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EDITORIAL

It appears that the 'retirement wave', which started in continental Europe already a few years back, has reached Ireland. Several of our distinguished colleagues retire in 2009 continuing what began a couple of years ago and will run on in the next years to come. The general expansion of European universities in the 1970's necessitated a larger body of academic staff who now have completed an accomplished career. Many of them will not leave (Research) Mathematics right away but the question does arise who will succeed them and when this shall happen. The economic situation may not favour rapid replacements and in addition it might be more difficult to find suitable successors who can fill the position of their predecessors and are not simply passing through.

Many of those who now retire considerably shaped the mathematical landscape in Ireland—the interview with Professor Laffey in this volume illustrates this very well. The quality and quantity of mathematical research output 'produced in Ireland' has increased tremendously over the past decades and no doubt there are many capable young mathematicians out there. This notwithstanding the Editor wonders whether the possibility of a 'vacuum', if even miniscule, looms. Can the small number of research notes written by Irish/Ireland based authors and submitted to the Bulletin *over the last few years be taken as an indication that few of us can 'afford' to publish in a small periodical which is not widely read? And thus have to look for activities outside of Ireland which can be more profitable for one's career? On the other hand, the large(r) number of international mathematical conferences organised at Irish universities may point in the opposite direction. But still ...*

—MM

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Joint Meeting of the

61st British Mathematical Colloquium and the

22nd Annual Meeting of the IMS

NUI Galway

April 6–9, 2009

The second joint meeting of the BMC and the annual IMS meeting was held at the National University of Ireland, Galway between 6 and 9 April 2009.

The plenary speakers comprised David Eisenbud (UC Berkeley), Ron Graham (UC San Diego), Ben Green (Cambridge), Rostislav Grigorchuk (Texas A&M) and Frances Kirwan (Oxford). A Public Lecture was delivered by Tom Körner (Cambridge). The twelve morning speakers were Jürgen Berndt (UCC), Tony Carbery (Edinburgh), Rod Gow (UCD), Martin Kilian (UCC), Ian Leary (Ohio), Tom Laffey (UCD), Martin Mathieu (Queen's University Belfast), Éamonn O'Brien (Auckland), Lars Olsen (St Andrews), Hinke Osinga (Bristol), Reidun Twarock (York) and Dominic Welsh (Oxford).

There were special sessions on *Computational Algebra* led by Eamonn O'Brien (Auckland) and Goetz Pfeiffer (Galway); on *Analysis* led by David Preiss (Warwick) and Ray Ryan (Galway); and on *Mathematics Education Research* led by Ken Houston (Belfast) and Rachel Quinlan (Galway). A large number of splinter groups and a postgraduate conference completed the programme.

All details, including titles and abstracts of talks, can be found at

`http://www.maths.nuigalway.ie/bmc2009/`

The meeting was supported by the London Mathematical Society, Science Foundation Ireland and the Irish Mathematical Society.

A Remark on the Global Lipschitz Regularity of Solutions to Inner Obstacle Problems Involving Degenerate Functionals of p-Growth

MARTIN FUCHS

Abstract. We extend some recent results of Jagodziński, Olek and Szczepaniak [*Irish Math. Soc. Bull.* **61** (2008), 15–27] on the Lipschitz character of solutions to inner obstacle problems associated to a uniformly elliptic operator to the case of nonlinear, degenerate operators.

In a recent paper Jagodziński, Olek and Szczepaniak [6] investigated the Lipschitz regularity of solutions to so-called inner obstacle problems extending earlier work of Jordanov [7]. By definition we are confronted with an inner obstacle problem if one or several side conditions imposed on the comparison functions are required to hold only on certain specified subregions of the domain of definition, where in case of several obstacles from above and/or below some natural conditions for compatibility have to be satisfied. The basic ideas for these kind of obstacle problems with obstacles defined only on a portion of Ω are explained in the textbook of Kinderlehrer and Stampacchia [8] (see pp. 137–139). The purpose of our short note now is the analysis of the inner obstacle problem for the p-energy functional

$$I[u,\Omega] := \frac{1}{p}\int_\Omega |\nabla u|^p \, dx \tag{1}$$

with arbitrary exponent $p \in (1,\infty)$. Of course we could also consider the variational inequality associated to the operator

$$Lu := -\sum_{i,j=1}^{n} \partial_i \left(A(x,\nabla u)a_{ij}(x)\partial_j u\right) ,$$

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Key words and phrases. Inner obstacle problems, degenerate functionals, Lipschitz regularity of minimizers.

where $a_{ij}(x)$ are smooth elliptic coefficients and where we have abbreviated $A(x,\nabla u) := (\sum\limits_{\alpha,\beta=1}^{n} a_{\alpha\beta}(x)\partial_\alpha u \partial_\beta u)^{\frac{p}{2}-1}$, but this would not lead to a deeper insight. Note that in the above cited papers the case $p=2$ is considered. For simplicity we also just discuss the case of one inner obstacle (from below): in the presence of several inner constraints the arguments of [6] have to be modified in an obvious way, which in particular means that we have to impose the same natural assumptions of compatibility on the functions acting as side conditions and on their domains of definitions as done in [6].

Next we give a precise formulation of our hypotheses concerning the data: let Ω denote a bounded, open set in $\mathbb{R}^n$ whose boundary can locally be represented as a graph of a function with Hölder continuous derivatives. Suppose further that ω is an open subset of Ω with $\partial\omega$ being of the same regularity as $\partial\Omega$ and such that $\overline{\omega} \subset \Omega$. Let us consider a function $\Psi \in C^{1,\alpha_1}(\overline{\omega})$ for some $\alpha_1 \in (0,1)$ and define the class of comparison functions

$$\mathbb{K} := \{w \in \overset{\circ}{W}{}^1_p(\Omega) : w \geq \Psi \quad \text{a.e. on } \omega\}\,, \tag{2}$$

where $\overset{\circ}{W}{}^1_p(\Omega)$ is the usual Sobolev space of functions vanishing on $\partial\Omega$ as introduced for example in [1]. Then we have following result:

Theorem 1. *The inner obstacle problem* $I[\,\cdot\,,\Omega] \to \min$ *in* $\mathbb{K}$ *with* $I[\,\cdot\,,\Omega]$ *and* $\mathbb{K}$ *being defined in* (1) *and* (2) *admits a unique solution* $u \in \mathbb{K}$. *The function u is globally Lipschitz, moreover we have* $u \in C^{1,\alpha}(\overline{\Omega} - (\partial\omega)_\varepsilon)$ *for some* $\alpha \in (0,1), (\partial\omega)_\varepsilon$ *denoting the set* $\{x \in \Omega : \operatorname{dist}(x,\partial\omega) < \varepsilon\}$.

Proof. Since $\mathbb{K} \neq \emptyset$, the existence and the uniqueness of a minimizer u is immediate, and clearly u is the solution of the variational inequality

$$\int\limits_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla(v-u)\,dx \geq 0 \quad \forall v \in \mathbb{K}\,. \tag{3}$$

As done in [6] we will reduce (3) to a global obstacle problem with a suitable constraint $\widetilde{\Psi} : \Omega \to \mathbb{R}$. To this purpose consider the minimization problem

$$I[w, \Omega - \overline{\omega}] \to \min,\ w \in \mathcal{C}\,, \tag{4}$$

where $\mathcal{C} := \{w \in W_p^1(\Omega - \overline{\omega}) : w|_{\partial\Omega} = 0 \text{ and } w|_{\partial\omega} = \Psi\}$, and let h denote the unique solution of (4). We further define

$$\widetilde{\Psi} := \left\{ \begin{array}{ll} h & \text{on } \Omega - \omega\,, \\ \Psi & \text{on } \omega \end{array} \right\} \in \overset{\circ}{W}{}_p^1(\Omega)$$

and introduce the "global" class $\widetilde{\mathbb{K}} = \{w \in \overset{\circ}{W}{}_p^1(\Omega) : w \geq \widetilde{\Psi} \text{ a.e. on } \Omega\}$ as well as the "global" problem

$$I[\,\cdot\,, \Omega] \to \min \ \text{ in } \widetilde{\mathbb{K}}\,. \tag{5}$$

If $\widetilde{u} \in \mathbb{K}$ denotes the unique solution of (5), we claim the validity of

$$u = \widetilde{u}\,. \tag{6}$$

In fact, $\widetilde{u}$ is admissible in (3), i.e. we have

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla(\widetilde{u} - u)\, dx \geq 0\,. \tag{7}$$

(5) is equivalent to the variational inequality

$$\int_\Omega |\nabla \widetilde{u}|^{p-2} \nabla \widetilde{u} \cdot \nabla(w - \widetilde{u})\, dx \geq 0 \quad \forall w \in \widetilde{\mathbb{K}}\,. \tag{8}$$

We like to insert u into (8), which means that we have to check that

$$u \geq h \quad \text{on } \Omega - \omega \tag{9}$$

holds. From (3) it follows

$$\int_{\Omega - \overline{\omega}} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi\, dx \geq 0 \quad \forall \varphi \in \overset{\circ}{W}{}_p^1(\Omega - \overline{\omega}), \varphi \geq 0\,, \tag{10}$$

whereas we get from (4)

$$\int_{\Omega - \overline{\omega}} |\nabla h|^{p-2} \nabla h \cdot \nabla \varphi\, dx = 0 \quad \forall \varphi \in \overset{\circ}{W}{}_p^1(\Omega - \overline{\omega})\,. \tag{11}$$

The function $\varphi := \max\{h - u, 0\}$ is in the space $\overset{\circ}{W}{}_p^1(\Omega - \overline{\omega})$ and for this choice of φ (10) and (11) imply $\int_M [|\nabla u|^{p-2} \nabla u - |\nabla h|^{p-2} \nabla h] \cdot \nabla(h - u)\, dx \geq 0$, $M := (\Omega - \overline{\omega}) \cap [h > u]$, which by the coercivity of the field $\mathbb{R}^n \ni \xi \mapsto |\xi|^{p-2}\xi$ immediately gives $\nabla \varphi = 0$ a.e. on $\Omega - \overline{\omega}$, i.e., $\varphi = 0$ on this set, so that (9) follows. But then we combine (7) with (8) choosing $w = u$ and arrive at (6).

From the works of e.g. Evans [5], Di Benedetto [3], Lieberman [9], Manfredi [11, 12] and Tolksdorf [14] we deduce that the solution h of problem (4) is of class $C^{1,\alpha}(\overline{\Omega}-\omega)$ for some $\alpha \in (0,1)$, in particular there is a finite constant K such that

$$|\nabla h| \leq K \quad \text{a.e. on } \Omega - \overline{\omega}\,. \tag{12}$$

According to our assumptions $|\nabla\Psi|$ is in the space $L^\infty(\omega)$ so that on account of (12) $\widetilde{\Psi}$ is Lipschitz on the whole domain Ω. (Note that we can not guarantee the validity of $\nabla h = \nabla\Psi$ on $\partial\omega$, which means that we do not know if $\widetilde{\Psi}$ is in $C^{1,\alpha}(\overline{\Omega})$.) This is enough to apply Theorem 1.2 of [2] with the result that the solution $\widetilde{u}$ of (5) and thereby u (recall (6)) is locally Lipschitz in Ω. The $C^{1,\alpha}$- regularity of u on the sets $\overline{\Omega}-(\partial\omega)_\varepsilon$ is consequence of the works of e.g. Choe and Lewis [4], Lieberman [10] or Mu and Ziemer [13]. This completes the proof of the Theorem, since obviously $|\nabla u| \in L^\infty_{\text{loc}}(\Omega)$ together with $u \in C^{1,\alpha}(\overline{\Omega}-(\partial\omega)_\varepsilon)$ implies $|\nabla u| \in L^\infty(\Omega)$. □

Remark 1. If we replace the energy from (1) by a more general functional $J[w,\Omega] := \int_\Omega f(\nabla w)\,dx$, where f should at least satisfy the hypotheses of Theorem 1.2 in [2], then again the global Lipschitz regularity of the minimizer will follow as soon as we can guarantee the global boundedness of $|\nabla h|$, h being the solution of problem (4) now formulated for the functional J.

Remark 2. If we assume that $\Psi \geq 0$ on $\overline{\omega}$ and if we consider the discontinuous obstacle

$$\hat{\Psi} := \begin{cases} \Psi & \text{on } \omega\,, \\ 0 & \text{on } \Omega-\omega\,, \end{cases}$$

then it is easy to see that the solution u of the problem

$$I[\,\cdot\,,\Omega] \to \min \text{ in } \mathbb{K}$$

coincides with the unique solution $\hat{u}$ of

$$\begin{aligned} &I[\,\cdot\,,\Omega] \to \min \text{ in } \hat{\mathbb{K}}\,, \\ &\hat{\mathbb{K}} := \{w \in \overset{\circ}{W}{}^1_p(\Omega) : w \geq \hat{\Psi} \quad \text{a.e. on } \Omega\}\,. \end{aligned}$$

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Martin Fuchs,
Department of Mathematics,
Saarland University,
P.O. Box 15 11 50,
D-66041 Saarbrücken, Germany
fuchs@math.uni-sb.de

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Applications of Prüfer Transformations in the Theory of Ordinary Differential Equations

GEORGE CHAILOS

Abstract. This article is a review article on the use of Prüfer Transformations techniques in proving classical theorems from the theory of Ordinary Differential Equations. We consider self-adjoint second order linear differential equations of the form

$$\mathcal{L}x = (p(t)x'(t))' + g(t)x(t) = 0,\ t \in (a,b). \qquad (\star)$$

We use Prüfer transformation techniques (which are a generalization of Poincaré phase-plane analysis) to obtain some of the main theorems of the classical theory of linear differential equations. First we prove theorems from the Oscillation Theory (Sturm Comparison theorem and Disconjugacy theorems). Furthermore we study the asymptotic behavior of the equation $(\star)$ when $t \to \infty$ and we obtain necessary and sufficient conditions in order to have bounded solutions for $(\star)$. Finally, we consider a certain type of regular Sturm–Liouville eigenvalue problems with boundary conditions and we study their spectrum via Prüfer transformations.

1. Introduction

In this review article we will present main theorems related to the study of the solutions of self-adjoint second order linear Differential Equations of the form

$$\mathcal{L}x = (p(t)x'(t))' + g(t)x(t) = 0,\ t \in (a,b), \qquad (1.1)$$

where $p(t) > 0$, $p(t)$ is absolutely continuous and $g(t) \in L^1(a,b)$ where a, b are elements in the extended real line. For this, we will develop and use the so called "Prüfer transformations" which are (roughly speaking) a generalization of the Poincaré phase plane

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analysis. The "Prüfer transformations" are general polar coordinate representations of the solutions of (1.1). The most common Prüfer transformation is

$$\begin{cases} x(t) = r(t)\sin\Theta(t) \\ x'(t) = \frac{r(t)}{p(t)}\cos\Theta(t). \end{cases} \tag{1.2}$$

Substituting (1.2) to (1.1) we obtain the Prüfer system

$$\begin{cases} r'(t) = (\frac{1}{p(t)} - g(t))r\sin\Theta\cos\Theta \\ \Theta'(t) = \frac{1}{p(t)}\cos^2\Theta + g(t)\sin^2\Theta. \end{cases} \tag{1.3}$$

In the second section (Oscillation Theory) we will use the transformed system (1.3) in order to prove main theorems from the Oscillation theory, like the Sturm Comparison Theorem, the Oscillation theorem, and Disconjugacy theorems. In the third section (Bounds of Solutions and Asymptotic Behavior) we will use the Prüfer transformation and the modified Prüfer transformation in order to study the asymptotic behavior of the equation (1.1) when $t \to \infty$ (without considering that $g(t) \in L^1(m)$). Moreover we will prove necessary and sufficient conditions in order to have bounded solutions for (1.1). Finally, in the last section (Spectral Theory) we will consider the regular Sturm–Liouville eigenvalue problem with boundary conditions,

$$\begin{cases} (p(t)x'(t))' + (\lambda r(t) - q(t))x(t) = 0, \quad t \in [a,b], \lambda \neq 0 \\ Ax(a) - Bx'(a) = 0 \\ \Gamma x(b) - \Delta x'(b) = 0. \end{cases} \tag{1.4}$$

We will use the Prüfer transformation to prove that there is an infinite number of eigenvalues of (1.4) forming a monotone increasing sequence with $\lambda_n \to \infty$, and that the eigenfunctions Φ_n corresponding to the eigenvalues λ_n have exactly n zeros in (a,b). Moreover, we will use the Prüfer transformed system to derive upper and lower bounds for the spectrum of (1.4).

At the end we present a list of references which were used in this article. The reader may refer to them for proofs that are not included in this paper.

I would like to note that I am particularly in debt to Professor D. Hinton for his constant willingness to discuss each step of this paper.

2. Oscillation Theory

In this section we will apply the Prüfer transformation on regular Sturm–Liouville problems in order to prove the Sturm Comparison theorem, the Oscillation theorem and "Disconjugacy" theorems.

Consider the equation of the form

$$\mathcal{L}x = (p(t)x')' + g(t)x = 0, \ \ t \in (a, b). \tag{2.1}$$

(Note that the equation $x'' + f(t)x' + h(t)x = 0$ can be transformed in the form of (2.1) by multiplying it with $e^{\int_0^t f(s)ds}$.) We assume that $p(t) > 0$ with p absolutely continuous and $g \in L^1(m)$.

In (2.1) we consider the substitution $y = p(t)x'$. From (2.1),

$$x' = \frac{y}{p(t)}, \qquad y' = -g(t)x. \tag{2.2}$$

If we use polar coordinates, $x = r(t)\sin\theta(t)$, $y = r(t)\cos\theta(t)$ on (2.2), and solve for r', θ', then we obtain the Prüfer system

$$r'(t) = \left(\frac{1}{p(t)} - g(t)\right) r \sin\theta\cos\theta \tag{2.3}$$

$$\theta'(t) = \frac{1}{p(t)}\cos^2\theta + g(t)\sin^2\theta. \tag{2.4}$$

In the sequel we use the above transformed system to prove the following theorems related to the solutions of (2.1). The first two results are from [6].

Theorem 2.5 (Oscillation Theorem)**.** *Suppose p_i', g_i are piecewise continuous functions in $[a, b]$, and $\mathcal{L}_i x = (p_i x')' + g_i x = 0$, $i = 1, 2$. Let $0 < p_2(t) \le p_1(t)$, $g_2(t) \ge g_1(t)$. If $\mathcal{L}_1\phi_1 = 0, \mathcal{L}_2\phi_2 = 0$, where ϕ_i are solutions of $\mathcal{L}_i x$, and $\omega_2(a) \ge \omega_1(a)$, where ω_i are solutions of (2.4), then $\omega_2(t) \ge \omega_1(t)$ $\forall t \in (a, b)$ (1). Moreover, if $g_2(t) > g_1(t), t \in (a, b)$, then $\omega_2(t) > \omega_1(t)$, $\forall t \in (a, b]$ (2).*

Proof. From (2.4), $\omega_i' = \frac{1}{p_i}\cos^2\omega_i + g_i\sin^2\omega_i$, $i = 1, 2$. We have

$$(\omega_2 - \omega_1) = (g_1 - \frac{1}{p_1})(\sin^2\omega_2 - \sin^2\omega_1) + h \quad (3),$$

where

$$h = \left(\frac{1}{p_1} - \frac{1}{p_2}\right)\cos^2\omega_2 + (g_2 - g_1)\sin^2\omega_2.$$

Note that $h \ge 0$.

If $\omega_2 - \omega_1 = u$, then by (3), $u' = fu + h$, where

$$f = \left(g_1 - \frac{1}{p_1}\right)(\sin\omega_1 + \sin\omega_2)\frac{\sin\omega_2 - \sin\omega_1}{\omega_2 - \omega_1}.$$

Hence, f is a piecewise continuous and uniformly bounded function. Since $h \geq 0$, $u' - fu \geq 0$. Set $F(t) = \int_t^a f(s)\,ds$. Then $e^F u' + F' e^F u \geq 0$, and by integrating this,

$$e^{F(t)}u(t) \geq e^{F(a)}u(a) \geq 0 \quad (4).$$

Now it is easy to see that the above proves (1).

Now suppose that (2) fails to hold. We show that there exist some $c > a$ such that $\omega_2(t) = \omega_1(t)$ $(a \geq t \geq c)$ (5).
Suppose not, then by (1) there exists a sequence $\{t_j\}_{j=1}^n$ such that a is a limit point of it with $\omega_2(t_j) > \omega_1(t_j)$, $j = 1 \dots n$. Now using (4) with a replaced by t_j, it follows that for $t > t_j$, $\omega_2(t) > \omega_1(t)$, $j = 1 \dots n$. With t_j arbitrarily close to a we have that (2) implies (5). This leads to a contradiction.

Using (5), (3) is true with $g_2 > g_1$ only if $\omega_2 = \omega_1 = 0$ $(mod\ \pi)$, and if $p_1 = p_2$ in (a,c). However, since ω_i, $i = 1 \dots n$, are solutions of (2.4), the case $\omega_1 = \omega_2 = 0$ $(mod\ \pi)$ in (a,c) is impossible. This proves (2) if $g_1 > g_2$, and concludes the proof of the theorem. □

Theorem 2.6 (Sturm Comparison)**.** *Suppose ϕ is a real solution of $(px')' + g_1 x = 0$ and ψ is a real solution of $(px')' + g_2 x = 0$, where $x \in (a, b)$. Let $g_1(t) > g_2(t)$ in (a,b). If t_1, t_2 are successive zeros of ϕ in (a,b), then ψ must vanish in some point in (t_1, t_2).*

Proof.

Claim 2.7. *$\phi(t)$ can vanish only where $\omega(t) = k\pi$, $k \in \mathbb{Z}$ (where $\omega(t)$ is a solution of (2.4).)*

Proof. For a solution ϕ of (2.1) there is a solution $r = \varrho(t), \theta = \omega(t)$ of (2.3), (2.4) respectively, where $\varrho^2 = (p\phi')^2 + \phi^2$, $\omega = \arctan\left(\frac{\phi}{p\phi'}\right)$. Since ϕ and ϕ' do not vanish simultaneously, it follows that $\varrho^2(t) > 0$, and without loss of generality we can assume that $\varrho(t) > 0$. A consequence of this is that $\phi(t) = \varrho(t)\sin\omega(t)$ can vanish only where $\omega(t) = k\pi$, $k \in \mathbb{Z}$. □

Now since $\cos^2\theta$, $\sin^2\theta$ are uniformly bounded, (2.4) has a solution over any interval on which $p > 0$ and p, g are piecewise continuous functions (Picard Theorem). Since the right hand side of (2.4) is differentiable in θ, it follows that the solution is unique in the usual sense. Now the proof of the theorem follows directly from the claim, the monotonicity of $\omega(t)$, and Theorem 2.5. $\square$

The following theorem is from [3].

Theorem 2.8 (Disconjugacy)**.** *Consider the problem*

$$\mathcal{L}x = (p(t)x'(t))' + g(t)x(t) = 0,$$

where $t \in [a, \infty)$, $x(a) = 0$, and without loss of generality $x'(a) > 0$. If $p(t), g(t)$ are continuous in $[a, \infty)$, and $\int_a^\infty \left(\frac{1}{p(t)} + |g(t)|\right) dt \le \pi$ with $p(t) > 0$, $t \in [a, \infty)$, then no nontrivial solution of $\mathcal{L}x = 0$ has two zeros in $[a, \infty)$.

Proof. Recall that $\theta'(t) = \frac{1}{p(t)} \cos^2\theta + g(t)\sin^2\theta$, $\theta(a) = 0$, and integrate it to obtain

$$\theta(s) = \int_a^s \left(\frac{1}{p(t)} \cos^2\theta(t) + g(t)\sin^2\theta(t)\right) dt \le \int_a^s \left(\frac{1}{p(t)} + |g(t)|\right) dt.$$

Hence,

$$\theta(s) < \int_a^\infty \left(\frac{1}{p(t)} + |g(t)|\right) dt \le \pi. \tag{2.9}$$

Since $\theta'(a) > 0$, $\theta(t) < \int_a^\infty \left(\frac{1}{p(t)} + |g(t)|\right) dt \le \pi$, $t \in [a, \infty)$. Since the zeros of $\mathcal{L}x = 0$ occur when $\theta(t) = k\pi$, $k \in \mathbb{Z}$, the above inequality proves the theorem. $\square$

Theorem 2.10. *Consider the problem $\mathcal{L}x = (p(t)x'(t))' + g(t)x(t) = 0$, $x(a) = 0$ $(x'(a) > 0)$, where $g(t) < 0$, $\forall t \in [a, \infty)$. Then the nontrivial solution of $\mathcal{L}x = 0$ has at most one zero in $[a, \infty)$.*

Proof. We know that $\theta(a) = 0$ and $\theta'(a) > 0$. Now the monotonicity of $\theta(t)$ implies that for some $b \in (a, \infty)$, $\theta(b) = \frac{\pi}{2}$. Since $\theta'(b) = \frac{1}{p(t)} \cos^2\theta(b) + g(t)\sin^2\theta(b) = g(b)$, we get that $0 < \theta(t) < \frac{\pi}{2}$ in (a, ∞). Therefore in (a, ∞) there are no zeros of any nontrivial solution of $\mathcal{L}x = 0$; since if they were any, they would occur at $\theta(t) = k\pi$. This concludes the proof. $\square$

In the following theorem, where its proof is taken from [9], we will make use of a modified Prüfer transformation in order to give an important result about the distance between two successive zeros of a fixed nontrivial solution of the equation

$$x''(t) + p_1(t)x'(t) + p_2(t)x(t) = 0, \tag{2.11}$$

where $p_1(t), p_2(t)$ are piecewise continuous, real valued functions in a closed interval.

Theorem 2.12. *Let a and b be consecutive zeros of a fixed nontrivial solution of (2.11), and let γ be a differentiable function defined on $[a,b]$. Set*

$$M_1 \equiv \sup_{a\le t\le b}(|2\gamma(t)-p_1(t)|),$$

$$M_2 \equiv \sup_{a\le t\le b}(|\gamma'(t)-p_2(t)-\gamma^2(t)+p_1(t)\gamma(t)|).$$

Then

$$b-a \ge 2\int_0^{\infty} \frac{ds}{1+M_1 s+M_2 s^2}.$$

Proof. Let x denote the solution referred to the statement of the theorem. Without loss of generality assume that $x(t)>0\ \forall t\in(a,b)$, and that $x'(a)>0,\ x'(b)<0$. Define the real valued functions R and Θ by the relations:

$$R\sin\Theta = x \quad (1)$$

$$R\cos\Theta = x'+\gamma x \quad (2),$$

where $R(t)>0$, $\Theta(t)\in[0,\pi]$, $t\in[a,b]$. We differentiate (1), (2) and substitute into (2.11). Then

$$R'\cos\Theta - R\Theta'\sin\Theta = R(\gamma-p_1)\cos\Theta + R(\gamma'-p_2-\gamma^2+p_1\gamma)\sin\Theta \quad (3)$$

$$R'\sin\Theta + R\Theta'\cos\Theta = R\cos\Theta - R\gamma\sin\Theta \quad (4).$$

We eliminate R' from (3),(4), and we have

$$\Theta' = \cos^2\Theta - (2\gamma-p_1)\sin\Theta\cos\Theta + (\gamma'-p_2-\gamma^2+p_1\gamma)\sin^2\Theta \quad (5).$$

From (1) we observe that the zeros of x occur when $\Theta(t)=k\pi$, and from (5) we note that Θ is increasing at $k\pi$ since $\Theta'(k\pi)=1$. Hence we can suppose that $\Theta(a)=0$ and $\Theta(b)=\pi$. We use the intermediate value theorem to obtain $t\in(a,b)$ such that $\Theta(t)=\frac{\pi}{2}$. Now let α denote the least such t. For $t\in(a,\alpha)$, $\sin\Theta$ and $\cos\Theta$ are both positive. Now we use (5) to get

$$|\Theta'| \le \cos^2\Theta + M_1\sin\Theta\cos\Theta + M_2\sin^2\Theta,$$

and so

$$\alpha - a \geq \int_o^{\frac{\pi}{2}} \frac{d\Theta}{\cos^2\Theta + M_1 \sin\Theta\cos\Theta + M_2 \sin^2\Theta} = \int_0^\infty \frac{ds}{1 + M_1 s + M_2 s^2} \quad (6).$$

Similarly, if β denotes the largest $t \in (a, b)$ such that $\Theta(t) = k\pi$, then

$$b - \beta \geq \int_0^\infty \frac{ds}{1 + M_1 s + M_2 s^2} \quad (7).$$

By combining (6) and (7) we have

$$b - a \geq 2 \int_0^\infty \frac{ds}{1 + M_1 s + M_2 s^2}. \qquad \square$$

By choosing appropriate values for γ and imposing certain conditions on $p_1(t)$ and $p_2(t)$, we can derive some very remarkable results.

Corollary 2.13. *If* $p_1 \equiv 0$, *then* $\sup_{a \leq t \leq b} |\int_a^t p_2(s)ds| \geq \frac{2}{b-a}$.

Proof. Set $\gamma \equiv \int_a^t p_2(s)ds$. Thus,

$$M_1 = 2 \sup_{a \leq t \leq b} \left| \int_a^t p_2(s)ds \right|,$$

$$M_2 = \sup_{a \leq t \leq b} \left| \int_a^t p_2(s)ds \right|^2.$$

Let $M = \sup_{a \leq t \leq b} |\int_a^t p_2(s)ds|$. By Theorem 2.12 we have

$$b - a \geq 2 \int_0^\infty \frac{dt}{1 + 2Mt + M^2 t^2} = \frac{2}{M}. \qquad \square$$

Corollary 2.14. *If* $p_1(t)$ *is differentiable then*

$$\sup_{a \leq t \leq b} \left| \frac{p_1'}{2} - p_2 + \frac{p_1^2}{4} \right| \geq \frac{\pi^2}{(b-a)^2}.$$

Proof. Choose $\gamma \equiv \dfrac{p_1}{2}$. Then $M_1 = 0$ and

$$M_2 = \sup_{a \leq t \leq b} \left| \frac{p_1'}{2} - p_2 + \frac{p_1^2}{4} \right|.$$

From Theorem 2.12,

$$b - a \geq 2 \int_o^\infty \frac{dt}{1 + M_2 t^2} = \pi \frac{1}{\sqrt{M_2}}. \qquad \square$$

Similarly we can prove that if p_1 is differentiable, then

$$\sup_{a \leq t \leq b} \left| \frac{p_1(a)}{2} - p_1(t) + \int_a^t \left(p_2(s) - \frac{p_1(s)^2}{4} \right) ds \right| \geq \frac{2}{b-a}.$$

(To see the above, choose $\gamma \equiv \frac{p_1(t)}{2} + \int_a^t (p_2 - \frac{p_1'}{2} - \frac{p_1^2}{4}) ds$ and apply Theorem 2.12.)

3. Bounds of Solutions and Asymptotic Behavior

In this section we will use the Prüfer transformation in order to study the asymptotic behavior of solutions of the equation

$$\mathcal{L}x = (p(t)x')' + g(t)x = 0 \ \text{ when } \ t \to \infty,$$

and we will prove that every solution of $\mathcal{L}x = 0$ is bounded if

$$\int_a^\infty \left| \frac{1}{p(t)} - g(t) \right| \, dt < \infty.$$

In the proof of the following theorem we will use Gronwall's Lemma.

Lemma 3.1 (Gronwall's Lemma)**.** *If u, v are real valued nonnegative functions in $L_1(m)$ with domain $\{t : t \geq t_0\}$, and if there exists a constant $M \geq 0$ such that for every $t \geq t_0$*

$$u(t) \leq M + \int_{t_0}^t u(s)v(s) \, ds,$$

then

$$u(t) \leq M \exp\left(\int_{t_0}^t v(s) \, ds \right).$$

The following theorem is from [3].

Theorem 3.2. *Every solution $x(t)$ of $\mathcal{L}x = 0$ satisfies the inequality*

$$|x(t)| \leq K \exp\left[\frac{1}{2} \int_a^t \left| \frac{1}{p(s)} - g(s) \right| \, ds \right],$$

where $t \in (a, \infty)$ and $K = \sqrt{x^2(a) + (p(a)x'(a))^2}$. Moreover, if

$$\int_a^\infty \left| \frac{1}{p(t)} - g(t) \right| \, dt < \infty,$$

then every solution of $\mathcal{L}x = 0$ is bounded.

Proof. We will use once more the transformed system

$$r'(t) = \left[\frac{1}{p(t)} - g(t)\right] r(t) \sin\Theta(t)\cos\Theta(t) \quad (1)$$

$$\Theta'(t) = \frac{1}{p(t)}\cos^2\Theta(t) + g(t)\sin^2\Theta(t) \quad (2),$$

where $x(t) = r(t)\sin\Theta(t)$, to conclude that

$$|x(t)| \leq |r(t)| \quad (3).$$

From (1) we have that

$$r(s) - r(a) = \int_a^s \left[\frac{1}{p(t)} - g(t)\right] r(t)\frac{1}{2}\sin 2\Theta(t)\,dt, \quad s \in (a,\infty).$$

Hence,

$$|r(t)| \leq r(a) + \frac{1}{2}\int_a^s \left|\frac{1}{p(t)} - g(t)\right| |r(t)|\,dt.$$

We apply Gronwall's Lemma in the above inequality and we get

$$|r(s)| \leq r(a)\exp\left(\frac{1}{2}\int_a^s \left|\frac{1}{p(t)} - g(t)\right|\,dt\right),$$

and so from (3),

$$|x(t)| \leq r(a)\exp\left(\frac{1}{2}\int_a^t \left|\frac{1}{p(s)} - g(s)\right|\,ds\right).$$

Now observe that if $\int_a^\infty |\frac{1}{p(t)} - g(t)|\,dt < \infty$, then

$$\exp\left[\frac{1}{2}\int_a^\infty |\frac{1}{p(t)} - g(t)|\,dt\right] \in \mathbb{R}.$$

Thus if $M = r(a)\exp(\frac{1}{2}\int_a^\infty |\frac{1}{p(t)} - g(t)|\,dt)$, then $M \in \mathbb{R}$ and $|x(t)| \leq M$. This shows that $x(t)$ is bounded. □

Now we will study the asymptotic behavior of the solutions of the equation

$$x''(t) + (1 + g(t))x(t) = 0, \tag{3.3}$$

where if x_0 is a fixed real number that is sufficiently "small" for large values of x, then $g(t)$ is a real continuous function for $x \geq x_0$. Now observe that (3.3) is of the standard form $\mathcal{L}x = (x'p)' + gx = 0$, where $p(t) \equiv 1,\ g(t) \mapsto g(t)+1$. In (3.3) we will use a modified Prüfer transformation by substituting $\Theta(t)$ with $\Theta(t)+1$. The transformed equations are

$$r'(t) = -g(t)r(t)\sin(t+\Theta(t))\cos(t+\Theta(t))$$
$$(t+\Theta(t))' = 1 + g(t)\sin^2(t+\Theta(t)),$$

which yield to

$$\frac{r'(t)}{r(t)} = -\frac{1}{2}g(t)\sin 2(t+\Theta(t)), \tag{3.4}$$

$$\Theta'(t) = \frac{1}{2}g(t)(1-\cos 2(t+\Theta(t)). \tag{3.5}$$

Using the above transformed system we will prove the following theorem which asserts that the fundamental system of solutions x_1, x_2 of (3.3) when $t \to \infty$ is

$$\begin{cases} x_1(t) = \cos(t) + o(1) & x_2(t) = \sin(t) + o(1) \\ x_1'(t) = -\sin(t) + o(1) & x_2' = \cos(t) + o(1). \end{cases} \tag{3.6}$$

The following theorem is from [8].

Theorem 3.7 (Asymptotic Behavior). *Let $g(t)$ be a real continuous function for $t \geq t_0$, where t_0 is a fixed real number, and assume that the following integrals exist:*

$$\begin{cases} \int_t^\infty g(s)\,ds,\ g_1(s) = \int_t^\infty g(s)\cos(2s)\,ds \\ g_2(t) = \int_t^\infty g(s)\sin(2s)\,ds,\ \int_{t_0}^\infty |g(t)g_j(t)|\,dt, \quad j = 1,2. \end{cases} \tag{3.8}$$

Then the equation $x'' + (1+g(t)x) = 0$ has a fundamental system of solutions satisfying (3.6).

Note that the above assumptions are certainly satisfied if $g \in L[t_0, \infty]$.

Proof. Step 1: We will show that for any nontrivial solution of (3.3) the corresponding $\Theta(t)$, $r(t)$, given by (3.4) and (3.5) respectively, tend to finite limits as $t \to \infty$. By using

$$g(t)\cos(2t)\cos(2\Theta) = -(g_1\cos 2\Theta)' - 2g_1\Theta'\sin 2\Theta$$

and

$$g(t)\sin(2t)\sin(2\Theta) = -(g_2\sin 2\Theta)' - 2g_2\Theta'\cos 2\Theta,$$

(3.5) can be written as:

$$\Theta'(t) = \frac{1}{2}g + \frac{1}{2}(g_1\cos 2\Theta)' - \frac{1}{2}(g_2\sin 2\Theta)' + g_1\Theta'\sin 2\Theta + g_2\Theta'\cos 2\Theta. \tag{3.9}$$

Since by (3.5) $|\Theta'| \leq |g|$, it follows from the hypothesis of the theorem that Θ' is integrable over $[x_0, \infty)$. Now from (3.9) we have

$$|\Theta'(t)| = \frac{1}{2}|g| + \frac{1}{2}\,|(g_1 \cos 2\Theta)'| + \frac{1}{2}\,|(g_2 \sin 2\Theta)'| + |g_1 g| + |g_2 g|.$$

From (3.8) and (3.9) we conclude that $\Theta(t)$ tends to a finite limit as $t \to \infty$. Similarly using the relations

$$g(t) \sin 2t \cos 2t = -(g_2 \cos 2\Theta)' - 2g_2 \Theta' \sin 2\theta$$

$$g(t) \cos 2t \sin 2t = -(g_1 \sin 2\Theta)' - 2g_1 \Theta' \cos 2\theta,$$

we can write (3.4) as

$$\frac{r'}{r} = (1/2)(g_2 \cos 2\Theta)' + (1/2)(g_1 \sin 2\Theta)' + g_2 \Theta' \sin 2\theta - g_1 \Theta' \cos 2\Theta. \tag{3.10}$$

Since $(\log r)' = \frac{r'}{r}$, from (3.4) we get $|(\log r)'| \leq |g|$, and so $(\log r)'$ is integrable over $[x_0, \infty]$. Now using (3.10) we obtain

$$|(\log r)'| \leq (1/2)|(g_2 \cos 2\Theta)'| + (1/2)|(g_1 \sin 2\Theta)'| + |g_1 g| + |g_2 g|,$$

and hence by (3.8), $\log r$ tends to a finite limit as $t \to \infty$. Therefore r has a positive (finite) limit. This concludes the proof of Step 1.

Step 2: Now we will show that two distinct solutions of (3.3) cannot tend to the same limit as $t \to \infty$. If we integrate (3.9), then by (3.5) we get

$$\Theta(t) = \Theta(\infty) + 1/2 \int_t^\infty g(s)\; ds + 1/2(g_1 \cos 2\Theta - g_2 \sin 2\Theta)$$
$$- 1/2 \int_t^\infty [g(g_1 \sin 2\Theta) + g_2 \cos 2\Theta)(1 - \cos 2(s + \Theta)]\; ds. \tag{3.11}$$

Now choose t_1 large enough such that $|g_1(t)| \leq 1/16$ for every $t \geq t_1$ and $\int_{t_1}^\infty |g_j g|\; ds \leq 1/16$, where $j = 1, 2$. If $\hat{\Theta}(t)$ is a solution of (3.5) with the same limit as Θ and $\Theta \neq \hat{\Theta}$, then if we subtract from (3.11) the corresponding relation with Θ replaced by $\hat{\Theta}$, we get that for $t \geq t_1$,

$$|\Theta(t) - \hat{\Theta}(t)| \leq 1/2 \sup_{s \geq t_1} |\Theta(s) - \hat{\Theta}(s)|.$$

Hence $\sup_{s \geq t_1} |\Theta(s) - \hat{\Theta}(s)| = 0$, and thus, $\Theta = \hat{\Theta}$. This is clearly a contradiction. Similarly we have $r(t) = \hat{r}(t)$. This conclude the proof of step 2.

In terms of (3.3), this means that for any nontrivial solution x there exist constants A, α, ($A > 0$, $0 \leq \alpha < 2\pi$) such that for $t \to \infty$

$$\lim_{t\to\infty} x(t) = A\sin(t+\alpha) + o(1),$$
$$\lim_{t\to\infty} x'(t) = A\cos(t+\alpha) + o(1).$$

Moreover, if x_1, x_2 are linearly independent solutions of (3.3), then the corresponding phase shifts α_1, α_2 cannot differ by an integer multiple of π. Consequently, by forming suitable combinations of x_1, x_2, we can obtain solutions with asymptotic behavior as it is described in (3.6). □

Examples 3.12. (1). Given the equation $y'' \pm ky = 0$, $k > 0$, from Theorem 2.6 with $p(t) = 1$, $g(t) = \pm k$, we get

$$|y(t)| \leq \sqrt{y^2(a) + (y')^2(a)}\,\exp\left(1/2|1 \pm k|\right).$$

(2). Consider the equation $(p(t)y')' + \frac{k}{p(t)}y = 0$, where $\frac{1}{p(t)} \in L^1(m)$ and $k > 0$. Then by Theorem 3.2, since

$$\int_a^\infty \left|\frac{1-k}{p(t)}\right| dt \leq |1-k| \int_a^\infty \frac{dt}{|p(t)|} < \infty,$$

we conclude that the solutions $y(t)$ are bounded.
(3). In this example we will illustrate an application of Theorem 3.7. We will study the asymptotic behavior of the equation

$$x''(t) + \left(1 + \frac{\sin 2t}{t}\right) x(t) = 0,\ t \geq 0,\ \lambda \in (\mathbb{Z} \setminus \{\pm 2\}).$$

Consider the function $g(t) = \frac{\sin \lambda t}{t}$, $\lambda \neq \pm 2$, $\lambda \in \mathbb{Z}$. Observe that $g(t) \notin L^1(m)$, but the hypothesis of the Theorem 3.7 are satisfied. Indeed,

$$\begin{aligned}
\int_a^\infty \left|\frac{\sin \lambda s}{s}\right| ds &\geq \int_a^\infty \frac{\sin^2 \lambda s}{|s|}\, ds \\
&= \int_a^\infty \frac{(1-\cos 2\lambda s)}{|2s|}\, ds \\
&= 1/2 \int_{2a}^\infty \frac{1-\cos u}{|u|}\, du \\
&= 1/2 \int_{2a}^\infty \frac{du}{u} - 1/2 \int_{2a}^\infty \frac{\cos u}{u}\, du. \qquad (3.13)
\end{aligned}$$

Moreover, $\int_{2a}^{\infty} \frac{du}{u} = \infty$ and $\int_{2a}^{\infty} \frac{\cos u}{u}\, du < \infty$, hence $\int_a^{\infty} |\frac{\sin \lambda s}{s}|\, ds = \infty$. This shows that $g(t) \notin L^1(m)$.

Observe that $\int_t^{\infty} g(s)\, ds = \int_t^{\infty} \frac{\sin \lambda s}{s}\, ds$ exists, since if $0 < s < t$,

$$\left| \int_s^t \frac{\sin \lambda s}{s}\, ds \right| \leq \left| \frac{\cos \lambda s}{\lambda s} - \frac{\cos \lambda t}{\lambda t} - \frac{1}{\lambda} \int_s^t \frac{\cos \lambda s}{s}\, ds \right|.$$

Thus,

$$\left| \int_s^t \frac{\sin \lambda s}{s}\, ds \right| \leq 1/|\lambda| \left(1/s + 1/t + \int_s^t \frac{1}{s^2}\, ds \right) = \frac{2}{|\lambda| s}.$$

Additionally,

$$g_1(t) = \int_t^{\infty} \frac{\sin \lambda s}{s} \cos 2s\, ds < \infty$$

(see [15], p.96 (15.34)) and

$$g_2(t) = \int_t^{\infty} \frac{\sin \lambda s}{s} \sin 2s\, ds < \infty$$

(see [15], p.96 (15.38)). Moreover, since $g^2 \in L^1(m)$, it is elementary to show that $\int_a^{\infty} |gg_j|\, dt < \infty$ for $a > 0,\ j = 1, 2$. This shows that the hypothesis of the theorem are satisfied, and thus the equation

$$x''(t) + (1 + \frac{\sin 2t}{t})x(t) = 0,\ t \geq 0,\ \lambda \in (\mathbb{Z} \setminus \{\pm 2\})$$

has a fundamental system of solutions satisfying (3.6).

Remark 3.14. If $\lambda = \pm 2$ we can easily see that

$$\int_a^{\infty} \frac{\sin^2 \lambda s}{s}\, ds = \int_a^{\infty} \frac{1 - \cos 4s}{2}\, ds = \int_a^{\infty} \frac{ds}{2} - \int_a^{\infty} \frac{\cos 4s}{2}\, ds,$$

where $\displaystyle\int_a^{\infty} \frac{ds}{2} = \infty$ and $\displaystyle\int_a^{\infty} \frac{\cos 4s}{2}\, ds < \infty \ \ \forall a \in [0, \infty)$. This shows that

$\displaystyle\int_a^{\infty} \frac{\sin^2 \lambda s}{s} ds = \infty$ and thus the integral $g_2(t) = \displaystyle\int_t^{\infty} \frac{\sin \lambda s}{s} \sin 2s\, ds$ does not exist and the Theorem 3.7 does not apply.

4. Spectral Theory

In this section we will consider the regular Sturm–Liouville eigenvalue problem with boundary conditions. We use once more "Prüfer transformation" techniques to obtain theorems concerning the spectrum of such problems.

Consider the system

$$\begin{cases} (p(t)x'(t))' + (\lambda r(t) - q(t))x(t) = 0, \ t \in [a,b], \\ Ax(a) - Bx'(a) = 0 \\ \Gamma x(b) - \Delta x'(b) = 0. \end{cases}$$

There is no loss of generality in assuming that $0 \leq |A| \leq 1$, $0 \leq B/p(a) \leq 1$ and $\frac{A^2+B^2}{p^2(a)} = 1$. This means that there is a unique constant α, $0 \leq \alpha \leq \pi$, such that the expression $Ax(a) - Bx'(a) = 0$ can be written as $(\cos\alpha)x(a) - (\sin\alpha)p(a)x'(a) = 0$. Similarly, there is a unique constant β, $0 \leq \beta \leq \pi$, such that $\Gamma x(b) - \Delta x'(b) = 0$ can be written as $(\cos\beta)x(\beta) - (\sin\beta)p(b)x'(b) = 0$. Hence the above system is equivalent to the following system

$$\begin{cases} (p(t)x'(t))' + (\lambda r(t) - q(t))x(t) = 0, \ t \in [a,b], \lambda \neq 0, \\ (\cos\alpha)x(a) - (\sin\alpha)p(a)x'(a) = 0 \\ (\cos\beta)x(\beta) - (\sin\beta)p(b)x'(b) = 0, \end{cases} \tag{4.1}$$

where λ is a real parameter and p', r, q are real and piecewise continuous functions in $[a,b]$ with $p > 0, r > 0$ in $[a,b]$. The values of λ for which the system (4.1) has a nontrivial solution are called eigenvalues and the corresponding (nontrivial) solutions, eigenfunctions.

Next, we present the most important theorem (which is taken from [4]) about the eigenvalues and the zeros of eigenfunctions of (4.1).

Theorem 4.2. *There is an infinite number of eigenvalues $\lambda_0, \lambda_1, \lambda_2, \ldots$ forming a monotone increasing sequence with $\lambda_n \to \infty$ as $n \to \infty$ of (4.1). Moreover, the eigenfunctions ϕ_n corresponding to λ_n have exactly n zeros in (a,b). Note that by Theorem 2.6 (Sturm Comparison) the zeros of ϕ_n separate those of ϕ_{n+1}.*

Proof. Let $\phi(t,\lambda)$ be the unique solution of the first equation of (4.1) which satisfies $\phi(a,\lambda) = \sin\alpha$, $\phi'(a,\lambda) = \cos\alpha$. Then ϕ satisfies the second equation of (4.1). Let $r(t,\lambda), \omega(t,\lambda)$ be the corresponding Prüfer transformations of $\phi(t,\lambda)$. The initial conditions are transformed to $r(a,\lambda) = 1, \omega(a,\lambda) = \alpha$. Eigenvalues are those values of

λ for which $\phi(t,\lambda)$ satisfies the third equation of (4.1). That is, are those values of λ for which $\omega(b,\lambda)=\beta+n\pi,\ n\in\mathbb{Z}$. By Theorem 2.5 (Oscillation) for any fixed $t\in[a,b]$, $\omega(t,\lambda)$ is monotone and increasing in λ. Note that $\omega(t,\lambda)=0\ (mod\ \pi)$ if and only if $\phi(t,\lambda)=0$. From $\theta'=\frac{1}{p}\cos^2\theta+(\lambda r-q)\sin^2\theta$ it is clear that $\theta'=\frac{1}{p}>0$ at a zero of ϕ, and hence $\omega(t,\lambda)$ is strictly increasing in a neighborhood of a zero.

Claim 4.3. *For any fixed $t=c,\ c\in[a,b],\ \lim_{\lambda\to\infty}\omega(c,\lambda)=\infty$.*

Proof. Since $\alpha\geq 0$ and since $\omega'>0$ for $\omega=0\ (mod\ \pi)$, $\omega(t,\lambda)\geq 0$. Thus it suffices to show that for some $t_0,\ \alpha<t_0<c$,

$$\lim_{\lambda\to\infty}[\omega(c,\lambda)-\omega(t_0,\lambda)]=\infty.$$

Let $t_0=\frac{a+b}{2}$, and P,Q,R be constants such that over (t_0,c), $p(t)\leq P,\ r(t)\geq R>0$ and $q(t)\leq Q$. Then the equation

$$Px''+(\lambda R-Q)=0 \tag{4.4}$$

with solution $\hat\phi$ satisfying $\hat\phi(t_0,\lambda)=\phi(t_0,\lambda)$, $\hat\phi'(t_0,\lambda)=\phi'(t_0,\lambda)$, has $\hat\omega(t_0,\lambda)=\omega(t_0,\lambda)$, and hence by Theorem 2.5

$$\omega(c,\lambda)-\omega(t_0,\lambda)\geq\hat\omega(c,\lambda)-\hat\omega(t_0,\lambda). \tag{4.5}$$

(4.4) implies that the successive zeros of $\hat\phi$ have spacing $\pi\sqrt{\frac{P}{\lambda R-Q}}$, and hence $\lim_{\lambda\to\infty}\pi\sqrt{\frac{P}{\lambda R-Q}}=0$. Then for any integer $j>1$, $\hat\phi$ will have j zeros between t_0 and c for λ large enough. Thus, $\hat\omega(c,\lambda)-\hat\omega(t_0,\lambda)\geq j\pi$. Since j is arbitrary, by (4.5), $\lim_{\lambda\to\infty}[\omega(c,\lambda)-\omega(t_0,\lambda)]=\infty$. This proves the claim. □

Claim 4.6. *For fixed $t=c,\ c\in(a,b]$, we have $\lim\limits_{\lambda\to-\infty}\omega(c,\lambda)=0$.*

Proof. We will use the equation $\theta'=\frac{1}{p}\cos^2\theta+(\lambda r-q)\sin^2\theta$. Choose $\delta>0$ sufficiently small such that $\alpha<\pi-\delta$. If $\delta\leq\omega\leq\pi-\delta$, $\lambda<0$, and if $0<P\leq p$, $0<R\leq r$, $Q\geq|q|$, then $\omega'<1/p-|\lambda|R\sin^2\delta+Q\leq-\frac{\alpha-\delta}{c-\alpha}<0$ whenever $\lambda<\left[\frac{\alpha-\delta}{\alpha-c}-Q-1/p\right]R\sin^2\delta<0$. Hence $\omega(c,\lambda)\leq\delta$ for $-\lambda$ sufficiently large. Since δ is arbitrary, $\lim\limits_{\lambda\to-\infty}\omega(c,\lambda)=0$, and this proves the claim. □

Now for $c=b$, $\lim_{\lambda\to-\infty}\omega(b,\lambda)=0$. Since $\beta>0$, and since $\omega(b,\lambda)$ is monotone and increasing in λ, it follows that there is a value $\lambda=\lambda_o$ for which $\omega(b,\lambda_o)=\beta$. Since $0\leq\alpha<\pi$ and $\beta\leq\pi$,

$0 < \omega(t, \lambda_o) < \pi$ in (a, b). Now from this we immediately obtain that the solution $\phi(t, \lambda_o)$ satisfies the third equation of (4.1) and in addition it does not vanish. Now let λ increase beyond λ_o. Then there is a unique λ_1 for which $\omega(b, \lambda_1) = \beta + \pi$. Clearly, $\phi(t, \lambda_1)$ satisfies the third equation of (4.1) and has exactly one zero in (a, b). If we continue in this manner, the n^{th} eigenvalue is determined by $\omega(b, \lambda_n) = \beta + n\pi$ and the n^{th} corresponding eigenfunction has exactly n zeros in (a, b). This concludes the proof of the theorem. $\square$

A Prüfer transformation, in combination with one dimensional Sobolev inequality, can be used to derive upper and lower bounds for the spectrum (set of eigenvalues) of regular self adjoint second order eigenvalue problems.

For the next theorem consider the following eigenvalue problem.

Let q be a real function in $L^s(a, b)$, $s \geq 1$, and let $\lambda_o < \lambda_1 < \lambda_2 < \dots$ and $\phi_o, \phi_1, \phi_2, \dots$ denote the eigenvalues and real orthonormal eigenfunctions (see Theorem 4.2) of

$$-y'' + q(x)y = \lambda y \quad y(a) = y(b) = 0. \tag{4.7}$$

We introduce the notation $f_+(x) \equiv \max(f(x), 0)$ and $f_-(x) \equiv f_+(x) - f(x)$ for a real function f.

The following theorem is taken from [5].

Theorem 4.8. *Let $\lambda < \lambda_n$. Then the eigenvalues of (4.7) satisfy the following inequality,*

$$\lambda_n \leq \lambda + \Big(\frac{\pi(n+1)}{2(b-a)} + \Big[\frac{(n+1)^2\pi^2}{4(b-a)^2} + \frac{\int_a^b (\lambda - q(x))_- \, dx}{b-a}\Big]^{1/2}\Big)^2 \quad (1),$$

which implies that

$$\lambda_n \leq \lambda + \frac{(n+1)^2\pi^2}{(b-a)^2} + 2\frac{\int_a^b (\lambda - q(x))_- \, dx}{b-a} \quad (2).$$

Proof. In the equation $-\phi_n'' + q\phi_n = \lambda_n \phi_n$ we apply the modified Prüfer transformation

$$\phi_n = r\sin\theta, \quad \phi_n' = \sqrt{\lambda_n - \lambda}\, r\cos\theta, \quad r(x) > 0.$$

This yields to

$$\theta' = \sqrt{\lambda_n - \lambda}\cos^2\theta - \frac{(q - \lambda_n)\sin^2\theta}{\sqrt{\lambda_n - \lambda}} = \sqrt{\lambda_n - \lambda} - \frac{(q - \lambda)\sin^2\theta}{\sqrt{\lambda_n - \lambda}} \quad (3).$$

From (3) we have,

$$\theta' \geq \sqrt{\lambda_n - \lambda} - \frac{(\lambda - q)_- \sin^2\theta}{\sqrt{\lambda_n - \lambda}} \geq \sqrt{\lambda_n - \lambda} - \frac{(\lambda - q)_-}{\sqrt{\lambda_n - \lambda}} \quad (4).$$

Since ϕ_n has exactly n zeros in (a,b) and vanishes at a, b, (see Theorem 4.2) we may take $\theta(a) = 0$ which implies that $\theta(b) = (n+1)\pi$. Now we integrate (4) to conclude that

$$(n+1)\pi \geq (b-a)\sqrt{\lambda_n - \lambda} - \frac{\int_a^b (\lambda - q(x))_- \, dx}{\sqrt{\lambda_n - \lambda}} \quad (5).$$

This is equivalent to $A \leq B\sqrt{A} + C$, where

$$A \equiv (\lambda_n - \lambda), \; B \equiv (b-a)^{-1}(n+1)\pi, \; C \equiv (b-a)^{-1} \int_a^b (\lambda - q(x))_- \, dx.$$

Hence $\sqrt{A} \leq [B + \sqrt{B^2 + 4C}]/2$, which is equivalent to (1). Moreover $\sqrt{B^2 + 4C} = B\sqrt{1 + 4C/B^2} \leq B(1 + 2C/B^2) = B + 2C/B$, since $\sqrt{1+x} \leq 1 + x/2 \;\; \forall x \geq 0$. Thus, $A \leq [B^2 + 2B(B + 2C/B) + B^2 + 4C]/4 \leq B^2 + 2C$, which is exactly (2). This concludes the proof. $\square$

Remark 4.9. The proof of Theorem 4.8 gives necessary and sufficient conditions for constructing q which will make (1) equality. Equality holds in (1) if and only if equality holds in (5). Consequently,

$$\int_a^b (\lambda - q(x))_- \sin^2\theta(x) \, dx = \int_a^b (\lambda - q(x))_- \, dx,$$

which implies that $(\lambda - q(x))_- \cos^2\theta(x) = 0, \; a.e[m]$. We integrate (3) and we get $(\lambda - q(x))_+ \sin^2\theta(x) = 0$, $a.e[m]$. Since $\theta'(x) > 0$, if $x = k\pi$, $(k \in \mathbb{Z})$, then $(\lambda - q(x))_+ = 0$, $a.e[m]$. Thus, $(\lambda - q(x)) = (\lambda - q(x))_-$ and $(\lambda - q(x))_- = 0 \; a.e \; on \; E$, where $E = \{x : \sin^2\theta(x) \neq 1\}$.

For example take $n = 0$ and $\lambda = 0$. Then $q(x) \geq 0$, and by (3) we have that

$$q(x) = \begin{cases} \lambda_o & \{x : \theta(x) = \pi/2\} \\ 0 & \text{elsewhere.} \end{cases}$$

In the following theorems we are trying to find, under certain general conditions on the coefficient q, a best possible (optimal) upper bound on the real parameter λ in order for the differential equation $y''(x) + (\lambda - q(x))y(x) = 0$, $x \in [a, \infty)$, to have a nontrivial solution in $L^2(a, \infty)$.

We consider the equation of the form

$$y'' + (f^2 + fg + fk)y = 0,\ x \in [a, \infty), \tag{4.10}$$

where all quantities are real, subject to the following:

(i) $f(x)$ is positive, locally absolutely continuous in $[a, \infty)$, and satisfies

$$\lim_{x\to\infty} f'(x)f^{-2}(x) = 0. \tag{4.11}$$

(ii) $g(x)$ is locally $L^1[a, \infty)$ and satisfies

$$\lim_{x\to\infty} g(x)f^{-1}(x) = 0, \tag{4.12}$$

$$k(x) \in\ L^1(a, \infty). \tag{4.13}$$

We define

$$\psi_1(x) = \sup_{t\geq x} |f'(t)/f^2(t)|, \tag{4.14}$$

$$\psi_2(x) = \sup_{t\geq x} |g(t)/f(t)|, \tag{4.15}$$

and we assume that

$$\psi_1^2 f,\ \ \psi_2^2 f \ \text{ are both in }\ L^1(a, \infty). \tag{4.16}$$

Then we have the following theorem (see [2]).

Theorem 4.17. *Let the above (4.11) to (4.16) conditions hold, and let y be a nontrivial solution for (4.10). Define $R(x)$ by*

$$R^2 = fy^2 + f^{-1}(y')^2 \quad R > 0. \tag{4.18}$$

Then for some constant A we have,

$$|\log R(x)| \leq A + 1/\pi \int_a^x f(t)(\psi_1(t) + \psi_2(t))\, dt \quad \forall x \in [a, \infty). \tag{4.19}$$

Proof. Consider the modified Prüfer transformation

$$y = Rf^{-1/2}\cos\theta,\ \ y' = -Rf^{1/2}\sin\theta. \tag{4.20}$$

Then we obtain the following differential equations for R, θ :

$$\theta' = f - (1/2)f'f^{-1}\sin 2\theta + g\cos^2\theta + k\cos^2\theta \tag{4.21}$$

$$R'R^{-1} = (1/2)f'f^{-1}\cos 2\theta + (1/2)g\sin 2\theta + (1/2)k\sin 2\theta. \tag{4.22}$$

We integrate (4.22) over (a, x) in order to obtain the bound in (4.19). The last term of (4.22), due to (4.13), yields a bounded integral.

Thus we only have to consider the other two terms. The first term in (4.22), using (4.14), yields to

$$\left|\int_a^x f'f^{-1}\cos 2\theta \; dt\right| \le \int_a^x \psi_1 f|\cos 2\theta| \; dt.$$

Now we use (4.21) to obtain,

$$\begin{aligned}\left|\int_a^x f'f^{-1}\cos 2\theta \, dt\right| &\le \int_a^x \psi_1\theta'|\cos 2\theta| \, dt \\ &\quad + \int_a^x \psi_1(1/2\psi_1 f + |g| + |k|) \, dt \\ &= \int_a^x \psi_1\theta'(2/\pi) \, dt \\ &\quad + \int_a^x \psi_1\theta'(|\cos 2\theta| - 2/\pi) \, dt \\ &\quad + \int_a^x \psi_1(1/2\psi_1 f + |g| + |k|) \, dt.\end{aligned}$$

Substituting θ' from (4.21), we have,

$$\begin{aligned}\left|\int_a^x f'f^{-1}\cos 2\theta \, dt\right| \le \int_a^x \psi_1 f(2/\pi) \, dt + \int_a^x \psi_1\theta'(|\cos 2\theta| - 2/\pi) \, dt \\ + (1 + 2/\pi)\int_a^x [(1/2)\psi_1^2 f + \psi_1|g| + \psi_1|k|] \, dt.\end{aligned} \tag{4.23}$$

The first integral on the right yields to the term ψ_1 (see (4.14)). Now we will prove that the other two terms in (4.23) are bounded. Indeed, for the case of the second integral in (4.23), we observe that ψ_1 is non-negative, non-increasing and that

$$\int_a^x \theta'(|\cos 2\theta| - 2/\pi) \; dt = \int_a^x (|\cos 2\theta| - 2/\pi) \; dt \tag{4.24}$$

is uniformly bounded for $x \ge a$. From the mean value theorem for integrals, and since $\int_a^x(\theta'|\cos 2\theta| - 2/\pi) \; dt$ is uniformly bounded, we obtain $\xi \in (a, x)$ such that

$$\int_a^x \psi_1\theta'(|\cos 2\theta| - 2/\pi) \; dt = \xi\int_a^x (\theta'|\cos 2\theta| - 2/\pi) \; dt \le C,$$

where C is a constant. Hence the second term in (4.23) is bounded. Now in the last integral in (4.23) all three summands of the integrand are in $L^1(a, \infty)$. For the first term this was assumed in (4.16). For the second term note that $|g| \leq \psi_2 f$, and again the result follows from (4.16). For the third term note that ψ_1 is bounded, and then use (4.13). Therefore $\int_a^x |f' f^{-1} \cos 2\theta| \, dt$ is bounded. Similarly we can prove that the second term in (4.22) yields to a bounded integral. (Just replace ψ_1 by ψ_2 and $\cos 2\theta$ by $\sin 2\theta$.) This concludes the proof of the theorem. □

The following theorem is from [2].

Theorem 4.25. *Let $r(x)$ be locally $L^1(a,\infty)$ and let $\lim_{x\to\infty} r(x) = 0$. Set $p(x) = \sup_{t\geq x} |r(x)|$ and assume that $p \in L^2(a,\infty)$. Moreover, let y be a nontrivial solution to $y'' + (\lambda - r(x))y = 0$, where $x \in [a,\infty)$, and define $R > 0$ by $R^2 = \lambda^{1/2} y^2 + \lambda^{-1/2}(y')^2$. Then for fixed $\lambda > 0$ and for some fixed constant $A > 0$, we have that*

$$|\log R(x)| \leq A + \frac{\lambda^{-1/2}}{\pi} \int_a^x p(x) \, dt \quad \forall x \in [a,\infty). \tag{4.26}$$

Proof. Apply Theorem 4.17 with $f = \lambda^{1/2}, g = -r\lambda^{-1/2}, k = 0$. The result now is immediate. □

Using the above theorem it is possible to prove that the constant $\lambda^{-1/2}\pi$ in (4.26) is the best possible, in the way that for any other constant $c < \lambda^{-1/2}\pi$ the inequality in (4.26) does not hold.

The Prüfer transformations are extremely useful in obtaining upper bounds for ratios of eigenvalues of certain Differential Operators. Before we close this section we will give a theorem describing the optimal bounds for ratios of eigenvalues of one dimensional Schrödinger Operator with Dirichlet boundary conditions and positive potential.

Theorem 4.27. *Let $H = -\dfrac{d^2}{dx^2} + V(x)$ be a Schrödinger Operator acting on $L^2(I)$, where $I \subset \mathbb{R}$ is a finite closed interval and where Dirichlet boundary conditions are imposed at both endpoints of I. Assume that $V \in L^1(I)$ and $V(x) \geq 0$ a.e on I. Then the ratio $\frac{\lambda_n}{\lambda_1}$, of the n^{th} eigenvalue of H to the first eigenvalue of H, satisfies the bound $\frac{\lambda_n}{\lambda_1} \leq n^2$. This bound is optimal, and for $V \in L^2(I)$ and $n > 1$ equality is obtained if and only if $V \equiv 0$ a.e. on I.*

The proof of the theorem is given in [1]. The Prüfer transformation needed for the proof is

$$\begin{aligned} y(x) &= r(x)\sin(\sqrt{\lambda}\theta(x)) \\ y'(x) &= \sqrt{\lambda} r(x)\cos(\sqrt{\lambda}\theta(x)). \end{aligned}$$

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George Chailos,
Department of Computer Science,
University of Nicosia,
Nicosia 1700, Cyprus
chailos.g@iunic.ac.cy

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The Imperfect Fibonacci and Lucas Numbers

JOHN H. JAROMA

ABSTRACT. A *perfect number* is any positive integer that is equal to the sum of its proper divisors. Several years ago, F. Luca showed that the Fibonacci and Lucas numbers contain no perfect numbers. In this paper, we alter the argument given by Luca for the nonexistence of both odd perfect Fibonacci and Lucas numbers, by making use of an 1888 result of C. Servais. We also provide a brief historical account of the study of odd perfect numbers.

1. Introduction

It has been shown as sufficient by Euclid and as necessary by Euler that an even number is perfect if and only if it is equal to

$$2^{p-1}(2^p - 1),$$

where $2^p - 1$ is prime. Primes of the form $2^p - 1$ are called *Mersenne primes*. They are named in honor of the 17th century priest, Fr. M. Mersenne (1588–1648), who claimed that such numbers are prime provided that $p \in \{2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257\}$ and are composite for all other values of $p \leq 257$. Although Mersenne's conjecture contained five mistakes, it had taken more than 300 years for mathematicians to discover them all.

Because of the Euclid–Euler defining characteristic of an even perfect number, the discovery of a new Mersenne prime is equivalent to the finding of a new even perfect number. As of 2008, only 44 Mersenne primes have been discovered. The four smallest were known at the time of Euclid. The prevailing conjecture is that there are infinitely many.

An equally, if not even more celebrated open problem is the question of whether or not an odd perfect number exists. It has remained unanswered for over two millennia. Nevertheless, a significant step toward a better understanding of them occurred in the 18th century when L. Euler provided us with their canonical form. In particular, he showed that if n is an odd perfect number, then it necessarily follows that

$$n = p^{\alpha} q_1^{2\beta_1} q_2^{2\beta_2} \cdots q_k^{2\beta_k} \tag{1}$$

where, $p, q_1, q_2, \ldots q_k$, are distinct odd primes and $p \equiv \alpha \equiv 1 \pmod 4$.

In his 1972 Ph.D. thesis, C. Pomerance asserted that the modern era of research on odd perfect numbers began with J. J. Sylvester [33], for in 1888 Sylvester published a series of papers that further qualified the structure that an odd perfect number must assume. Specifically, he demonstrated that such a number has at least four distinct prime divisors. Sylvester also established a lower bound of eight on the number of distinct prime factors that an odd perfect number can have provided that it is not divisible by three [43]. In addition, he showed that no odd perfect number is divisible by 105 [43]. Furthermore, before that year was over, Sylvester also improved the unrestricted bound on the number of distinct prime divisors of an odd perfect to five [44].

Sylvester offered the reader some of his thoughts regarding the existence of an odd perfect number in [42] when he equated the question to *a problem of the ages comparable in difficulty to that which previously to the labours of Hermite and Lindemann ... environed the subject of the quadrature of the circle.* He contended that odd perfect numbers do not exist and in [41] declared that *... a prolonged meditation on the subject has satisfied me that the existence of any one such — its escape, so to say, from the complex web of conditions which hem it in on all sides — would be little short of a miracle.*

Recently, Luca showed that perfect numbers do not exist among either the Fibonacci numbers $\{F_n\} = \{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\}$ or the Lucas numbers $\{L_n\} = \{1, 3, 4, 7, 11, 18, 29, 47, 76, 123, \ldots\}$ [23]. His argument for the odd perfect number showed that if either F_n or L_n is odd perfect, then $n = p$, where p an odd prime. He then proved that F_p is not an odd perfect number by quoting an earlier result of his that asserts $\sigma(F_n) \leq F_{\sigma(n)}$, $\forall n \geq 1$ [24].[1] For the case of the Lucas numbers, Luca demonstrated that $\sigma(L_p) < 2L_p$, for all primes $p \geq 2$.

[1]If we let n be any positive integer, then $\sigma(n)$ denotes the sum of the positive divisors of n.

We later summarize Luca's argument for the nonexistence of an even perfect Lucas number, as well as show there are no even perfect Fibonacci numbers by recapping the solution of [31]. Upon doing this, we offer a proof, apart from the one given by Lucas, demonstrating the impossibility of either an odd perfect Fibonacci or Lucas number. Our main tool will be an 1888 theorem of C. Servais that places an upper bound on the least prime divisor of an odd perfect number [38].

Before this is accomplished, we present the following account of the study of odd perfect numbers.

2. Brief Study of Odd Perfect Numbers

This section briefly recaps and updates the history of the study of odd perfect numbers offered in [9].

Approximately twenty three hundred years ago, Euclid showed in Proposition 36 of Book IX of his *Elements* that a number of the form $2^{n-1}(2^n - 1)$ is perfect provided that $2^n - 1$ is prime. Four hundred years later, Nicomachus of Gerasa continued the study of perfect numbers in his *Introductio Arithmetica*. Unfortunately, all of his assertions, including the declaration that all perfect numbers are even, were given without proof. Nonetheless, his conjectures were taken as fact for centuries.

It appears that the first mathematician of note to suggest that an odd perfect number exists was R. Descartes. In a letter to Mersenne dated November 15, 1638, he announced that he could demonstrate that every odd perfect number must be of the form ps^2, where p is a prime. Furthermore, he stated that he saw no reason to prevent the existence of an odd perfect number and cited the example of $p = 22021$ and $s = 3 \cdot 7 \cdot 11 \cdot 13$ as evidence. For, ps^2 would be an odd perfect number provided one pretends that 22021 is prime.

In 1832, B. Peirce studied existence criteria from a different perspective by establishing a lower bound of four on the number of distinct prime divisors that an odd perfect number can have [32]. We remark that the credit for this important discovery seems to have eluded Peirce, being often misdirected to either Sylvester [43] or to Servais [39]. Both of these mathematicians independently proved the same result more than fifty years later (See [43] and [39].) In fact, even L. E. Dickson neglected to credit Peirce with this important discovery in his magnum opus, *History of the Theory Numbers* [7].

However, as noted earlier, Sylvester did break new ground in 1888 by improving the said lower bound to five [44]. The year 1888 also saw Servais placing an upper bound of $k+1$ on the least prime divisor of an odd perfect number with k distinct prime divisors [38].

In 1913, Dickson demonstrated that for any integral value of k, there are only finitely many odd perfect numbers with k components[2] [8]. He proved this as a corollary to a similar result for odd primitive non-deficient numbers. By definition, such numbers necessarily contain all the odd perfect numbers[3]. A significant aspect of Dickson's paper is that one may now conduct a search for the an odd perfect number with k components by initially listing out all of the finitely many primitive odd non-deficient numbers associated with that k-value and then checking for those among them are equal to the sum of their proper divisors. Alas, the approach is not feasible for most values of k, for the resulting lists quickly become intractably large.

In 1925, I. Gradstein advanced the lower bound on the number of distinct prime divisors on an odd perfect number to six [11]. In 1949, H. J. Kanold revisited the four-divisor case and published a proof of the same result given in earlier years by Peirce, Servais, Sylvester, and Dickson [27]. The significance of this paper extended considerably beyond the stated result for in it, Kanold demonstrated that the largest prime divisor of an odd perfect number must exceed 60. This marked the first theorem of its kind. Moreover, it represented the initial contribution to a class of propositions that would ultimately be suggested by Pomerance some twenty-five years later.

More specifically, because the approach to the odd perfect number question from the perspective of Dickson's paper is impractical, it was necessary to seek out alternative approaches to studying the structure of an odd perfect number. In 1974, and in addition to showing that an odd perfect number must have at least seven distinct prime divisors[4], Pomerance proposed a class of theorems for consideration: *An odd perfect number is divisible by j distinct primes* $> N$ [33].

Furthermore, along this line of thought, Kanold's 1949 result of $j = 1$ and $N = 60$ was improved for the $j = 1$ case to $N = 11200$

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[2]For example, that the components of (1) are $p^{\alpha}, q_1^{2\beta_1}, q_2^{2\beta_2}, \dots q_k^{2\beta_k}$.

[3]A *deficient number* is any integer n with $\sigma(n) < 2n$. So, n is a *non-deficient number* if $\sigma(n) \geq 2n$. Dickson had called a number *primitive non-deficient* provided that it is not a multiple of a smaller non-deficient number.

[4]In 1974, N. Robbins independently proved the same result [36].

by P. Hagis, Jr. and W. McDaniel [15] in 1973. Two years later, they bettered their own result by exhibiting $N = 100110$ [16]. In 1975, Pomerance became the first to illustrate a case for $j > 1$ upon showing $j = 2$ and $N = 138$ [34].

In 2008, T. Goto and Y. Ohno showed that an odd perfect number must have a factor greater than 10^8. This improved the 2003 result of P. Jenkins who demonstrated that the largest prime divisor of an odd perfect number exceeds 10^7 [22], which augmented the earlier lower bound of 10^6 discovered by Hagis and G. Cohen in 1998 [4].

In 1999 and 2000, D. Iannucci published initial results on the second and third largest prime divisors of an odd perfect number. He proved they exceed 10000 and 100, respectively [19], [20]. It appears that these remain as the best such estimates to date.

The study of odd perfect numbers has also included attempts to provide a bound on its magnitude. In 1908, A. Turăninov established a lower bound of 2000000. The current best estimate of this kind has been given by R. P. Brent, Cohen, and H. J. J. te Riele in 1991 [1], which showed that any odd perfect number is necessarily greater than 10^{300}. This result was achieved by developing an algorithm which demonstrated that if there exists an odd perfect number n then $n > K$, upon which they applied the algorithm to $K = 10^{300}$.

In 1994, R. Heath-Brown proved that if n is odd and $\sigma(n) = an$, then $n < (4d)^{4^k}$, where d is the denominator in a and k is the number of distinct prime factors of n [18]. In particular, if n is an odd perfect number then n has an upper bound of 4^{4^k}. This represents an improvement over the previous best estimate of $n < (4k)^{(4k)^{2^{k^2}}}$, which was given by Pomerance in 1977 [35]. Heath-Brown has remarked that his bound is still too large to be of practical value. Nevertheless, we remark that when his bound is viewed in conjunction with the lower bound of 10^{300}, Sylvester's result that every odd perfect number has at least five distinct divisors follows immediately; that is, $10^{300} < n < 4^{4^k}$ implies $k > 4.48$. In 1999, R. J. Cook improved Heath-Brown's result to $n < (2.124)^{4^k}$ [6]. In 2003, P. Nielsen further reduced it to 2^{4^k} [29].

Presently, the best result for the least number of distinct prime divisors that an odd perfect number can have is nine. This was recently discovered by Nielsen [30]. It is a long-awaited improvement over the bound of eight established independently in 1979 by E. Z. Chien [3] (who published nothing of his work) and by Hagis in 1980 [12]. Hagis's original proof contained almost two hundred manuscript pages. We note that Cohen and Sorli in 2003 described an algorithmic approach for showing that if there exists an odd perfect number, then it has t distinct prime factors [5]. In that work, they also discussed the algorithm's applicability to the case $t \geq 9$. Nielsen's demonstration ultimately avoids previous computational results for odd perfect numbers.

Some obtained estimated on the total number of prime divisors that an odd perfect number can have. In 1986, M. Sayers showed that such a number necessarily has at least 29 such factors [37]. This was later improved to 37 by Iannucci and Sorli [21]. The best estimate to date appears to be 75, given recently by Hare [17].

The best improvement to Sylvester's bound on the number of distinct prime factors of an odd perfect number not divisible by three now stands at twelve which also appears in Nielsen's paper [30]. The best previous estimate of eleven had been obtained independently in 1983 by both Hagis [13] and M. Kishore [28].

Finally, we point out that a study of odd perfect numbers from a somewhat different perspective was initiated in 1937 by R. Steuerwald upon showing that not all the β_i's in Euler's canonical form given by (1) can all be equal to one. This continued in 1941 when Kanold showed that neither may all of the β_i's be equal to two nor may one of the β_i's be equal to two while all the rest are equal to one [**26**]. In 1972, Hagis and McDaniel proved in [**14**] that not all the β_i can be equal to three. In 2003, Iannucci and Sorli showed that an odd perfect number cannot be divisible by three if for all i, $\beta_i \equiv 2 \pmod 3$ or $\beta_i \equiv 2 \pmod 5$ [**21**].

3. F_n and L_n are not Even Perfect

In this section, we present previously developed arguments for the nonexistence of even perfect Fibonacci and Lucas numbers. As previously noted, if $2^p - 1$ is prime, then a necessary and sufficient condition for an even number N to be perfect is that it is necessarily of the form

$$N = 2^{p-1}(2^p - 1). \tag{2}$$

Now, to show that a Fibonacci number cannot be even perfect, we refer Padwa's 1972 solution of the problem posed by R. Whitney [31]: *Prove that there are no even perfect Fibonacci numbers.* For the Lucas numbers, we illustrate Luca's proof given in [23].

Theorem 1. *There are no even perfect Fibonacci or Lucas numbers.*

Proof. Let $N = 2^{p-1}(2^p - 1)$ be a perfect number.
Case 1. (Whitney/Padwa) Assume that N is a Fibonacci number. Since all even perfect numbers are given according to (2), this implies that all even perfect numbers greater than 28 are also divisible by 16. Now, the only Fibonacci numbers divisible by 16 are also divisible by 9. Thus, a Fibonacci number cannot be of the form $2^{p-1}(2^p - 1)$. Therefore, no Fibonacci number is even perfect.
Case 2. (Luca) Assume that N is a Lucas number. First, if $p = 2$ then $N = 6$, and, if $p = 3$ then $N = 28$. Since both of these are not Lucas numbers, it is without loss of generality that we assume $p > 3$. In light of (2), this implies that $8 \mid L_k$. However, this is impossible, as no Lucas number is divisible by 8. Therefore, there are no even perfect Lucas numbers. □

4. Generating F_n and L_n from the Lucas Sequences

Before we proceed with our demonstration that F_n and L_n cannot be odd perfect numbers, we will need to view these numbers as iterations of a specific Lucas and companion Lucas sequence, respectively.

To this end, let P and Q be relatively prime integers. The *Lucas sequences* are defined recursively by

$$U_{n+2}(P,Q) \;=\; PU_{n+1} - QU_n, \; U_0 = 0, \; U_1 = 1, \; n \in \{0, 1, \ldots\}. \quad (3)$$

Similarly, the *companion Lucas sequences* are

$$V_{n+2}(P,Q) \;=\; PV_{n+1} - QV_n, \; V_0 = 2, \; V_1 = P, \; n \in \{0, 1, \ldots\}. \quad (4)$$

We point out that the Fibonacci numbers, $\{F_n\}$, are produced by the Lucas sequence $\{U_n(1,-1)\}$ and the Lucas numbers $\{L_n\}$ are generated by the companion Lucas sequence $\{V_n(1,-1)\}$. Furthermore, since (3) and (4) are linear they are solvable. In particular, for $n \in \{0, 1, \ldots\}$,

$$F_n \;=\; U_n(1,-1) \;=\; \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]; \quad (5)$$

$$L_n \;=\; V_n(1,-1) \;=\; \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n. \quad (6)$$

5. F_n and L_n are not Odd Perfect

Our proof of the nonexistence of Fibonacci and Lucas odd perfect numbers relies on the following proposition of Servais which places an upper bound on the least prime divisor of an odd perfect number [38].

Theorem 2. (Servais) *The least prime divisor of an odd perfect number with k distinct prime factors does not exceed k.*

The next result demonstrated by Luca in [23] will also be utilized in our argument.

Lemma 1. *Let either F_n or L_n be an odd perfect number. Then, n is prime.*

In a given Lucas sequence, the *rank of apparition* of p is the index of the first term that contains p as a divisor. A prime factor of either U_n or V_n is *primitive* provided that its rank of apparition is n. Such a factor is called *intrinsic* if it divides n. Otherwise, it is said to be *extrinsic.*

The following three lemmas come from either [2] or [25].

Lemma 2. *The odd extrinsic factors of U_n are of the form $rn \pm 1$.*

Lemma 3. *The odd extrinsic factors of V_n are of the form $2kn \pm 1$.*

Lemma 4. *Assume that $p \nmid PQ$ and let ω denote the rank of apparition of an odd prime p in the sequence $\{U_n(P,Q)\}$. Then, $p \mid U_n$ if and only if $n = k\omega$.*

Lemma 5 is based on results found in [2].

Lemma 5. *For any $p \neq 5$, both $p \nmid F_p$ and $p \nmid L_p$.*

We are now ready to offer a proof of the nonexistence of odd perfect Fibonacci numbers and odd perfect Lucas numbers. Both cases are demonstrated at once.

Theorem 3. *There are no odd perfect Fibonacci or Lucas numbers.*

Proof. By Lemma 1 and the fact that $F_2 = 1$, $L_2 = 3$, $F_5 = 5$, and $L_5 = 11$ are all not perfect, it suffices to consider only the case where $n = p$ is an odd prime not equal to 5. Now, for the sake of contradiction, let's assume that there exists a prime p for which either F_p (L_p) is an odd perfect number. Let d be the least prime divisor of F_p (L_p). Since the index p of F_p (L_p) is prime, it follows from Lemma 4 that d is a primitive prime factor of that term. Furthermore, since $p \neq 5$, it follows by Lemma 5 that every prime factor of F_p (L_p) is extrinsic. As F_3 and L_3 are respectively, the only even Fibonacci and Lucas numbers of prime index, we conclude that for all $p > 3$, every primitive factor of F_p (L_p) is odd. Since d and p are odd, it then follows from Lemma 2 and Lemma 3 that $d = rp \pm 1 = 2kp \pm 1$. Hence, $d \geq 2p - 1$. Moreover, Theorem 2 tells us that F_p (L_p) has at least $2p-1$ distinct prime factors. Therefore, utilizing (5), (6), and Lemma 3, we obtain

$$2\left(\frac{1+\sqrt{5}}{2}\right)^{2p-1} > \left(\frac{1+\sqrt{5}}{2}\right)^{2p-1} + \left(\frac{1-\sqrt{5}}{2}\right)^{2p-1} = L_{2p-1} > L_p \geq$$
$$\geq (2p-1)(2p+1)(4p-1)(4p+1)$$
$$\ldots [2(2p-2)-1][2(2p-2)+1][2(2p-1)-1],$$

where the last product consists of $2p-1$ terms, which for all p, is impossible. □

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John H. Jaroma,
Department of Mathematics and Physics,
Ave Maria University,
Ave Maria, FL 34142, USA
john.jaroma@avemaria.edu

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Hurling's Mathematical Story

ROBIN HARTE

Abstract. GAA people know about matrices.

South of Thurles, Hurling followers know all about Ring Theory [2], and know they know about it, but they also know a little about Matrix Theory, and possibly do not know they know about it. A goal is worth 3 points,

$$\text{goal} = 3 \text{ point} \tag{1}$$

and hence a score line of say $(2,5)$ is an example of a *row matrix*:

$$\begin{pmatrix} 2 & 5 \end{pmatrix} \begin{pmatrix} \text{goal} \\ \text{point} \end{pmatrix} = 2 \text{ goal} + 5 \text{ point} = \big((2)(3)+(5)(1)\big)\text{point} = 11 \text{ point}. \tag{2}$$

There is more matrix multiplication latent in this:

$$\begin{pmatrix} \text{goal} \\ \text{point} \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \text{point} \tag{3}$$

and

$$\begin{pmatrix} 2 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = (2)(3) + (5)(1) = 11. \tag{4}$$

In fact equation (1) has not always been true: back in 1894 ([1], Chapter II) a goal was worth "any number of" points. The matrix theory pattern however would survive a finite change in the relationship between goals and points.

Potentially Hurling followers might also learn about Infinitesimals. There is at present no concept of "tie breaker", or "penalty shoot out", in Gaelic games; in soccer however there is an advantage, in two-leg encounters, awarded to "away goals". In a similar way one could imagine, in drawn encounters, offering an added advantage to goals over points. For example the score lines (1,8), (2,5) and

(3,2) all currently contribute the same number, 11, points, and the suggestion would be that

$$(3,1) < (1,8) < (2,5) < (3,2) < (2,6): \tag{5}$$

the goal advantage would only kick in when the number of points was equal. Since this pattern is to persist for unimaginably large score lines, this is achieved by replacing (1) by

$$\text{goal} = (3+\varepsilon)\text{point}, \tag{6}$$

where ε is an "infinitesimal" [3]. Similarly, back in 1894 a point was an infinitesimal goal:

$$\text{point} = \varepsilon \text{ goal}. \tag{7}$$

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Robin Harte,
School of Mathematics,
Trinity College Dublin,
Dublin 2
rharte@maths.tcd.ie

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An Interview with Professor Thomas J. Laffey

GARY MCGUIRE

Introduction

On the occasion of his 65th birthday I conducted an informal interview with Tom, to gather some memories and stories. This is the second recorded interview with Tom; the first can be found online.

Background

GMG: *From what age did you get interested in mathematics?*

TL: I come from a background with no tradition in education. My parents were farmers in County Mayo. Very few members of the family had gone to second level education. My father and mother thought education was a way of getting away from farming, so they encouraged all their children in that respect. I remember a neighbour of mine was interested in reading, and had some books. People around there were quite poor and books were not common, so this neighbour had some influence on me. We lived near the Mayo-Galway border. I remember the man who ran the local post office was a (Gaelic) football fanatic. When Mayo got knocked out of the championship before Galway, as usually happened, he would switch his support to Galway.

After primary school my parents hoped I would get a scholarship for secondary school. During my first year in secondary school I got the scholarship, which supported me for the remainder of my secondary school days. During that first year I was actually put into second year classes, which helped me get the scholarship. At the end of third year I did the Inter Cert, and I did very well, particularly in mathematics and latin. Because the principal felt I was a bit young to start the Leaving Cert, and there was no transition year at that time, the principal told me to do the Inter Cert again. So I did the Inter Cert a second time, which involved different books and plays in subjects like English. Then we got a new school principal, who was more interested in encouraging students to go to third level. I think he came from O'Connell's school in Dublin. He encouraged me to do honours maths in the Leaving Cert. This was unusual at the time, only a small number of schools offered honours maths. I was the first student ever from my school to do honours maths in the Leaving Cert.

The new principal gave me a lot of books from the school library. These books were very old, like Crystal's algebra and Hall and Stevens' Geometry. They had been in the school since the school was founded. He also arranged for a brother who was an honours maths teacher to come on five Saturdays over the two years, to give me a few classes. Mostly I learnt the material myself from reading the books. It taught me that having several sources is a useful thing, because sometimes one book has a very nice treatment of one thing, whereas it's a different book that has the best treatment of the next result.

I also practiced on the past papers, a bit like students do now. During school time, I was the only one doing honours maths so during pass maths class the teacher used to make me do the problems on the board while he went off and took a break. There was always the threat of him coming back, so there were no discipline problems.

GMG: *So is it fair to say you are self-taught?*

TL: Yes, you could say that, at that level, in mathematics and physics anyway. One new thing for me was learning mathematics through English, because all subjects were taught through Irish at that time. Because the books were written in English, I was now learning mathematics through English. This had a funny consequence later at UCG, when I was a senior undergraduate. At that time the head of the Commerce faculty was a gaelgóir and insisted that all subjects were taught and examined through Irish. First year students had to do maths. During the actual exam, some people (usually senior undergraduates or MSc students) had to be available to answer student queries and orally translate Irish phrases in the paper for students who didn't understand the Irish. I was one of the few students who knew the technical terms in both English and Irish, because of reading the books in English and being taught through Irish.

I was also self-taught in physics, because the teacher concentrated more on experiments but I was more interested in the theoretical parts.

GMG: *It's quite an achievement to learn the honours maths syllabus by yourself, and obtain one of the top marks in the country. If that principal had not encouraged you to do honours maths, where would you be now?*

TL: Perhaps I would have become a primary school teacher, because that was a realistic option at the time. It was something you could do without having the money.

GMG: *You went on to university then.*

TL: My parents could not have afforded to send me to university. I ended up getting a state scholarship to university, and I went to UCG, which is now named NUI Galway. There were around 20 of those scholarships at the time, over the whole country, and usually they went to students from the top fee-paying schools. They were not dependent on the income of the parents. My principal was congratulated because no-one from a school like mine had won a state scholarship before. I also got a Mayo county council scholarship, of which there were two, and these were dependent on parents income. However the state scholarship was more lucrative, so I took that one.

Maths and classics counted slightly more towards the state scholarship than other subjects, 600 as opposed to 400 for other subjects. I was able to get very high marks in those two subjects, so this helped me get the state scholarship. Latin at that time was quite technical and mathematical, in a way, so it suited me. You had to make everything grammatically correct, and there were certain rules that had to be obeyed. They gave certain rhythms, various numbers of shorts and longs. Recently I heard that the old testament when spoken in Aramaic has this kind of a rhythm to it. Since students could score close to full marks in maths and latin, being good in those helped win the scholarship. Many of the scholars studied maths, but also some went into classics, one example being Tom Mitchell, the former provost of TCD. He was a few years ahead of me.

SECONDARY SCHOOL MATTERS

GMG: *You have been been involved in the BT Young Scientists exhibition for some years now, as a reviewer and as a judge. What are your thoughts on it?*

TL: It generates great excitement for the children. It teaches them that academic subjects can be exciting. It seems that children do a lot of memorization in school, but unfortunately they also do this in mathematics. So one of the main benefits is that secondary school students get a feeling for what it is like to find out something for themselves, and they may discover that the actual process of discovery is exciting and enjoyable. This is probably more true for group projects than individual projects – more excitement is generated.
It is also important for the country, because the winner competes in a European competition and some international competitions. This gives important visibility to Ireland. For example, this year's individual winner has won prizes sponsored by the American Mathematical Society and the Indian Mathematical Society at the Intel Science Talent Search in the USA.
Mathematics projects have been both group and individual. The number of mathematics projects was extremely small for a period in the 1990s, and there was no mathematician on the judges panel then. If a mathematics project was a contender for a prize, they used to call in a mathematician to be sure of the quality (for the reputation of the competition and the sponsors). At that time I was normally the person who was called. That was before the year 2000, approximately. I have been a judge since about 2000. Fergus Gaines was a judge back in the 1980s.

GMG: *Tony Scott, who is in Experimental Physics in UCD, was a founder of the Young Scientists competition I believe.*

TL: Yes, it was started by Tony Scott and the late Fr. Tom Burke in the 1960s, and they obtained sponsorship from Aer Lingus for about the first 30 years. What happened was that Fr. Tom Burke visited Los Alamos lab to do research on sabbatical, and he invited Tony Scott to visit him there for the summer, as they did research together. He asked Tony to come early to see a science exhibition they had there, and then he and Tony had the idea that they might run such a thing in Ireland. That's how it started.

GMG: *And I believe that one year a student entered a mathematics project that would have won him a Fields medal?*

TL: Yes, a student entered a project claiming to have proved the conjecture about primes in arithmetic progressions, a result that was recently proved by Green and Tao. Not only that, he also claimed to have proved a conjecture of Hardy and Littlewood, that $\pi(x+y) \leq \pi(x) + \pi(y)$. Both proofs were incorrect, however the project was very good and showed talent. He had a number of correct lemmas that were quite sophisticated, and he ended up winning the prize. I recall being asked whether these results would be publishable, if they were correct. I replied that not only were they publishable, they would win him a Fields medal!

GMG: *Were you involved the year that Sarah Flannery won the Young Scientists competition with her cryptography project?*

TL: No, in fact I was in Portugal that year doing some research. I recall being shown a Portuguese newspaper whose front page headline was about an Irish genius who was going to be a millionaire with a new encryption system. A footnote on that article thanked Le Monde for allowing them to use their article, and translate it.
The consultant for that project was a visitor in Trinity College, I believe, who said it would be very good work if correct, and could not find any mistakes at the time. Her father, David Flannery, asked me to go over it before Sarah entered the European Young Scientists competition. I noticed that you could break it without much trouble, because she had a 2×2 non-derogatory matrix A and another matrix B that commuted with A. If you apply the theorem that B must be a polynomial in A, you can break the system. I wrote to David Flannery and told him this, he came up to Dublin and we talked about it. In the European competition Sarah also presented the attack.

GMG: *I understand that politicians played a role in the first participation of Ireland in the International Mathematical Olympiad. Could you tell me about the background to that?*

TL: Ireland first participated in 1988, as it eventually turned out. That year the event was held in Australia. But let's go back a bit. In fact, it all began with Con O Caoimh, a secondary school teacher in

Cork who later went into the Department of Education and became chief inspector for mathematics. In the 1970s he ran a competition in Cork with tricky maths problems. He got the best students to take part in the American high school mathematics competition, long distance. Finbarr Holland in UCC thought it would be great to run such a competition nationwide. He discussed this with me, and I said I would be happy to help out. Fergus Gaines was also very supportive. Starting in 1978 the three of us ran that competition. Subsequently we always talked about having Ireland compete in the International Mathematical Olympiad (IMO), because you would hear about this a lot at conferences and so on. People would be talking about a great mathematician Joe Bloggs, and they would say that Joe represented Russia or whatever country he came from in the IMO when he was younger, and won a gold medal. So we thought it would be good for mathematics in Ireland if we entered the IMO.

Moving into the 1980s then, the country was in recession and times were hard. We felt our only hope was to wait until the IMO was being held in a nearby country. It was due to be held in Germany in 1989, which we could have travelled to by boat and train. Flying was out of the question, it was too expensive.

Completely out of the blue, a man named Peter O'Halloran, who was the the chief organiser of the IMO to be held in Australia in 1988, wrote a letter which ended up on Finbarr Holland's desk somehow. It was originally written to the government or the academy, I think. Anyway, O'Halloran said that he had managed to get the Australian government to name the IMO as an official event of the Australian bicentennial, to be held in 1988. You see, O'Halloran was in a university in Canberra, which is a small town consisting mainly of politicians and academics at the two universities there. The academics there have great access to the politicians. So O'Halloran was able to get great political and financial support for the IMO, he had great people skills. The point is that O'Halloran said it was very important that Ireland participate in the IMO in Australia, because of all the historical and cultural links between the two countries.

Peter O'Halloran managed to get the Australian ambassador to Ireland on board, and asked him to tell every Irish minister he met that Ireland should particiapte in the IMO. The Irish Minister for Education at the time was Patrick Cooney, from Athlone, who had replaced Gemma Hussey in 1986. It turned out that he was a good friend of the Australian ambassador! Peter Barry was also contacted, he was Minister for Foreign Affairs at that time, in Garrett Fitzgerald's Fine Gael-Labour government. So we met both these ministers and the ambassador in one week. They were worried about the finances, but nevertheless they agreed to support an Irish team to go to the IMO in Australia. They also set up a committee, which still exists, to regulate Ireland's participation in the IMO in future years. The first such committee had myself, Finbarr Holland, Con O Caoimh, and Bill Nolan from the Department of Foreign Affairs.

However, there was then a change of government in 1987 when Charlie Haughey became Taoiseach. Immediately after the election, financial cutbacks came to the fore, and we feared that the IMO funding would be lost. Luckily, Peter O'Halloran had convinced the Australian Minister of Culture that it was important for Ireland to participate in the IMO, and this Minister was on an opportune visit to Dublin. He was speaking at a dinner where Brian Lenihan (Minister for Foreign Affairs after Peter Barry) was present. During his speech he said that Australia wanted an Irish team to take part in the IMO! Coming from outside, this was very powerful. It all worked very well and that's how our politicians came to support the IMO. The new government agreed to send an observer to the IMO in Cuba in 1987, to see how it all worked, and Finbarr went there.

It is curious that Ireland was in dire straits economically at the time, and a trip to Australia was about as expensive as you could get, maybe 1300 pounds, and yet it happened. It's probably still the most expensive IMO we have ever participated in!

GMG: *Looking back, after 20 years of participation, do think it has been good for Ireland to participate in the IMO?*

TL: Yes, I think it has. I remember one time a teacher told me that the great thing about olympiad training is that his best students have actually met questions that they don't immediately know the answer to. Those students are used to knowing the answer to every question that is asked in school, and knowing it immediately. So for these students to actually struggle with a question, to meet a question that challenges them, he felt it was a good thing for them and stopped them boasting.

We still find it difficult to get people to come for the training, but for those who come it is very beneficial.

I think it plays an important role for the country, in that it ensures visibility. Every European country enters a team, and just about every so-called "advanced" country is involved.

Siddartha Sen once said to me that the olympiad training is having a curious social consequence. One finds that many of the top students know each other from olympiad training, and this includes students from all universities and all subjects! The students who have participated in olympiad training tend to be determined and hard-working people, and go on to be top of their class in university.

From the point of view of winning medals, we are not in the professional leagues. We win some medals, I think we have won about 8 so far. We have been successful with setting questions, in that we have had four questions appear on the actual IMO exam.

GMG: *Have you participated much on committees related to secondary school mathematics?*

TL: Yes, back in the late 1960s and early 1970s, there was a national committee formed to discuss and implement a change of syllabus – this was the era of so-called "new mathematics." Professor Gormley, who was head of department in UCD, was on the committee, he was the NUI rep. After he died in 1973 I was his replacement for a few years on the review committee. At that time there was a lot of discussion about the geometry, and what should be on the higher level syllabus. There were a lot of teachers reps. The university people were quite conservative and did not show the same excitement as the teachers for new mathematics. Within the academic community of mathematicians opinion was divided. I remember Stephen O'Brien who was at UCD was in favour of it, and Con O Caoimh whom I mentioned earlier was also in favour of it. Professor Timoney, Professor Gormley, David Lewis and myself were more sceptical. The teachers put this down to us being in a university environment, and not having to go in and teach thirty 15-year olds every day. The main university contribution was that the influence of very non-intuitive geometry was diminished from the first draft, which I was happy about.

GMG: *What other suggestions did you make?*

TL: I remember for example that the university people were in support of having a lot of material on solvability of linear systems of equations. On the other hand, we suggested that finite group theory be dropped because the amount covered in school was so little, and in university we start from scratch anyway. However the teachers organizations sent word back that they did not want linear systems of equations, but they did want the group theory. I think this reflected the fact that the group theory scope was so limited, so the number of questions that could be asked was very small, whereas there would be huge scope for different questions on linear systems of equations.

One thing teachers were anxious to have was that the exam adhere to the syllabus very strictly. It was claimed that Con O Caoimh used to use the exam to move the syllabus in a direction he thought was beneficial. For example, he might put in a question with a 2×2 matrix A, and he would ask students to find a number λ and numbers x and y so that A times the vector $(x\ y)$ is equal to λ times $(x\ y)$. This can of course be solved by writing down the equations and bashing it out. There would be complaints that eigenvalues and eigenvectors were not on the syllabus, but this was his way of introducing it. The top schools would then learn eigenvalues and eigenvectors for the next years exam, because their teacher would prepare them for it. In this way the syllabus had grown a little bit. The teachers unions used to object to this, because it was not adhering to the syllabus.

There used to be a university representative on the Department of Education Inter Cert advisory board, but this had stopped by the 1970s.

GMG: *I gather there was once a stolen Leaving Cert exam paper.*

TL: In the 1990s the Department of Education used to send copies of the Leaving Cert exam to the universities for approval. This is because the exam was acting as a matriculation exam, after the NUI matriculation exam was abolished. So for some years I had to approve the draft copy and make comments, and I made a trip to the Department of Education. I was the NUI representative delegated to get the copy of the Leaving cert exam in advance.

One year a different NUI representative kept his copy of the paper in the small safe in his university. A thief broke in over a weekend and took the safe away. It was later found elsewhere on campus and it had been broken into. The contents had disappeared. I doubt whether the thief was interested in reading mathematics, but anyway, the exam was considered to be compromised and had to be set again.

It was not the fault of the NUI representative of course, he had taken every precaution.

I also used to sit on the Mathematics committee of the Royal Irish Academy. The RIA used to have three committees, two for mathematics and one for mechanics, but now they are merged into one, a committee for mathematical science. One job of the committee is to suggest names for the Hamilton lecture, which takes place each year on 16 October. This year (2009) we will have Zelmanov speaking.

Your Mathematical Career

GMG: *You embarked on your mathematical career in Sussex, where you received your Ph.D. under Ledermann.*

TL: The maths department in Galway where I did my undergraduate degree, was headed by Sean Tobin. We got a very strong training in algebra, probably the strongest in the country at that time (the 1960s). It was different in Cork where the strength was in analysis. Because of Sean Tobin, if you did your degree in Galway and you wanted to do a PhD, you went off somewhere to do a PhD in group theory. A number of people did that. Tobin himself was a student of Higman. Students used to enjoy the group theory, it was elegant and also tricky stuff. Students used to enjoy the trickiness of it. Well, some students anyway!

After I did the masters in Galway, I was an assistant lecturer there for a year. During that year I was thinking of doing further postgraduate work, and I needed funding. I wrote to Walter Ledermann in Sussex, and he invited me for an interview. I got the train to Dublin, the boat to Holyhead and the train to London and then to Brighton. I met Ledermann and I got an offer of a grant from Sussex.

I also wrote to Philip Hall in Cambridge, because I had heard his name in lectures from Sean Tobin when he discussed Hall subgroups and things. Hall wrote back to me saying he would nominate me for a Gulbenkian fellowship. I had never heard of a Gulbenkian fellowship before, but anyway he said he would nominate me. However the decision wouldn't be made for 3 months. I did not know if I would get that fellowship, so I accepted the offer from Ledermann. Had I known the system in Cambridge, and the status of Philip Hall, I would have known that whoever Hall nominated was very likely to get that fellowship. I'm very sorry I didn't keep that letter from

Hall. The letter I got from him talked about all the things in group theory that we could work on, and it was very interesting.

GMG: *How were things at Sussex?*

TL: It was very nice at Sussex, and Ledermann was a very nice person to work with. I was the only finite group theorist, besides Ledermann, but there were some infinite group theorists like Dunwoody. Very soon after going to Sussex there was a Fulbright fellow there from America named Bob Bumcrot, who was joining in the algebra seminars we had every week. He had done his PhD in finite geometries with Marshall Hall. I remember Bumcrot being a bit discouraging about finite group theory, because he said it was so difficult. His reason for thinking this was as follows. He thought that Marshall Hall was the greatest genius he could ever imagine. He used to talk about how Marshall Hall could fill up the blackboard with equations and then start proving things on the spot. But then Marshall Hall came back from a conference on finite group theory once and told Bumcrot that he had met Thompson and people like that, and that they were so clever and how he couldn't imagine anyone being as clever as those guys. So I didn't find this very encouraging as I was starting out in finite group theory! He kept talking about how clever all these guys were.

I felt really well off at Sussex at that time, because the grant was very good. I didn't have to pay any tax on that grant, and I was paid almost the same as the lecturers after you deducted tax from their pay. Sussex was a new university, and I was one of the first PhD graduates from there in mathematics.

GMG: *What mathematicians have been a big influence on you?*

TL: While I was in Sussex, E.C. Dade was visiting London from Caltech for six months and he was giving a course on representation theory. Ledermann arranged for me to meet him and attend that course, and that's really where I learnt representation theory. Dade used to give two hour lectures non-stop, going like lightning.

At one stage Ledermann was very worried, because he had met Fröhlich who told him that Dade's lectures were difficult to understand. So Ledermann came in to reassure me that I shouldn't worry if I found the lectures very hard.

Then Ledermann discovered that Feit was coming to Warwick to visit. He arranged for me to meet with Feit and to discuss what I might work on. So I did that and we talked about various problems. I met Feit about 20 years later, and he remembered having met me that time in Warwick, which I was surprised about.

Olga Taussky was a very big influence on me. When I came to UCD in 1968 there was virtually no algebra here, the only person doing algebra was Fergus Gaines. David Lewis was more interested in geometry then. I decided I would learn the matrix theory that Fergus Gaines knew, and then I could work with him. He had done his PhD at Caltech under Olga Taussky. So I spent some time learning matrix theory, which also involves learning some ring theory.

I spent the year 1972–73 in Northern Illinois University, west of Chicago. John Lindsey was there, he was a student of Feit and he was also a Putnam fellow. He is one of these people who solves most of the American Mathematical Monthly problems. Also Selfridge was there, he was in number theory, as head of department. And Harvey Blau and Henry Leonard were there, who were group theorists. There were a few visitors there that year, like John Brilhart. Paul Erdös visited for a term that year. Derrick Lehmer was there, he was interested in computation and prime numbers, like Brilhart. There was also a very active seminar in semigroups, led by McFadden and McAlistair, both originally from Northern Ireland.

I remember that year there was a guy working in differential equations named Zettl who used to come around to the algebraists asking matrix theory questions that arose in his work. To my surprise, I found that I knew more matrix theory than most of the algebraists there, and I was able to answer some of this questions. I started to think more about matrix theory at that stage, and gradually moved into matrix theory.

To go back to Olga Taussky, I started to look at questions that Olga has asked, which Fergus told me about. I used to write to her quite a bit. She was motivated by questions in matrix theory and number theory, so she had a big influence on me and I spent a lot of time on that stuff. One thing about Olga Taussky was that she knew everybody. So if I was at a meeting and she was there, she would introduce me to all these people and tell them what I was doing. I remember one meeting in Pittsburgh where I met Harold Stark and Peter Lax. She was great like that, for introducing people around the place. She knew all the big name people. I remember one time

I was in Caltech, and John and Olga had Charles Fefferman and his wife and me for dinner. I knew her husband John Todd well, but not very much mathematically. His niece Jenny teaches in UCD.

I got more into linear algebra and matrix theory from about 1980 onwards, which led me to non-negative matrices and that's a topic I have been interested in for about the last ten years.

Mathematics In General

GMG: *How do you think mathematics has changed as we moved from the 20th to the 21st century?*

TL: One of the big changes that has happened since the 1960s is the development of algorithmic mathematics, and applications of mathematics. One was always aware of applications of differential equations, but the applications of discrete mathematics has tremendously influenced mathematics over the last twenty years. Bourbaki is dead now. It's more old fashioned than the mathematics of 100 years ago, in a sense. There was obviously motivation there, but they didn't stress the motivation, they derived mathematics from the axioms. We were trained in this way in the sixties, we were trained to do the definition and theorem and proof. We didn't get the intuition when we were trained, it wasn't considered the right thing to do. Also there was no reason for bringing in applications. The idea that applications might actually help you or give you an insight was not at all emphasized in pure mathematics. The idea was that pure mathematics was superior in the sense that it was totally rigorous. That whole world has changed. There is an experimental side to mathematics now that wasn't present when I was young. You can do lots of experiments to see if an equation has a solution, or something like that.

So I've seen a big change in the philosophy of the subject. The biggest change is that there is an interest in applications of the subject, and that is happening at every level of the subject. Even at the Fields medal level, people like Terry Tao are interested in applications of their work. That was not true 30 or 40 years ago.

Another aspect of mathematics that has changed is its unification, the development of machinery that unifies algebra and analysis and number theory.

There are also a lot more people involved in mathematics research now, there are a lot more papers being published each year.

Another thing I find interesting is the success there has been in solving famous problems. In the last forty or so years, a number of problems have been solved like the Poincaré conjecture and Fermat's Last Theorem. There are also other big results, like the classification of finite simple groups, which as far as anybody knows is complete. The easiest ones to understand are problems in number theory, questions to do with the distribution of primes. And there are totally new areas like coding theory. I remember the first time I heard about that was in a talk by Thompson, and he was talking about the projective plane of order 10 existence question, and he had derived properties of the code. He also had proved things about the automorphism group of the code. Eventually the computer was able to complete it, using the work of Thompson and others. But that's a new kind of thing, algebra being applied. Another lecture I remember is one at the BMC by Conway who talked about Leech lattices and Golay codes.

One area I would like to have known more about at an earlier stage in my life is engineering, and how these things, particularly algebraic things, are used in engineering. In the 1990s there were articles in several journals about how algebra was being used, how things like finite fields and difference equations were now useful. We discussed offering coding theory and finite fields to the engineering students here.

GMG: *How has mathematics changed in Ireland?*

TL: There are a lot more people now doing mathematics in Ireland. The amount of research in the country has enormously increased, even in the last ten years. When I started, there was very little research done in Ireland. Around 1970, say, there were a couple of people doing some research in TCD. In Galway there was some research in group theory, which is the area they focused on. Cork had some research in complex analysis. There were a few people getting research off the ground in UCD. Now this picture has changed enormously.

GMG: *What do you think of the funding situation in Ireland?*

TL: One of the problems facing mathematics is to get more people to do it. Having money to fund people to do PhDs is a big change that happened in Ireland, and this is good for getting people to study mathematics. It is good for academic staff here to have PhD students around – in the early days we didn't have that. We didn't have money to support students, except for the little money they could get from teaching. Now we have more funding. It's great to have postdocs and PhD students around our universities. It's great to be able to bring people in from abroad as well. I think we should continue to send some students abroad for their PhD. I do think that our best students should get some experience abroad, not necessarily at PhD level, it could be at postdoc level. It's good to go to a big centre abroad. Before, this was forced on us; the only way we could get funding was to go abroad, so most people went to the US for their PhD.

I do worry sometimes about how all these PhDs will get jobs. The theory is that Ireland will develop an economy and an industry that can hire PhDs in mathematics, like the US has. The present economic situation is a bit of a setback to that. In some places, like in Toronto for example, they prefer to hire PhDs in mathematics to work in their financial centre. In Germany some Deutsche Bank people said that the maths PhDs were the most useful people they hired, and mathematicians have been using that there to get more people to study maths. Apparently it has been successful there.

GMG: *Tom, thank you very much.*

Gary McGuire,
School of Mathematical Sciences,
University College Dublin,
Dublin 4, Ireland
gary.mcguire@ucd.ie

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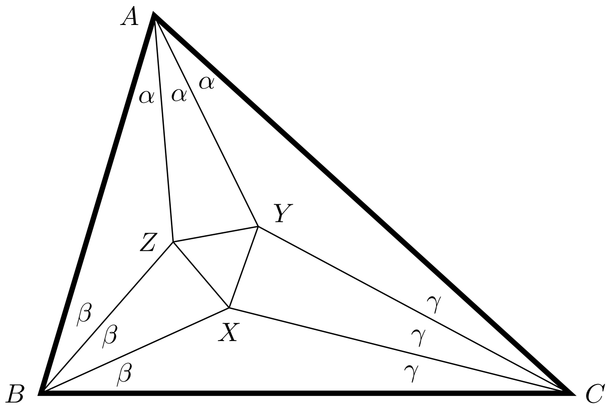
MacCool's Proof of Morley's Miracle

M. R. F. SMYTH

One of the most beautiful results in plane geometry is known as *Morley's Miracle* (1899). In essence it states that the triangle XYZ in the figure below is always equilateral. It features prominently on the front cover of the popular work [2] but is "still not as well-known as it deserves to be" [3]. The excellent web article [1] continues to track its development and also hosts a wide variety of proofs. None of the early proofs was easy but since 1990 elementary ones have emerged which are *backward* in the sense that they start from the equilateral triangle and eventually reconstruct the original. Finding a *direct* proof that matches them in brevity and simplicity has always been an elusive goal [3].



So I was amazed to find just such a proof in MacCool's notebooks and indeed it was so short that I nearly missed it. At first glance he seemed to be merely doodling, but moments later he had finished the proof and was working on something completely different.

Those readers who haven't heard of MacCool's notebooks may be surprised to learn that I am still less than halfway through the first one. Translation from the Ogham script is proving a long slow process and I am deeply indebted to one correspondent who reviewed and improved upon my original efforts, often spotting intricacies that I had overlooked. Although the gist of his arguments is always clear MacCool delights in recording only a minimum of information, and this particular proof was little more than a sketch decorated with jottings of line segments and angles. Like all the rest so far it is based solely on straight line geometry and similar triangles, but anyone interested in more advanced concepts may be pleased to know that diagrams containing circles begin to appear early in book two.

In his doodle the unit of measure is the perpendicular DX, and the lengths of BX and CX are s and s'. E and F are points on BC where $\angle BXE = \angle FXC = 60^o$. P and P' are where $BP = s$ and $CP' = s'$ and S is constructed so that $BS = s$ and $\angle SBX = 120^o$. This makes the four marked angles 60^o (even if ΔABC is obtuse). The rest of his construction is self-explanatory.
Now by (vi) and (vii) $2ST = 2SU + 2UT = s + 2\left(s - 2s^{-1}\right) = 3s - 4s^{-1}$ and $\Delta BQV \sim \Delta BDX$ yields $VQ = 1 - 4s^{-2}$ so $PQ = PV + VQ = 3 - 4s^{-2}$ thus

$$2ST = sPQ.$$

Then by (iv) and (v)

$$AY = \left(\frac{AC}{s'}\right)\left(\frac{PS}{ST}\right) = \frac{2AC.PS}{ss'PQ}.$$

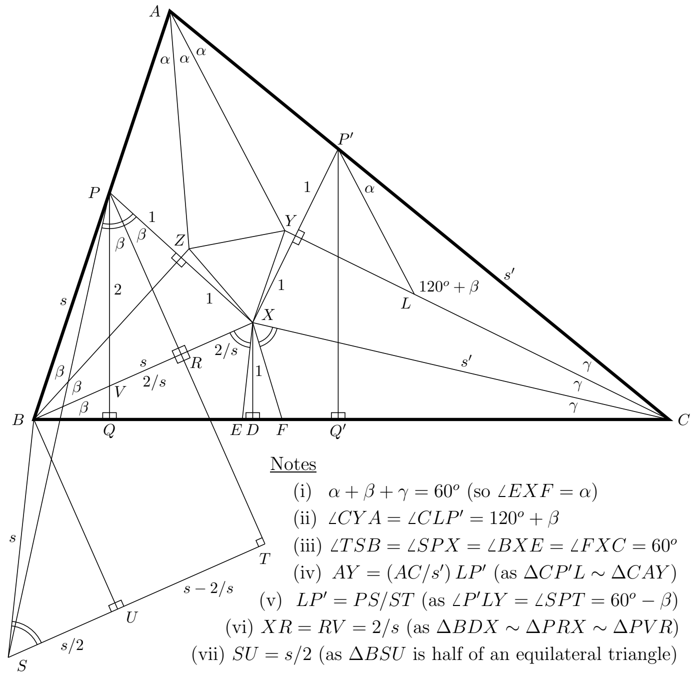
If W is the midpoint of PS then since ΔBSP is isosceles ΔBWP and ΔEDX will have identical angles, hence $\Delta BWP \sim \Delta EDX$ giving $XE = 2s/PS$. Therefore

$$XE.AY = \frac{4AC}{s'PQ}$$

and, by symmetry,

$$\frac{XE.AY}{XF.AZ} = \left(\frac{4AC}{s'PQ}\right)\left(\frac{sP'Q'}{4AB}\right) = \frac{sAC.P'Q'}{s'AB.PQ} = 1$$

because $PQ(AB/s) = P'Q'(AC/s')$ is the height of ΔABC. However this means $AZ : AY = XE : XF$ and as $\angle ZAY = \alpha = \angle EXF$



then $\Delta AZY \sim \Delta XEF$. Hence $\angle YZA = \angle FEX = 60^o + \beta$ and $\angle AYZ = \angle XFE = 60^o + \gamma$. Analogous arguments for ΔBXZ and ΔCYX show $\angle ZXB = 60^o + \gamma$, $\angle CXY = 60^o + \beta$ and $\angle BZX = \angle CYX = 60^o + \alpha$. All the angles in the doodle may now be deduced in terms of α, β, γ and it transpires that every angle of ΔXYZ is 60^o.

Here are some comments on the proof leading to a slight variation that may help to make it more intuitive. The underlying idea is to treat it as a series of left/right linkages. The results that $2ST = sPQ$ and $XE = 2s/PS$ are clearly "internal" to the left hand side. On the other hand LP' has a foot in each camp since it can be expressed both in terms of objects from the left PS/ST and objects from the right $s'AY/AC$. Equating these expressions gives a "cross-linkage" $XE.PQ = 4AC/(s'AY)$ and its companion $4AB/(sAZ) = XF.P'Q'$ which may then be combined to form the complicated looking quotient above. Even MacCool seems to have been shocked by the final devastating cross-linkage $PQ.AB/s = P'Q'.AC/s'$ which reduces this quotient to unity. After that the rest is plain sailing.

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M. R. F. Smyth,
15 Harberton Avenue,
Belfast BT9 6PH
malcolm.smyth@ntlworld.com

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