative answer from the mathematics community to this question is to the work of the Mathematics work group of the "Tuning Educational Structures in Europe" project. The Tuning Project was established in 2000 with the goal of connecting the political objectives of the 1999 Bologna Declaration (such as "Adoption of a system of easily readable and comparable degrees") to institutions of higher education and to educational structures, at the subject level. Mathematics was one of nine subject areas considered in Phases I and II of Tuning, which was a pilot project running from 2000 to 2004. Although the inception and development of this project was linked to the Bologna

Process, it was an academic project rather than a political one : its participants were groups of academics within subject areas, whose tasks included "developing reference points for common curricula on the basis of agreed competences and learning outcomes as well as cycle level descriptors". The Mathematics group included 14 academic mathematicians from 14 European countries, and its outputs included a "Summary of Outcomes" document that is available on the Tuning webpages [10], and a discussion document entitled "Towards a common framework for Mathematics degree in Europe" [6], which appeared in the Newsletter of the European Mathematical Society in 2002. The point here is not to comment on the Bologna or Tuning projects but to suggest that the contents of these documents, arising from the discussions of a group of mathematicians assembled from different institutions and countries and from their wider consultations with colleagues, might be reasonably construed as reflecting "typical" views of mathematicians on what mathematics curricula should aim to achieve.

Both documents identify the following "key skills" that "may be expected of any mathematics graduate":

- the ability to conceive a proof;
- the ability to model a situation mathematically;
- the ability to solve problems using mathematical tools.

The "Summary of Outcomes" document also proposes the following descriptors for students graduating with a primary degree in Mathematics; these were identified following a survey of the views of a wider group of academics.

Students will be able to

- show knowledge and understanding of basic concepts, principles, theories and results of mathematics;
- understand and explain the meaning of complex statements using mathematical notation and language;
- demonstrate skill in mathematical reasoning, manipulation and calculation;
- construct rigourous proofs;
- demonstrate proficiency in different methods of mathematical proof.

These expectations seem to be generally consistent with statements made by institutions and their mathematics units on learning outcomes, objectives, and value of their primary degree programmes in mathematical subjects. A quick search returned the following items and many others of a similar nature. The specific items listed below come from six universities in four predominantly English speaking countries including Ireland.

- [Graduates will be able to] demonstrate in-depth knowledge of Mathematics, its scope, application, history, problems, methods, and usefulness to mankind both as a science and as an intellectual discipline.
- [You will learn] how to analyse and solve problems of a quantitative nature and to communicate the results clearly.
- [You will have the] ability to follow complex mathematical arguments and to develop mathematical arguments of your own.
- The increased analytical ability, comprehension of abstract concepts and creative thinking that you gain from studying mathematics are highly valued in the business, industrial, social and academic worlds.
- [Graduates will have] proficiency in the comprehension and writing of mathematical proofs. They will be able to write well–organized, grammatically correct, and logically sound mathematical arguments.
- [Students will have the opportunity to develop the ability] to think critically about solutions and to defend an intellectual position.

These extracts are selected from a scattered assortment of sources, and are unlikely to have any special relevance for any department or programme. However such themes as “analytical thinking”, “reasoning with abstract concepts”, “proficiency with proof”, “usefulness”, “creative thinking”, “communication of mathematical ideas” and “problem solving” seem to be widely recurrent in statements made by higher education institutions in advertising their programmes to prospective students and in communicating their expectations to current students. Experience shows that abilities of these sorts are not easily learned, not easily taught and not easily assessed. Our expectations of ourselves and our students are high, if we are serious about presenting a curriculum that will both demand and support thinking that is variously or simultaneously rigourous, complex, abstract, creative, useful and effective for communicating ideas. What students need in order to achieve such learning is described by the mathematics education theorist David Tall in [13] in the following terms :

> What is essential [for students] is an approach to mathematical knowledge that grows as they grow: a cognitive approach that takes account of the development of their knowledge structure and thinking processes. To become mature mathematicians at an advanced level, they must ultimately gain insight into the ways of advanced mathematicians but, en route, they may find a stony path that will require a fundamental transition in their thinking processes.

Curriculum is the mechanism through which all of this is supposed to be achieved. Assessment is the means by which the success of the individual student's engagement with the curriculum is supposed to be measured.

3. Some questions

The following quotation, from the essay "On proof and progress in mathematics" [14], by the geometer and Fields medallist William Thurston, is proposed as a focal point for a discussion about the reality of our curricula. Thurston's widely cited article first appeared in the Bulletin of the American Mathematical Society in 1994, and was republished in 2006 in the volume "18 Unconventional Essays on the Nature of Mathematics", edited by Reuben Hersh. Positing that the business of mathematicians is "to advance human understanding of mathematics", Thurston asks (as part of a wider discussion) "How is mathematical understanding communicated?". His answer includes the assertion that communication of technical mathematical ideas particular to a narrow research specialization is remarkably efficient and reliable among expert practitioners in that specialization. However, he also charges that communication across subdisciplinary boundaries or to more general (mathematical) audiences is "often dysfunctional", citing experiences of colloquium talks where "most of the audience" is "lost within the first 5 minutes". The following is what he has to say about what happens in classrooms.

> This pattern is similar to what often holds in classrooms, where we go through the motions of saying for the record what we think the students 'ought' to learn, while the students are busy with the more fundamental issues of learning our language or guessing at our mental models. Books compensate by giving samples of how to solve every type of homework problem. Professors compensate by giving homework and tests that are much easier than the material 'covered' in class, and then grading the homework and tests on a scale that requires little understanding. We assume that the problem is with the students rather than with communication: that the students either just don't have what it takes, or else just don't care.
>
> Outsiders are amazed at this phenomenon, but within the mathematical community, we dismiss it with shrugs.

This is not a flattering description of our curricula in action, either as experienced by students or as presented by lecturers. It is offered as a general description of what "often holds", not as a comment about individual lecturers or individual students. It is doubtful whether an international community of practitioners of mathematical instruction at third level really exists (although this may eventually change, for example as the Bologna process rolls on). Our practice in the educational area of our work is organized within local structures; discussion of educational issues within subjects does not necessarily carry easily across institutional or national boundaries. However, the existence of a genuine worldwide community of research in the mathematical sciences can be reasonably asserted. Members of this community collaborate on research projects, they run societies and journals, they gather at conferences and they maintain and share repositories such as *Mathematical Reviews*. The members of this research community are, by and large, the same people who are responsible for the design and operation of mathematics curricula in the world's institutions of higher education. By referring to "the mathematical community", Thurston lays his charge squarely at all of our doors. As mentioned in the introduction and again in Section 4 below, critical scrutiny of our collective performance as educators seems to be an activity more often located outside our community than within it (at least if we are to judge from research literature on the subject). Thurston's charge however is levelled from within. How would we answer it, if we had to? Here are some questions.

(1) Is Thurston's description accurate?
(2) If not, how can we refute it? Would our refutation satisfy an "outsider amazed at this phenomenon" if such persons exist as alleged?
(3) If the answer to Question 1 is (partly) *yes*, is this ok?
(4) If so, how can we justify it?
(5) If our answer to Question 3 is *no, we would much prefer if things were different*, can we change the situation? What would we need? What would it take?

One pedagogical tool that is cited and used by many practitioners and theorists in higher education is the model of *constructive alignment* proposed by the Australian educational psychologist John Biggs [5]. This model is founded on the premise that what the student learns depends more on *what the student does* than on what the instructors do or on any other factor. It is widely accepted also that "from the students' point of view, assessment always defines the actual curriculum" [11]. So : if assessment determines what students do, and what students do determines what students learn, Biggs basically proposes a deliberate and visible alignment of assessment tasks, learning activities and learning outcomes, so that the activities in which students are prompted to engage by the threat/promise of impending assessment are activities that cannot avoid addressing the learning outcomes. This summary description of Biggs's model may appear at first glance to come straight from the "stating the obvious and giving it a name" school of pedagogical theory, but at least two observations can be made. First, the idea of constructive alignment calls for a shift of attention from syllabus content and assessment requirements to *what the student does*, and it calls for explicit attention to and emphasis on learning outcomes, but "more as a function of students' activities than of their fixed characteristics" [5]. Second, what Thurston seems to be describing is a pattern of curriculum in action that is (in his view) not aligned in the sense of Biggs. According to his description, statements of supposed learning outcomes are offered in a tone more of resignation than (even) aspiration, with a sense of “going through the motions”. Meanwhile, students and instructors alike are complicit in a spoken or unspoken understanding about how the assessment will operate, with the result that what students do is imitate worked examples from an exhaustive supply. Similar general observations have been made by many authors. A 2007 study [4] of 16 examination papers from calculus courses in four Swedish universities found that 70% of the examination tasks could be completed using only “imitative reasoning” and that 15 of the papers could be passed using only reasoning of this kind. Michèle Artigue comments in [1] that

> A good number of scholastically well-adapted students succeed, including at the university, more by learning to decode the terms of the didactic contract and by conforming to it than by really learning mathematics . . . it is not easy to construct learning situations where we can ensure that students' success implies real mathematical engagement.

It is my opinion that if we wanted to assemble a case for a negative answer to Question 1 above, we would have no difficulty finding compelling evidence. We are all involved in many examples of courses where the evident patterns of instruction and student activity bear little resemblance to Thurston's description and are demonstrably connected to the graduate attributes described in Section 2. The prosecution might argue though that such courses are more typical of the later stages of degree programmes, by which point few of the students from the first year class remain, most having opted for another path after perhaps being alienated by experiences of the nature described by Thurston.

The environment in which our curricula operate is limited and constrained in many ways. In most of our institutions, we have no choice but to teach our first year students in large groups, in which attention to each student's pace of progress along Tall's “stony path” is just not possible. Funding for such supports as part time teaching is being cut at a time when our students are more numerous and their needs more diverse than ever before. Modularized programmes oblige us to package our subjects into compact and separate chunks, primarily and probably unavoidably along syllabus lines. Another feature of modularization is that summative assessment is frequent and also highly compartmentalized, with examination papers typically corresponding to courses of 24 or 36 lectures that take place over only 12 weeks. Insistent reliance on old exam papers appears to be a deeply ingrained feature of Irish students' behaviour, especially in the early years of their third level education. Radical departure from tradition in assessment practice is not always an appealing prospect for lecturers, no matter how sincere their desire to improve the quality of learning. It's a risky business and the consequences of failure are serious, for students and for us. Our teaching work frequently involves a negotiation of many institutional and disciplinary priorities and pressures, and tensions between them. We are competing against other disciplines for students. We want to know that our curricula are effective at developing such sweeping attributes as "creative mathematical thinking", but we have to assess our students' learning on a compartmentalized basis, and in a pressurized environment that is very different from the one in which our own mathematical creativity primarily finds expression.

4. Can research in Mathematics Education help?

The question of whether research in Mathematics Education can help individual lecturers faced with specific teaching challenges is separate from the question of whether the Mathematics Education research community can help the community of mathematics lecturers with its task of designing and delivering effective curricula. The first question, obviously, is for any lecturer who feels so inclined to investigate for himself or herself: different lecturers who do so may reach different conclusions. There are no theorems in Mathematics Education. The Winter 2002 issue of this bulletin contains an account [7] by Maria Meehan of how consultation of the Mathematics Education research literature *can* help a lecturer to develop insight into the nature and causes of apparent obstacles to student learning. Lecturers contemplating potential or actual difficulties in the teaching and learning of specific syllabus items may find in this literature a useful strategy, a perceptive discussion of the problem, or at least an assurance that the difficulty is not imagined or invented. The 1991 volume *Advanced Mathematical Thinking* [13] challenged what I thought I understood about how people learn our subject and persuaded me that Mathematics Education research has something of interest to say.

On the second question above, opinion seems to be divided. A considerable amount has been written on the subject of the relationship between the Mathematics and Mathematics Education research communities (see, for example, Part VI of [12]). A strong advocate for a strengthening of this relationship is the American algebraist Hyman Bass, who makes the following argument in [3].

> The emergence of a highly competitive and technological world economy has fundamentally enlarged the demands on mathematics education. We now seek, for the broad workforce, levels of scientific and technical competence and literacy that approach what was formerly deemed appropriate only for a select and specialized student population . . . When large numbers of students fail and/or leave mathematical study, which is the gateway to such competence and literacy, this is judged now to be the failure – not of the students – but of the educational system.

Bass goes on to propose that a reconsideration by mathematical scientists of their role as educators is needed, and he argues for programmes of professional development of academic mathematicians as teachers, with the mentorship of "education professionals". He concedes that this is not a popular view in the academic mathematical community and "is not an easy proposition", but argues

> Much remains to be done to establish contexts for respectful communication and professional collaboration between mathematical scientists and education professionals. . . . This is ultimately a two-way street, along which mathematical scientists can contribute to the disciplinary strengthening of school programs and teaching practice, while the teacher and education research communities can elevate the pedagogical consciousness and competence of academic mathematical scientists.

It hardly needs to be said that not every academic mathematician shares the conviction of Bass on the potential impact and mutual benefit of a closer alliance between the two communities. For balance (or at least opposition), here is the view of another prominent algebraist, Shimshon Amitsur. These words are from an interview with the mathematics educationalist Anna Sfard, carried out in 1994 and documented in [12].

> To strengthen the status of research in mathematics education, one has to prove its usefulness. The onus of proof is on the researchers themselves. They have to show that they have a theory of mathematical thinking which convincingly explains observed phenomena. Only when they can provide such a theory will mathematics education turn into a true academic discipline. Not even one day earlier. [1]

More recently, the UK-based mathematics educationalist Elena Nardi has written extensively on what she refers to as "the often difficult relationship between the communities of mathematics and mathematics education". Her 340-page book "Amongst Mathematicians" [8], which appeared in 2008, is a study of teaching and learning of mathematics at undergraduate level, based on a wide range of written and audio data from both lecturers and students and supported by the author's professed "fundamental underlying belief" that "development in the practice of university-level mathematics teaching is manageable, and sustainable, if driven and owned by the mathematicians who are expected to implement it". In a related work [9], Nardi and Paola Iannone report on a study of the views of 20 research mathematicians on the relationship, and its potential, between the two communities. While the strong call for collaboration in these works and others may well be echoed by many mathematicians, [8] and [9] seem to be cooperative rather than truly collaborative projects. They are studies *of* the attitudes and practices of members of one community, *by* members of the other. While the "two-way street" envisaged by Bass is a notion with obvious appeal, it remains to be seen whether the "fragile, crucial" relationship discussed in [9] can evolve into a genuine partnership.

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The Reflective Property of a Parabola

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1. Introduction

Thanks to my Mathematics teacher, An tUasal Concubhair Ó Caoimh, I left school knowing some of the properties of the conics. However, it wasn't until much later that I came to appreciate their relevance in our daily lives. Ironically, when Mathematics teachers are nowadays expected to teach real-world applications of their subject, and we're surrounded by those of the parabola especially, and, to a lesser extend, by those of the ellipse, conics have no place in the school curriculum. It is this irony which prompted this note, whose purpose is to remind readers of the reflection property possessed by the parabola, and to point out that no other curve shares this feature, something that may not be widely known.

2. A Reflection Property of the Parabola

Recall that a parabola is a set of points that are equidistant from a fixed point (called its *focus*), and a fixed line (called its *directrix*). We may suppose, for definiteness, that the focus is $(a, 0)$, where $a > 0$, and that the directrix has equation $x + a = 0$. Then, in order that the point (x, y) be on the resulting parabola, it is necessary and sufficient that

$$\sqrt{(x-a)^2 + y^2} = |x + a|,$$

i.e., that $y^2 = 4ax$. This is the equation of the parabola with which we'll work from now on.

The following is a basic property of any parabola, and is the key idea behind such everyday objects as lamp-shades, satellite dishes, car headlamps and hearing aids.

Theorem 1. *Let $P = (at^2, 2at)$ be any point on the parabola $y^2 = 4ax$, and let $F = (a, 0)$. Let the tangent to the parabola at P meet the x-axis at Q. Then $|PF| = |FQ|$.*

Proof. If $t = 0$, P coincides with the origin and the result is clear. So, suppose $t \neq 0$. Then $y = x/t + at$ is the equation of the tangent at P. This intersects the x-axis at $Q = (-at^2, 0)$. Hence

$$|QF| = |-at^2 - a| = a(1 + t^2).$$

But,

$$\begin{aligned} |FP| &= \sqrt{(at^2 - a)^2 + (2at - 0)^2} \\ &= \sqrt{(a^2t^4 - 2a^2t^2 + a^2 + 4a^2t^2} \\ &= \sqrt{(a^2t^4 + 2a^2t^2 + a^2} \\ &= \sqrt{(at^2 + a)^2} \\ &= a(1 + t^2). \end{aligned}$$

Thus, $|PF| = |FQ|$. □

Corollary 1. *$\angle FPQ = \angle PQF$.*

This means that the tangent at any point P of a parabola is equally inclined to the line joining P to its focus and the axis of the parabola. As a consequence, Heron's Reflection Principle comes into play, and we have the following fundamental fact: a light ray travelling towards a parabola from right to left along a straight line parallel to the axis of the parabola is reflected through its focus. Conversely, a light ray emitted from a source situated at the focus of a parabola is reflected by the parabola along a line parallel to its axis.

Shouldn't this simple fact be taught in school?

3. This Reflection Property Characterizes the Parabola

Could a satellite dish have been invented without knowing the properties of the parabola that the Greeks discovered over two thousand years ago?

If we think of such a dish as being a surface of revolution obtained by rotating a smooth function about the positive x-axis, this question is answered by the following result.

Theorem 2. *Suppose $a > 0$, f is continuous on $[0, \infty)$, with $f(0) = 0$ and $f(a) \neq 0$, and differentiable on $(0, \infty)$, with $f' \neq 0$ there. Suppose, in addition, that the tangent to any point $P \in f$ is equally inclined to the line joining P and $(a, 0)$, and the x-axis. Then*

$$f^2(x) = 4ax, \ \forall x \in [0, \infty).$$

Proof. Let $t > 0$. The tangent at $(t, f(t))$ has equation $y - f(t) = f'(t)(x - t)$. This meets the x-axis at $(t - \frac{f(t)}{f'(t)}, 0)$. By hypothesis, the distance between this point and $(a, 0)$ is the same as the distance between the latter and $(t, f(t))$. Thus

$$|t - a - \frac{f(t)}{f'(t)}| = \sqrt{(t-a)^2 + f^2(t)}.$$

Equivalently

$$-2\frac{(t-a)f(t)}{f'(t)} + \frac{f^2(t)}{f'^2(t)} = f^2(t).$$

In other words, for all $t > 0$,

$$f^2(t) = f^2(t)f'^2(t) + 2(t-a)f(t)f'(t),$$
$$f^2(t) + (t-a)^2 = \big(f(t)f'(t) + (t-a)\big)^2.$$

Let

$$g(t) = f^2(t) + (t-a)^2, \quad t \geq 0,$$

so that $g'(t) = 2\big(f(t)f'(t) + (t-a)\big)$, $t > 0$. We deduce that g satisfies the differential equation

$$g'^2(t) = 4g(t),\ 0 < t < \infty,\ g(0) = a^2.$$

However, $g > 0$, and so $h = \sqrt{g}$ is differentiable on $(0, \infty)$, and satisfies the simpler non-linear differential equation

$$h'^2(t) = 1,\ 0 < t < \infty,\ h(0) = a.$$

But $h'^2 - 1 = (h'-1)(h'+1)$, and so, for each $t > 0$, either $h'(t) = 1$ or $h'(t) = -1$. Since every derivative satisfies the Intermediate Value Property, this means that if h' takes the values -1 and $+1$, it must, perforce, take the value 0, which it patently does not! Conclusion: either $h' \equiv 1$ or $h' \equiv -1$. In other words, for all $t \geq 0$, either $h(t) = t + a$, or $h(t) = -t + a$, i.e., either $g(t) = (t+a)^2$ or $g(t) = (-t+a)^2$. Thus,

$$f^2(t) + (t-a)^2 = (t+a)^2, \text{ or } f^2(t) + (t-a)^2 = (t-a)^2.$$

The second of these possibilities is plainly untenable. Hence $f^2(t) = 4at$, $\forall t \geq 0$. In other words, f is the upper or lower branch of the parabola $y^2 = 4ax$. $\square$

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Ireland's Participation in the 51st International Mathematical Olympiad

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The International Mathematical Olympiad (IMO) is the oldest and largest among all international academic competitions for secondary school students. The first IMO was held in Romania in 1959, with 7 participating countries. At the end of the 1970s, about 20 countries were regularly participating in the IMO. Since then, the number of participating countries monotonically increased to over 100 in Bremen last year.

From 2^{nd} until 14^{nd} July 2010, the 51^{st} International Mathematical Olympiad took place in Astana (Kazakhstan). With 523 participants (47 of whom were girls) from 97 countries, the size of this IMO, which was the first ever held in central Asia, was similar to its two predecessors.

The Irish delegation consisted of six students (see Table 1), the Team Leader, Bernd Kreussler (MIC Limerick) and the Deputy Leader, Gordon Lessells (UL).

Name	School	Year
Colin Egan	Clonkeen College, Blackrock, Co. Dublin	6^{th}
Dmitri Tuchapsky	Christian Brothers College, Cork	6^{th}
Vicki McAvinue	St. Angela's Secondary School, Waterford	5^{th}
Mel O'Leary	Lucan Community College, Co. Dublin	6^{th}
Owen Binchy	Coláiste Iognáid, Galway	6^{th}
Kieran Cooney	CBS Charleville, Co. Cork	5^{th}

TABLE 1. The Irish contestants at the 51^{st} IMO

1. Team selection and preparation

The IMO is the most prestigious mathematical problem solving contest for second level students in the world. Participation in this event is already considered to be a great honour. In order to be able to gain any marks at the IMO exams, it is not sufficient, even for the brightest of students, to rely solely on the Leaving Certificate Mathematics Syllabus (or the equivalent in other countries). Some countries are able to identify the most talented students at a very early age and organise long-term training programmes for them, whereas others offer very intense training programmes for the brightest of their students which have been selected through a nationwide multiple round contest.

In Ireland, we are currently able to identify students with exceptional mathematical talent in three different ways: top performance in the Junior Certificate Examination in Mathematics, or excellent results in the PRISM competition or recommendation by a person who has identified this talent (maths teachers, school principals).

At five different locations all over Ireland (UCC, UCD, NUIG, UL and NUIM), mathematical enrichment programmes are offered to the students who came to our attention through one of these sources. These classes run each year from December/January until April and are offered by volunteer academic mathematicians from these universities or nearby third-level institutions.

This year, participation in the training programme was down considerably. In previous years, but, unfortunately, not this year, the Department of Education and Skills and the State Examinations Commission provided information which enabled school principals to identify those, if any, of their students who were among the best performers in the country in Junior Certificate Mathematics. This practice was very helpful in attracting healthy numbers of good students, and we hope that it will be possible to have the practice restored in coming years. We then contacted schools directly, but this is less efficient, because school principals often do not know whom they should nominate and might be too busy to send on this information to mathematics teachers.

In order to give the students the opportunity to gain additional competition experience, the centre in Limerick organised the first Limerick Mathematical Olympiad, which was held on 4$^{\text{th}}$ March, 2010. Later, the winner of this Olympiad secured a place on the

Irish IMO team. The hope is to make this an annual event with increasing participation which may help to awake the interest of a greater number of students in the enrichment programme or in mathematics in general. A similar event takes place in Galway, to mark the end of the NUI Galway Mathematics Enrichment course, and to give students a chance to participate in a mathematical problem-solving contest before the IrMO. As in Limerick, the winner of this year's Galway Olympiad became a member of Ireland's IMO team.

The selection contest for the Irish IMO team is the Irish Mathematical Olympiad (IrMO), which was held for the 23$^{\text{rd}}$ time on Saturday, 24$^{\text{th}}$ April, 2010. The IrMO contest consists of two 3-hour papers on one day with five problems on each paper. The participants of the IrMO, who normally also attend the enrichment classes, sat the exam at the same time in one of the five centres. This year, a total of 43 students took part in the IrMO. The top performer is awarded the Fergus Gaines cup; this year this was Colin Egan. The best six students (listed in order in Table 1) were invited to represent Ireland at the IMO in Astana.

As in previous years, the final preparation of our contestants for the IMO took place in special training camps in Limerick. Because the IMO in Astana started more than one week earlier than usual, there was not much time available between the end of the Leaving Certificate Examinations and the departure date for our students. Therefore, instead of a week-long camp we organised two shorter camps for the students. The first of these was held at MIC Limerick on 8/9 June 2010. As the LC Examinations started on these days, four of the six team members could not participate. In fact, this camp could be seen as the start of the preparation for IMO 2011. The participants included the two team members who did not sit their LC Examinations this year and six of the best students (chosen on the basis of their performance at the IrMO) who will be eligible to participate in future IMOs. The second camp, at which the six members of the Irish IMO team participated, was held at the University of Limerick from 28 to 30 June 2010. The camps were organised as usual in a very efficient way by Gordon Lessells. This year, the sessions with the students were directed by Mark Burke, Mark Flanagan, Eugene Gath, Donal Hurley, Kevin Hutchinson, Bernd Kreussler, Tom Laffey, Jim Leahy and Gordon Lessells.

Thanks to an initiative of Stephen Buckley, communication among students as well as between them and their trainers was facilitated by a bulletin board. This web-based tool with restricted access was used to discuss attempts of solutions of olympiad-style problems between the weekly enrichment sessions and also between the IMO preparation camps.

2. Jury meetings – the problem selection

The Jury of the IMO, which is composed of the Team Leaders of the participating countries and a Chairperson who is appointed by the organisers, is the prime decision making body for all IMO matters. In particular it is responsible for choosing the six contest problems out of a shortlist of 30 problems provided by a problem selection committee, also appointed by the host country. This year's Chairman of the Jury was Professor Yerzhan Baissalov.

In the recent past, the location at which the Jury of the IMO resides was kept as secret as possible as one of the measures which should help to ensure that the contest problems will not become publicly known before the exams take place. This year, it was known in advance that the Jury members were supposed to arrive in Almaty (known as Alma-Ata until 1993), at a distance of about 950 km from Astana. Almaty was the capital of Kazakhstan until 1998 and is still an important cultural, scientific and business centre of Kazakhstan.

The Jury meetings which took place on 3–5 July were held in the Wellness Complex "Alatau" outside of Almaty, at the foothills of the Trans-Ili Alatau Mountains, the northernmost part of the Tien Shan mountain system. This beautiful and quiet location was the ideal place to think about the proposed contest problems. As my flight had arrived in Almaty shortly after midnight on the 2 July, I had plenty of time to get familiar with the shortlisted problems without being spoilt by the official solutions which were handed out on the next day after lunch.

After having obtained the official solutions, time was less plentiful, because during the Jury meetings on Sunday, 4^{th} July, the six contest problems had to be selected. Based on the longstanding experience of many Jury members and the calm direction of the Chairman, this task was completed before 4 p.m. on Sunday.

During the evening meeting, the final formulation of the six problems was discussed so that representatives of the language groups of the five official languages (English, French, German, Russian and

Spanish) were able to produce their versions over night. After approving these five versions at a Jury meeting on Monday morning, the rest of the day was available for the translation of the contest problems into 50 languages, in addition to the five mentioned before.

Because the Jury was to be flown to Astana, all exam papers had to be printed before departure. This task was finished in time for the very early departure to the airport at 3 a.m. on Monday, 6 July. The reason why the organisers were keen to finish the printing before departure is that even with the use of the PDF format, it may happen after changing to another PC, that problems are arising in viewing or printing a document which contains locally unusual fonts. As usual in recent years, the support provided by Matjaž Željko from Slovenia was invaluable for all IT related tasks at this IMO.

3. The problems

First Day. Problem 1. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that the equality

$$f(\lfloor x \rfloor y) = f(x)\lfloor f(y) \rfloor$$

holds for all $x, y \in \mathbb{R}$. (Here $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z.) *(France)*

Problem 2. Let I be the incentre of triangle ABC and let Γ be its circumcircle. Let the line AI intersect Γ again at D. Let E be a point on the arc BDC and F a point on the side BC such that

$$\angle BAF = \angle CAE < \tfrac{1}{2}\angle BAC.$$

Finally, let G be the midpoint of the segment IF. Prove that the lines DG and EI intersect on Γ. *(Hong Kong)*

Problem 3. Let $\mathbb{N}$ be the set of positive integers. Determine all functions $g : \mathbb{N} \to \mathbb{N}$ such that

$$\big(g(m) + n\big)\big(m + g(n)\big)$$

is a perfect square for all $m, n \in \mathbb{N}$. *(United States of America)*

Second Day. Problem 4. Let P be a point inside the triangle ABC. The lines AP, BP and CP intersect the circumcircle Γ of triangle ABC again at the points K, L and M respectively. The tangent to Γ at C intersects the line AB at S. Suppose that $SC = SP$. Prove that $MK = ML$. *(Poland)*

Problem 5. In each of six boxes $B_1, B_2, B_3, B_4, B_5, B_6$ there is initially one coin. There are two types of operation allowed:

Type 1: Choose a nonempty box B_j with $1 \leq j \leq 5$. Remove one coin from B_j and add two coins to B_{j+1}.

Type 2: Choose a nonempty box B_k with $1 \leq k \leq 4$. Remove one coin from B_k and exchange the contents of (possibly empty) boxes B_{k+1} and B_{k+2}.

Determine whether there is a finite sequence of such operations that result in boxes B_1, B_2, B_3, B_4, B_5 being empty and box B_6 containing exactly $2010^{2010^{2010}}$ coins. (Note that $a^{b^c} = a^{(b^c)}$.) *(Netherlands)*

Problem 6. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers. Suppose that for some positive integer s, we have

$$a_n = \max\{a_k + a_{n-k} \mid 1 \leq k \leq n-1\}$$

for all $n > s$. Prove that there exist positive integers ℓ and N, with $\ell \leq s$ and such that $a_n = a_\ell + a_{n-\ell}$ for all $n \geq N$. *(Iran)*

4. The contest

The six Irish contestants, accompanied by the Deputy Leader, Gordon Lessells, arrived in Astana in the very early morning of Sunday, 4$^{\text{th}}$ July. The early arrival one day ahead of schedule did help to minimise the costs for the flights and was very helpful for adjusting to the local time which is 5 hours ahead of Irish Summer Time.

Since 1998 Astana is the capital of Kazakhstan. An incredible feast of architecture has tripled the area covered by this city compared with its size before 1997. The contrast between the original part of Astana (known as Akmola between 1992, when Kazakhstan became independent, and 1998) and the newly created City could not be bigger. Many buildings in the old town look very run down and do not seem to be maintained properly anymore. At the same time, the contemporary architecture of the new City is very shiny, with a great variety of forms, wide roads, expensive ultramodern shopping centres and the palace of the president, the pyramid shaped Palace of Peace and Reconciliation, the Bajterek tower (the new symbol of Astana) and a few more structures in exact alignment.

The Opening Ceremony took place on Tuesday morning at the Palace of Independence in Astana. In addition to the traditional parade of all participating teams and the usual speeches, we saw a nice and colourful presentation of traditional and modern music and dance from Kazakhstan.

On Tuesday afternoon the students were driven to their new accommodation in the Youth Centre "Baldauren", situated on the shore of Lake Shchuchye, 240 km north of Astana in a very scenic part of Kazakhstan. The six-hour bus trip with dinner at 10.30 p.m. was not the ideal preparation for the first exam next day. Baldauren was the venue for the exams and the centre for a variety of recreational and sight-seeing activities after the exams were over. The facilities available were of a high standard and much appreciated by the students. The Irish Team guide, Zuzaira, was particularly helpful throughout this time. The students returned to Astana for three nights at the end of the Olympiad.

On the day the students left for Astana, the members of the Jury checked in to Hotel "Duman" which the Students and Deputies had vacated in the morning. For the afternoon and evening, meetings were scheduled for the discussion of the detailed marking schemes for all six contest problems. Due to delay with the check-in procedure, the meetings were postponed to 8 p.m. in the evening. The meeting was adjourned at 10:15 p.m., because everybody was too tired from the very early start into this day. The finalisation of the marking schemes of three problems was postponed to the next morning, directly after the questions-and-answers session. The problem captains indicated that they intend to follow the marking schemes very strictly.

The two exams took place on the 7^{th} and 8^{th} of July, starting at 9 o'clock each morning. There were three venues for the exams, all located in the "Baldauren" resort. On each day, $4\frac{1}{2}$ hours were available to solve three problems. During the first 30 minutes, the students were allowed to ask questions if they had difficulties in understanding the formulation of a contest problem. A scan of these questions was sent to the hotel "Duman", where the printout was given to the relevant Team Leader. Each question was discussed and answered in front of the Jury, so that equal standards were applied to all contestants. The approved answer was then sent back electronically to the exam location. The questions did not indicate that there was any major source of confusion in the formulation of the problems. With 26 questions on the first day and about 55 on day two, these were unusually short Q&A sessions.

On the evening of the 8th of July an extraordinary Jury meeting took place at which the team of the Democratic Peoples Republic of Korea was disqualified for this year.

5. Marking and Coordination

The first scripts had arrived late Wednesday evening. A first reading on that night revealed that only Colin's solution of Problem 1 could be a candidate for full marks. Unfortunately, it turned out later that he didn't finish one case completely and so narrowly missed an Honourable Mention.

After the end of the second exam on Thursday, the Deputy Leaders were transferred to the hotel "Duman". Because the venue for the exams and the students' accommodation were far away from our hotel, there was no way to meet the students. The first time the leaders were able to briefly see their students was on Monday, 12th July, during an impressive presentation of traditional Kazakh equestrian sports at the hippodrome "Kazanat", not far from Astana.

Gordon arrived at about 8 p.m. on Thursday. All our coordination sessions were scheduled for Friday and Saturday – the first one on Friday morning at 9 o'clock. This was a very tight programme and we were painfully missing the helping hand of an Official Observer during two almost sleepless nights.

The marking of the scripts at the IMO is undertaken by two independent groups. One group consists of the Team Leader, the Deputy Leader and, if available, the Official Observer. The second group consists of the coordinators, who were appointed by the local organisers. The two groups met according to a tight schedule which was distributed on the morning of the second exam day.

The coordination is a very important part of the IMO, because it is the well established method to fairly mark the scripts of the students from so many different nations. It is important for the representatives of the teams to be well prepared for each of the half-hour meetings with the coordinators. For example, a detailed explanation of each step of a student's solution could be necessary or there might be the need to show that filling the gaps in a solution, which was not covered by the agreed marking scheme, does not require deep ideas and is easily carried out.

We found that the very young coordinators were very knowledgeable and always well prepared. In general they were very strictly adhering to the marking scheme and did not award points too generously. Other Team Leaders have experienced the coordination process in the same way so that we were sure that the points were distributed with equal measure in a fair way.

Name	P1	P2	P3	P4	P5	P6	total	ranking
Colin Egan	6	0	0	0	0	0	6	446
Owen Binchy	0	0	0	0	6	0	6	446
Kieran Cooney	0	0	0	0	2	0	2	485
Mel O'Leary	2	0	0	0	0	0	2	485
Vicki McAvinue	0	1	0	0	0	0	1	494
Dmitri Tuchapsky	0	0	0	0	1	0	1	494

TABLE 2. The results of the Irish contestants

Table 2 shows the results of the Irish contestants. A comparison with the results achieved by all other contestants reveals a weakness of our students in geometry (Problems 2 and 4). However, the Irish contestants have shown their true potential on Problem 5, where they scored well above average. To understand these anomalies, one has to take into account that Problem 5 does not fit into any standard pattern and so creativity was more important than experience. On the other hand, to be good in solving geometry problems requires a lot of experience which normally is acquired through continuous training during a couple of years.

The Jury tries to choose the problems in such a way that Problems 1 and 4 are easier than Problems 2 and 5. Problems 3 and 6 are usually designed to be the hardest problems. Table 3 shows that this year's results fit very well into this pattern, whereby Problem 5 was a bit harder than anticipated.

If the medal cut-offs are taken as an indicator, it seems that achieving high marks at this IMO has been harder than in previous years. But this might also be related to a less generous marking process. Gold medals were awarded to 47 students who scored at

	P1	P2	P3	P4	P5	P6
0	39	223	429	84	353	471
1	17	93	48	2	85	14
2	27	23	10	10	26	4
3	16	8	4	47	3	3
4	34	2	4	2	0	0
5	35	4	4	4	2	6
6	54	2	2	2	11	4
7	295	162	16	366	37	15
average	5.453	2.586	0.464	5.348	0.930	0.368

TABLE 3. For each problem, how many contestants have got how many points

least 27 points, 104 students with points in the range 21–26 got silver medals and each of the remaining 115 students who scored at least 15 points was awarded a bronze medal.

Although the IMO is a competition for individuals only, it is interesting to compare the total scores of the participating countries. This year's top teams were from China (197 points), Russia (169 points) and the USA (168 points). Ireland, with 18 points in total, finished in 90$^{\text{th}}$ place. Only one student, Zipei Nie from China, achieved the perfect score of 42 points. The detailed results and statistics can be found on the official IMO website `http://www.imo-official.org`.

6. EUROPEAN GIRLS' EVENT

The Leader of the UK team, Geoff Smith, organised a meeting on Sunday, 11$^{\text{th}}$ July, where he announced the launch in 2012 of a European Girls' Event. The original plan, he said (after apologising not to be a woman), was to have a Girls' Mathematical Olympiad. There was an email discussion among some European Team Leaders about this idea before they travelled to the IMO. This original idea was received with some scepticism. A basis of this scepticism might be that in most European countries separate girls' education is not common.

The discussion showed that an initiative which could help to increase female participation in the IMO and to encourage more girls to engage in mathematical problem solving is very welcome. On the other hand, to create a girls only competition similar to the IMO could be seen to be discriminatory. It seemed that the consensus was that the main focus of such an event should be on educational activities which boost the participants' abilities in competitions like the IMO and which help to develop girls' interest in mathematics. Part of such a training camp-like event could well be a competitive exam.

What the exact flavour of it will be depends on the organisers of the first event of this kind. According to Geoff, Murray Edwards College Cambridge and the UK Mathematical Trust will host it in Easter 2012. The hope is that this will be the start of an annual event, held in different European countries in the future.

7. Outlook

The closing ceremony took place on Tuesday, 13$^{\text{th}}$ July, at the Palace of Independence in Astana. Speeches were given, the medals were awarded and folklore and contemporary music performances in between made it a nice event. At the end, the IMO banner was passed on to the delegation from the Netherlands, where the 52$^{\text{nd}}$ IMO will be held from 16$^{\text{th}}$ until 24$^{\text{th}}$ July 2011.

The next countries to host the IMO will be

2011	Netherlands	`http://www.imo2011.nl/`
2012	Argentina	

Standard procedure would have been to formally decide this year about the location of the IMO 2013. However, no such decision could be made, because no country has expressed its interest in organising the IMO that year.

At one of the Jury meetings, Nazar Agakhanov, Leader of the team from Russia, was elected as new chairman of the IMO Advisory Board. The outgoing chair, József Pelikán from Hungary, who has served for the past eight years, was thanked for his work with standing ovation.

During the final Jury meetings a discussion was started about reforming some IMO procedures. This includes the problem selection process and the regulations to deal with allegations of fraud. Within the next 11 months these issues should be discussed by all those interested in IMO matters.

8. Conclusions

Due to the recession, the funding provided by the Department of Education and Skills was not sufficient to send a full team of six students. For the first time in 14 years, no funding was available to send an Official Observer who accompanies and supports the Team Leader. The work done by the Leader and Deputy Leader, in particular when marking the scripts of the students is enormous. There is a real difference if an observer is available and if so, this helps to ensure that the best possible results for the Irish students could be achieved. Without an observer there is just not enough time to go through all solutions of all our contestants with the same care. Even in the problem selection stage, the helping hand of an observer is very useful to get a better picture of the difficulties involved in the shortlisted problems. Therefore, efforts have to be increased to get better funding in the future.

The results of the Irish Team show that our students have much less experience in problem solving than the majority of the contestants from other nations. This becomes obvious by looking at Problem 4, for which 366 students (more than 70 percent) got the full score of 7 points. Among the top 50 countries, only Denmark with 15 points has scored less than 26 for Problem 4 as a team. There are only two other teams who did not score any points for Problem 4.

Improved Irish performance at the IMO requires a solid foundation in basic geometry. We have very good geometry teachers in all the five training centres in Ireland, but our students are exposed to problems from elementary geometry too late. Building up experience in solving mathematical problems, not only from geometry, needs time. The way forward seems to be to get students in their Junior Cycle interested in such activity. The PRISM competition (`http://www.maths.nuigalway.ie/PRISM/`) is a promising and valuable activity which may help to attract younger students.

Because almost all participating countries have performed better than our team on the two geometry questions, we need to find answers to the following questions. Can we improve the training in geometry in our enrichment programmes? Is the weakness of our students in geometry caused by the Irish school curriculum or how the curriculum is put into practice? What else could be the underlying reasons?

It would be alarming if the cause would be found in relation to the curriculum or its implementation, because elementary geometry is a traditional subject trough which primary and secondary school children could be and indeed were successfully introduced to clean logical thinking, mathematical proof and problem solving. As there are almost no prerequisites needed in order to do geometry, this is the ideal subject through which such skills can be developed from an early age.

On the positive side, it is a very encouraging fact that three of our students scored on Problem 5. Only 14 of the top 50 countries have got more than the 9 points our team has scored for this problem. I take this as an indication that it is the experience and not the talent or creativity our students are lacking.

At a time where interest in mathematics and science courses at third level institutions is on the increase, it should be possible to get secondary school teachers interested in fostering their mathematically talented students. To enhance skills related to logical thinking, mathematical proof and problem solving, certainly would be a very valuable contribution to the education of future generations of students who wish to study mathematics at university.

I suggest that we should try to initiate a platform or network for those teachers who are willing to run regular training sessions with focus on mathematical problem solving in their schools for students in their Junior Cycle. The trainers from the five training centres could then support these teachers with their experience.

Those students who really engage in mathematical problem solving activities will experience the satisfaction of success whenever they solve a problem having spent some hours tackling it. Such satisfaction could be an important motivation in continuing to think independently about mathematical problems. And this is what a successful IMO participant as well as a good student of mathematics needs.

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At UL: Mark Burke, Eugene Gath, Bernd Kreussler, Jim Leahy and Gordon Lessells.

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