**Irish Mathematical Society**

*Cumann Matamaitice na hÉireann*

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**Contents**

## EDITORIAL

*Do you publish in the "list"? The "list" here refers to the ISI or Thomson Scientific impact factor list, which ranks a number of mathematical journals (and many other scientific periodicals, altogether more than 9,000) according to their impact factor. For a particular journal and year, the journal impact factor is computed by calculating the average number of citations to articles in the journal during the preceding two years from subsequent articles published in the collection of indexed journals in that given year. (It should be immediately noted that Thomson Scientific indexes less than half the mathematics journals covered by Mathematical Reviews and Zentralblatt.) Originally intended "not to be used without careful attention to the many phenomena that influence citation rates, as for example the average number of references cited in the average article. The impact factor should be used with informed peer review."* [Thomson]*, the impact factor has nowadays become one of the most important (sometimes the sole) bibliometric data on which the quality of a journal—and by extrapolation, the quality of the articles and their authors—are judged.*

*Research funding bodies, such as governments and research councils, increasingly rely on what they deem to be simple and objective criteria to assess the quality of research. One is expected to publish in* good *journals. This is decisive for the award of a grant and for promotion prospects. Committees base their judgement of what is* good *on these bibliometrics rather than on "subjective" assessment methods such as peer review. Mostly because this approach is easier to understand and capable of handling large numbers of applications. However, there are serious problems with an oversimplified methodology to assess mathematical research—this has been pointed out by many before and is very impressively demonstrated in a detailed report commissioned by the IMU, which can be found at*

`http://www.mathunion.org/fileadmin/IMU/Report/CitationStatistics.pdf`

*The main Canadian research council NSERC write in their guidelines "Selection committees and panels are advised by NSERC to neither rely on numbers of publications in their assessment of productivity nor create or use lists of 'prestigious' or 'unacceptable' journals in their assessment of quality. The quality of the publication's content is the determining factor, not that of the journal in which it appears, and the onus is on the applicant to provide convincing evidence of quality." and "The ultimate tests of quality of any research contribution or publication are its significance and use by other researchers and end-users, and the extent to which it influences the direction of thought and activity in the target community."*

*Of course, it is each of us own decision where we submit our papers, and there are manifold reasons for choosing a particular journal. Even if oneself does not feel "bound" to the list, maybe your co-author insists on publishing in a periodical that it highly ranked. Maybe your head of department.*

*As editor of the* Bulletin *I am glad that this journal is not in the list. And I hope that, in the medium and long term, enough people who have a say will agree with the IMU's statement that "Research is too important to measure its value with only a single coarse tool." and "If we set high standards for the conduct of science, surely we should set equally high standards for assessing its quality."*

*—MM*

## NOTICES FROM THE SOCIETY

### Officers and Committee Members

| | | |
|---|---|---|
| **President** | Dr J. Cruickshank | Dept. of Mathematics<br>NUI Galway |
| **Vice-President** | Dr S. Wills | Dept. of Mathematics<br>NUI Cork |
| **Secretary** | Dr S. O'Rourke | Dept. of Mathematics<br>Cork Inst. Technology |
| **Treasurer** | Dr S. Breen | Dept. of Mathematics<br>St Patrick's College<br>Drumcondra |

Prof S. Dineen, Dr S. Breen, Prof S. Buckley, Dr T. Carroll, Dr J. Cruickshank, Dr B. Guilfoyle, Dr R. Higgs, Dr C. Hills, Dr P. Kirwan, Dr N. Kopteva, Dr M. Mathieu, Dr S. O'Rourke, Dr C. Stack, Dr N. O'Sullivan, Prof R. Timoney, Prof A. Wickstead, Dr S. Wills

### Local Representatives

| | | |
|---|---|---|
| **Belfast** | QUB | Dr M. Mathieu |
| **Carlow** | IT | Dr D. Ó Sé |
| **Cork** | IT | Dr D. Flannery |
| | UCC | Prof. M. Stynes |
| **Dublin** | DIAS | Prof Tony Dorlas |
| | DIT | Dr C. Hills |
| | DCU | Dr M. Clancy |
| | St Patrick's | Dr S. Breen |
| | TCD | Prof R. Timoney |
| | UCD | Dr R. Higgs |
| **Dundalk** | IT | Mr Seamus Bellew |
| **Galway** | UCG | Dr J. Cruickshank |
| **Limerick** | MIC | Dr G. Enright |
| | UL | Mr G. Lessells |
| **Maynooth** | NUI | Prof S. Buckley |
| **Tallaght** | IT | Dr C. Stack |
| **Tralee** | IT | Dr B. Guilfoyle |
| **Waterford** | IT | Dr P. Kirwan |

## Applying for I.M.S. Membership

1. The Irish Mathematical Society has reciprocity agreements with the American Mathematical Society, the Irish Mathematics Teachers Association, the New Zealand Mathematical Society and the Real Sociedad Matemática Española.

2. The current subscription fees (as from 1 January 2009) are given below:

| | |
|---|---|
| Institutional member | 160 euro |
| Ordinary member | 25 euro |
| Student member | 12.50 euro |
| I.M.T.A., NZMS or RSME reciprocity member | 12.50 euro |
| AMS reciprocity member | 15 US$ |

   The subscription fees listed above should be paid in euro by means of a cheque drawn on a bank in the Irish Republic, a Eurocheque, or an international money-order.

3. The subscription fee for ordinary membership can also be paid in a currency other than euro using a cheque drawn on a foreign bank according to the following schedule:

   If paid in United States currency then the subscription fee is US$ 30.00.
   If paid in sterling then the subscription is £20.00.
   If paid in any other currency then the subscription fee is the amount in that currency equivalent to US$ 30.00.

   The amounts given in the table above have been set for the current year to allow for bank charges and possible changes in exchange rates.

4. Any member with a bank account in the Irish Republic may pay his or her subscription by a bank standing order using the form supplied by the Society.

5. Any ordinary member who has reached the age of 65 years and has been a fully paid up member for the previous five years may pay at the student membership rate of subscription.

6. Subscriptions normally fall due on 1 February each year.

7. Cheques should be made payable to the Irish Mathematical Society. If a Eurocheque is used then the card number should be written on the back of the cheque.

8. Any application for membership must be presented to the Committee of the I.M.S. before it can be accepted. This Committee meets twice each year.

9. Please send the completed application form with one year's subscription to:

   The Treasurer, I.M.S.
   Department of Mathematics
   St Patrick's College
   Drumcondra
   Dublin 9, Ireland

# IRISH MATHEMATICAL SOCIETY
## President's Report 2008

**Introduction:** I have delegated a large number of my duties to members of the Committee, whom I gratefully thank. This ensures that such members maintain an active interest in the Society, without hopefully being over-burdened with administrative duties. In particular attendance at the Committee meetings in December 2007 and September 2008 was notably high.

**Meetings:** I represented the IMS at the first meeting of Presidents of European mathematical societies in Marseilles in April, which was hosted by the European Mathematical Society. It was of note that increasingly new employees in mathematics are not joining their respective national societies, despite the low cost.

I also attended the Council Meeting of the European Mathematical Society in Utrecht in July. At this meeting Kracow was elected as the host for the 2012 European Congress of Mathematics.

Ireland at the moment only has 16 members of the European Mathematical Society, which will cost €24 to join next year as a reciprocity member from the IMS. The EMS reciprocity fee can be paid to the Treasurer of the IMS at the same time as your normal IMS membership fee, or it can be paid directly via the EMS website. Benefits of EMS membership now include free access to Zentralblatt (for the next two years).

**Local Representatives:** The local representatives of the IMS have been refreshed, mostly with present members of the Committee. Their job is to recruit members from their respective institutions.

**Membership communication:** An insert was placed in one of the Bulletins this year for members to update their details including e-mail addresses. The number of e-mail addresses received was about two-thirds of the membership. My thanks to Gordon Lessells for taking on the administration of this task and his continued help in the production and distribution of the Bulletin.

**Treasurer:** My thanks to Sinead Breen for being the Society's Treasurer this year, a job she executed without any fuss. She will investigate the possibility of making membership payments by electronic transfer, bearing in mind the potential for fraud. She is also considering a new policy for the allocation of funds to support conferences.

**Links:** The Society now has a formal reciprocity agreement with the New Zealand Mathematical Society. My thanks to Niamh O'Sullivan, who is currently looking after all reciprocity matters.

**Website:** The IMS website `http://www.maths.tcd.ie/pub/ims/` continues to be maintained and regularly updated by Richard Timoney, my thanks to him. We have decided to discontinue the general IMS e-mailing facility through Trinity as it is no longer used.

**SFI Mathematics Initiative:** As members are aware this scheme is not running this year, and is unlikely to be resurrected given the present economic situation.

**Bulletin:** Thanks to Martin Mathieu for continuing to act as Editor of the Bulletin and in particular in producing issue 61, Summer 2008. Issue 62 should be distributed in January 2009. Professor Boland (UCD) has given the Society a nearly complete run of the Bulletin, which Martin is keeping on behalf of the Society.

**Annual conferences:** The annual conference and AGM was successfully held in Cork IoT on September 1st and 2nd with a wide variety of talks. My thanks to Shane O'Rourke for not only organizing the conference, but also acting as Secretary of the Society.

**Fergus Gaines' Cup:** The Irish Mathematical Society awards the Fergus Gaines' Cup annually to the best performer in the Irish Mathematical Olympiad. I awarded the cup in St Patrick's College on 15th November 2007 to Stephen Dolan, and on 6th November 2008 to Galin Ganchev in the presence of the Bulgarian ambassador. I also made two special presentations on 6th November 2008 on behalf of the Society to Professor Tom Laffey (UCD) and Mr Gordon Lessells (UL) for their long and continued service to Irish Mathematical Olympiad training. Thanks were expressed to St Patrick's College and Maurice O'Reilly for hosting and arranging the event respectively.

**Teaching:** A policy document on service teaching of mathematics was formally approved in September with one minor amendment and has been posted on the IMS website as a position paper:
`http://www.maths.tcd.ie/pub/ims/business/2008.09.02.serviceteaching.pdf`

**Public Relations:** Brendan Guilfoyle is the Public Relations Officer of the Society. His name together with that of Ann O'Shea (as a reserve) have been given to IBEC, the Irish Times, the Irish Independent, and Engineer's Ireland.

**National Science and Engineering Commemorative Plaques Committee:** Maurice O'Reilly was appointed as the representative of the IMS on this Committee.

**Institute of Technology members:** Jim Cruickshank has been chairing the sub-committee on IoT members and reported on its progress in September. This report has been posted on the IMS website as a discussion document:
`http://www.maths.tcd.ie/pub/ims/business/2008-09-01-IoT.pdf`

**Conference support:** The Society supported the following conferences in 2008:

- International Workshop on Multi-Rate Processes & Hysteresis, UCC, 31 March–5 April
- Complex Function Theory Meeting, UCC, 18 April
- Operator Theory & Operator Algebras in Cork (in memory of Gerard Murphy), UCC, 7–9 May
- Third Conference on Mathematics Service Teaching, WIT, 26–27 May
- 23rd British Topology Meeting, QUB, 25–27 August
- Instructional Workshop on Subfactors & Planar Algebras, QUB, 26–28 August

**David Armitage:** There was a dinner in Belfast to mark the retirement of David Armitage (a former President of the Society). I expressed the best wishes of the Society to David by e-mail and Maurice O'Reilly represented the Society at the dinner and spoke on its behalf.

**Irish Mathematical Trust:** Stephen Buckley is the Society's representative in this regard. There was one meeting on this topic in May, which was attended by Maurice O'Reilly in the absence of Stephen.

**RIA Mathematical Sciences Committee:** Stephen Buckley has kindly volunteered to be the Society's next representative on this committee.

**Future meetings:** Future planned meetings of the IMS are as follows:

- April 6-9 2009, NUI Galway: Joint meeting of the British Mathematical Colloquium & the IMS
- September 2010, Dublin IT: IMS Conference & AGM
- September 2011, University of Limerick: IMS Conference & AGM
- September 2012, IT Tallaght: IMS Conference & AGM

A special arrangement will have to be made for the AGM in 2009, since it must be held after July 31st.

Russell Higgs
President of the Irish Mathematical Society
5th December 2008

**Minutes of the Meeting**
**of the Irish Mathematical Society**
Annual General Meeting
2nd September 2008

The Irish Mathematical Society held its Annual General Meeting from 12:00 to 13:00 on Tuesday 2nd September at Cork Institute of Technology. There were 22 members present at the meeting.

**1. Minutes**

The minutes of the meeting of September 2007 were approved and signed.

**2. Matters arising**

The President clarified that the new membership rate in sterling would be £20 rather than £18 in light of the adverse conversion rate to the €.

**3. Correspondence**

There was only routine correspondence.

**4. New Members**

*Institutional member*: St Patrick's College, Drumcondra.
*Ordinary members*: Mary Hanley, Conor Muldoon, Norah Daly, David Barrett, Christina Naughton, Francesca O'Rourke, Kevin Jennings, Garry Plunkett, Claas Roever, Gavin Bradley, David Henry, George Wyatt, Sarah Mitchell, Kevin Murphy.
*Student member*: Jo Anne McLoughlin (NUIG).

**5. President's Report**

The President presented an interim report on issues that have arisen this year. He thanked Sinéad Breen for her efficient work as Treasurer over the year, and Richard Timoney for maintaining the IMS website. He also thanked Gordon Lessells and Martin Mathieu, respectively, for their sterling work on producing and editing the Bulletin.

**6. Treasurer's Report**

The Treasurer presented her report for 2007. It shows a surplus of 2,142.07€. It was noted that there were relatively few applications for funding conferences this year. The report was approved.

**7. The Bulletin**

The Editor reported that Volume 61 is ready and is being distributed. He noted that there were few research articles from Irish mathematicians, though he received many submissions from other parts of the world, and he encouraged members to submit articles to the Bulletin. The section on announcements of conferences will be discontinued as the Society's on-line diary is a better medium. He suggested that the section on Departmental News be revived, and should include retirements and new appointments.

**8. Election to Committee**

The following were elected unopposed to the committee:

| Committee Member | Proposer | Seconder |
|---|---|---|
| J. Cruickshank (President) | R. Higgs | M. Mathieu |
| S. Wills (Vice President) | J. Cruickshank | S. O Rourke |
| C. Stack | R. Higgs | R. Ryan |

As editor, M. Mathieu will be invited to committee meetings. The total number of years each existing member will have been on the committee as of 31 December 2008 will be: J. Cruickshank (6), T. Carroll (5), N. O'Sullivan (5), R. Timoney (4), R. Higgs (4), N. Kopteva (3), B. Guilfoyle (3), S. Breen (2), S. O'Rourke (2).

The following will then have one more year of office: T. Carroll, N. O'Sullivan.

**9. EMS Membership**

The President noted that there were currently only 16 members of the European Mathematical Society with addresses in Ireland. He encouraged IMS members to join the EMS, noting that the membership rate for existing IMS members will be 24€ next year.

**10. David Armitage retirement**

M. O Reilly represented the Society at a dinner to mark David Armitage's retirement, and conveyed the Society's thanks for his service to it.

**11. Visit of John Curran**

It was noted that John Curran, a member of the New Zealand Mathematical Society, was visiting Ireland, and it was suggested that he be invited to speak at seminars.

**12. Adoption of Service Teaching Report**

This report was formally adopted by the meeting without objection. Cora Stack had made a written submission suggesting that the report could have a wider scope, and could go further to emphasise the benefits of a rounded mathematical education. Maurice O Reilly emphasised the importance that the report be implemented. The President agreed to disseminate the report to the heads of Irish universities and IoTs.

He thanked the Committee for Service Teaching of Mathematics and the IMS Subcommittee for Educational Issues for their work.

**13. Any Other Business**

There were four other items of business.

(i) **Irish Mathematical Trust** Maurice O Reilly and Stephen Buckley attended a meeting to examine the possibility of setting up an Irish Mathematical Trust. Such a body would play a leading role in second-level mathematics competitions including publicity. A group is looking into the financial, legal and other arrangements required for setting up the Trust.

(ii) **Institute of Technology Subcommittee** Jim Cruickshank chaired a subcommittee to consider the role that the IMS can play for mathematicians in the IoT sector. He presented his report to the meeting. The report highlights the fact that there are many mathematicians in the IoT sector with PhDs but who are no longer active in research or who feel isolated, and who might be encouraged or supported by greater interaction both with the university sector and between institutes. It was noted however that many mathematicians in the IoT sector are not interested in research, seeing their role as primarily a teaching role. Such mathematicians might welcome a greater emphasis on mathematics education in the IMS meetings. Finbarr Holland suggested the possibility of IoTs and universities in different regions jointly organising colloquia.

It is anticipated that there will be an ongoing discussion on this matter.

(iii) **Future Meetings** There will be a joint BMC/IMS meeting in NUI Galway on 6-9 April 2009. It was suggested that the next AGM take place in Dublin, possibly in TCD, after 31 July.

(iv) **Thanks to Outgoing President** Russell Higgs was warmly thanked by the meeting for his tireless work as President of the Society over the last two years.

Shane O Rourke
CIT.

These minutes still need to be approved by the next AGM.

**PROGRAMME**
**21st SEPTEMBER MEETING**

CIT Cork
1–2 September 2008

**Monday, 1 September**

| | |
|---|---|
| 10:00–10:45 | Registration and coffee |
| 10:45–11:00 | Opening remarks |
| 11:00–11:50 | **Daphne Gilbert (DIT)**<br>*Singular Sturm–Liouville boundary value problems on the line* |
| 12:00–12:25 | **Patrick Quill (CSO)**<br>*An Input-Output model of the Irish economy* |
| 12:30–12:55 | **Sander Zwegers (UCD)**<br>*Mock modular forms* |
| 13:00–14:00 | Lunch |
| 14:00–14:50 | **Bill Lynch (NCCA)**<br>*Project Maths—developments in post-primary mathematics education* |
| 15:00–15:25 | **Jerome Sheahan (NUIG)**<br>*Linking probability and statistics* |
| 15:30–16:00 | Coffee |
| 16:00–16:50 | **Martin Bridson (Oxford)**<br>*The geometry and complexity of finitely presented groups* |
| 17:00–18:00 | IMS Committee Meeting |
| 19:30 | Conference Dinner |

**Tuesday, 2 September**

| | |
|---|---|
| 9:30–9:55 | **Robin Harte (TCD)**<br>*Where algebra and topology meet:*<br>*a cautionary tale* |
| 10:00–10.25 | **Cora Stack (IT Tallaght)**<br>*On the powers of a nilpotent algebra* |
| 10:30–11:00 | Coffee |
| 11:00–11:50 | **Niall Smith (CIT)**<br>*Castles, martians and algorithms* |
| 12:00–13:00 | IMS Annual General Meeting |
| 13:00–14:00 | Lunch |
| 14:00–14:50 | **Conor Houghton (TCD)**<br>*Understanding spike trains* |
| 15:00–15:25 | **Ciaran Mac an Bhaird (NUIM)**<br>*The use of technologies to create an*<br>*active mathematical environment for students* |
| 15:30–15:55 | **Martin Kilian (UCC)**<br>*On the Lawson conjecture* |
| 16:00–16:30 | Coffee |

# The IMS September Meeting 2008 at CIT, Cork
# Abstracts of Invited Lectures

## Singular Sturm - Liouville Boundary Value Problems on the Line

Daphne Gilbert (DIT)

We consider the relationship between the asymptotic behaviour of solutions of the singular Sturm-Liouville equation and spectral properties of the corresponding self-adjoint operators. In particular, we review the main features of the theory of subordinacy by considering two standard cases, the half-line operator on $L_2([0,\infty))$ and the full-line operator on $L_2(R)$. It is assumed that the coefficient function $q$ is locally integrable, that 0 is a regular endpoint in the half-line case, and that Weyl's limit point case holds at the infinite endpoints. We note some consequences of the theory for the well-known informal characterisation of the spectrum in terms of bounded solutions. We also consider extensions of the theory to related differential and difference operators, and discuss its application in conjunction with other asymptotic methods to some typical problems in spectral analysis.

## An Input-Output Model of the Irish Economy

Patrick Quill (CSO)

This talk presents the supply and use framework applied by the national accounts section of CSO. This framework is employed to balance different measures of GDP, to give meaningful estimates of transactions within the economy in a given year as well as to integrate national accounts aggregates with business survey results. A matrix transformation converts the use table into an input-output table. Methods used for expanding the number of rows and columns are examined. The Leontief Inverse is defined and interpreted. Techniques for analysing input-output tables over time are discussed.

## Mock modular forms

Sander Zwegers (UCD)

Ramanujan wrote his last letter to Hardy in January of 1920, a few months before his death, telling him about a new class of functions he had discovered which he called “mock theta functions”. Many people have studied these functions, including famous mathematicians like Watson, Selberg and Andrews, who found many wonderful identities concerning them. However, for a long time, no natural definition was known that described what these functions are intrinsically and hence could give a natural explanation for the identities between them. More recently, an interpretation was found for these mock theta functions within the theory of modular forms, which enables us to give a natural definition.

This interpretation has opened the way to further progress and to the construction of infinitely many new examples.

In this talk we will describe Ramanujan’s original examples and the nature of their modularity, and discuss some of the further progress that has been made.

## Project Maths - developments in post-primary mathematics education

Bill Lynch (NCCA)

Following a review of post-primary mathematics education, changes in mathematics syllabuses at second level get under way in a small group of schools from September 2008, with roll-out to all schools commencing in September 2010. Project Maths is a new initiative in curriculum development, which sees a phased, incremental approach to syllabus revision in tandem with, and informed by, professional development and support for teachers in introducing a changed approach to the teaching and learning of mathematics. The project is aimed at both Junior Certificate and Leaving Certificate mathematics and also takes into consideration the links between mathematics in the primary school and that in the first year at second level.

This presentation looks briefly at the background to Project Maths and the proposals that have been adopted for its introduction. The structure and format of the revised syllabuses at different levels are described, together with the revised assessment arrangements that are seen as key to reinforcing and supporting changed methodology in the classroom. The changed approach sees greater emphasis being placed on the students understanding of the mathematical concepts involved, together with a focus on the development of problem-solving skills and strategies rather than reliance on rote learning of procedures. An overview is given of the timescale for the project, with detail being provided on the two revised syllabus strands which are being introduced in 24 schools from September. Finally, the programme of teacher professional development and support which has been put in place is described, as well as an outline of the resources being developed to support the project.

**Linking Probability and Statistics**

JEROME SHEAHAN (NUIG)

Probability and statistics books introduce, but generally treat in an isolated manner, terms like 'sample space', 'population', 'random variable' and 'distribution'. By differentiating and linking these concepts, we hope to de-mystify and unify the undergraduate and graduate level teaching of the (scientifically opposite) fields of probability and statistics. On the way, we give recent developments on a number of issues in mathematical modelling, including the question of whether probability is the only way of modelling random variation, and we give the latest results on a famous probability problem with an unexpected answer.

**The geometry and complexity of finitely presented groups**

MARTIN BRIDSON (OXFORD)

I'll begin with a general discussion about why one should regard groups as objects that belong not so much to algebra as to mathematics as a whole. I'll discuss why finite presentability is a natural constraint to impose on groups, and I will explain why Dehn's decision problems are so fundamental to the understanding of groups. Following a brief sketch of the universe of finitely presented groups, I'll focus on a fascinating class of groups closely related to free groups – limit groups – where one finds particularly deep connections between geometry, topology, algebra and logic.

### Where algebra and topology meet: a cautionary tale

Robin Harte (TCD)

In a sense the Kuratowski axioms reduce topology to algebra. In another sense one of the cornerstones of Banach algebra theory ushers in a curious topology for rings.

### On the powers of a nilpotent algebra

Cora Stack (IT Tallaght)

An algebra $R$ over a field $K$ is said to be nilpotent if $R^n = 0$ some $n \geq 1$. It is widely accepted that the structure of nilpotent algebras is not well understood. A better understanding of the structure is crucial if further significant breakthroughs are to be made in the theory of these algebras. Questions in nilpotent algebras have also a very important bearing on other questions in more general ring theory, group theory, coding theory etc. In this talk I will discuss and prove some recent results in the structure theory by considering certain relationships between the various powers $R^i$ of the algebra $R$.

### Castles Martians and Algorithms

Niall Smith (CIT)

In 2007 Blackrock Castle in Cork opened its interactive science center to the public. Based upon the theme of "The Search for Life in the Universe" and called "Cosmos at the Castle", the award-winning center informs visitors about our present knowledge of the universe, and examines the likelihood that we may not be alone. The Castle also houses Ireland's first robotic observatory, operated by researchers' from CIT's Astronomy & Instrumentation Group. The group recently launched its PlanetSearch Programme. This talk will summarise the Blackrock Castle Observatory project, describe our experiences to date and our plans for the future. The project website is `http://www.bco.ie`

### Understanding spike trains

Conor Houghton (TCD)

Axons connect neurons; axons are thin, membrane-walled tubes the interior fluid of which is at a lower voltage to the exterior. Axons support the propagation of what are called spikes, brief voltage pulses of stereotypical profile and amplitude. In the brain information propagates between neurons in the form of spike trains, sequences of spikes. It is not known in any detail how information is coded in spike trains, this is a difficult problem because the spike trains themselves are unreliable, the same stimulus acting on a neuron leads to different spike trains from trial to trial. Here I will describe how defining a metric on the space of spike trains can help determine properties of the information coding.

### The use of technologies to create an active mathematical environment for students

Ciaran Mac an Bhaird (NUIM)

One of the main challenges in third level Mathematics education is how to address the issue of the weak mathematical background of incoming students. The numbers of students with poor understanding of core mathematical material seems to be constantly increasing. Recent reports have expressed concern with the mathematical competences of Irish students at second level (State Examinations Commission, 2005; NCCA, 2006), and low attainment in Mathematics is often cited as a contributing factor in low enrollment and low retention rates in science and technology courses (Task Force on the Physical Sciences, 2002).

Students have widespread access to complex technologies including advanced computer software, state of the art mobile phones and ipods. We should take advantage of their interest in such technologies, and incorporate as much Mathematics as possible into similar environments.

Such initiatives aim to equip students with the mathematical skills they need to succeed at university. The Mathematics Support Centre (MSC) and the Department of Mathematics in NUI Maynooth are actively engaged in introducing new methods of mathematical teaching including the use of podcasting, screencasting and touchscreen technologies. Students, especially weaker students, have been shown to respond very positively to these innovations. We will discuss all the feedback from these developments, as well as the challenges that face anyone hoping to follow a similar path. There will also be a brief demonstration of some of the software and equipment that we use.

### On the Lawson Conjecture

Mrtin Kilian (UCC)

While there are no compact minimal surfaces in Euclidean 3-space, Lawson showed in 1970 that the curvature of the 3-sphere allows for embedded compact minimal surfaces of arbitrary genus. In particular, in collaboration with Hsiang he investigated minimal tori in the 3-sphere, and conjectured that the only embedded minimal torus in the 3-sphere is a torus which possesses a 2-parameter family of isometries, the so-called Clifford torus. In recent work in collaboration with M.U. Schmidt, I proved that Lawson's conjecture indeed holds, and in this talk I will give an outline of the proof, which uses modern methods from the theory of integrable systems.

# ANNOUNCEMENTS OF MEETINGS AND CONFERENCES

This section contains the announcement of the annual meeting of the IMS and closely related conferences (satellites) as supplied by organisers. The Editor does not take any responsibility for the accuracy of the information provided.

## *Joint Meeting of the 61st British Mathematical Colloquium and the 22nd Annual Meeting of the IMS*

**NUI Galway**

**April 6–9, 2009**

The second joint meeting of the BMC and the annual IMS meeting, following on to the first such meeting at Queen's University Belfast in 2004, will be held at the National University of Ireland, Galway between 6 and 9 April 2009.

The speakers comprise David Eisenbud (Berkeley), Ben Green (Cambridge), Ron Graham (San Diego), Rostislav Grigorchuk (Texas A&M) and Frances Kirwan (Oxford), together with twelve Morning Speakers. A Public Lecture will be held by Tom Koerner (Cambridge). There will be a Special Session on *Computational Algebra* led by Eamonn O'Brien (Auckland) and Goetz Pfeiffer (Galway) and a Special Session on *Analysis* led by David Preiss (Warwick), Sean Dineen (Dublin) and Ray Ryan (Galway). There will be opportunities to present talks at various splinter groups.

The meeting is supported by the London Mathematical Society and the Irish Mathematical Society.

Full information on registration and the programme is available at

`http://www.maths.nuigalway.ie/bmc2009/`

## *3rd International Workshop on Elementary Operators and their Applications (ElOp2009)*

**Queen's University Belfast**

**April 14–17, 2009**

As a satellite to the BMC2009 (see the announcement above), the third international workshop on elementary operators and applications will be held in the Pure Mathematics Research Centre of Queen's University, Belfast between April 14-17, 2009. The workshop is organised by Dr. Martin Mathieu and supported by the London Mathematical Society and the Irish Mathematical Society.

Graduate students studying in the RoI or the UK can be supported; for details please contact `elop2009@qub.ac.uk`

Full information on registration, the programme and a list of speakers is available on the conference website

`http://elop2009.awardspace.co.uk/`

# Abstracts of PhD Theses at Irish Universities 2008

## Methods of Ascent and Descent in Multivariable Spectral Theory

Derek Kitson
*dk@maths.tcd.ie*

This is an abstract of the PhD thesis *Methods of ascent and descent in multivariable spectral theory* written by Derek Kitson under the supervision of Professor Richard M. Timoney at the School of Mathematics, Trinity College Dublin and submitted in June 2008.

In this thesis the classical notions of ascent and descent for an operator acting on a vector space are extended to arbitrary collections of operators. The resulting theory is applied to the study of joint spectra for commuting tuples of bounded operators acting on a complex Banach space. Browder joint spectra are constructed and shown to satisfy a spectral mapping theorem.

For a set $A$ of operators on a vector space $X$ we define the *ascent* $\alpha(A)$ and *descent* $\delta(A)$ as the smallest non-negative integers such that

$$N(A) \cap R(A^{\alpha(A)}) = \{0\} \text{ and } N(A^{\delta(A)}) + R(A) = X$$

where $N(A)$ denotes the joint null space, $R(A)$ the joint range space and $A^k$ the set of all products of $k$ elements. We show that the collection $A$ has finite ascent and finite descent if and only if there exist $A$-invariant subspaces $X_1, X_2$ with $X = X_1 \oplus X_2$ such that the restriction of $A$ to $X_1$ satisfies a nilpotent condition while $A$ restricted to $X_2$ satisfies a bijectivity condition. Moreover, the ascent and descent of $A$ are necessarily equal and determine $X_1$ and $X_2$ uniquely:

$$X_1 = N(A^r) \text{ and } X_2 = R(A^r) \text{ where } r = \alpha(A) = \delta(A).$$

For commuting $n$-tuples $\mathbf{a} = (a_1, \ldots, a_n)$ of bounded operators on a complex Banach space we define a Browder joint spectrum

$$\sigma_b(\mathbf{a}) = \{\lambda \in \mathbb{C}^n : \mathbf{a} - \lambda \notin \mathcal{B}\}$$

where $\mathcal{B}$ is the collection of commuting Fredholm $n$-tuples with finite ascent and finite descent. This Browder joint spectrum is smaller than the Taylor-Browder spectrum of [1] but contains the upper and lower semi-Browder spectra of [3]. We show that this Browder joint spectrum is compact-valued, has the projection property and consequently satisfies a spectral mapping theorem:

$$\sigma_b(f(\mathbf{a})) = f(\sigma_b(\mathbf{a}))$$

for all mappings $f$ holomorphic on the Taylor spectrum of $\mathbf{a}$. We also give a characterisation

$$\sigma_b(\mathbf{a}) = \bigcap_{\mathbf{r} \in \mathcal{R}} \sigma_\pi(\mathbf{a}+\mathbf{r}) \cup \sigma_\delta(\mathbf{a}+\mathbf{r})$$

where $\sigma_\pi$ and $\sigma_\delta$ denote respectively the joint approximate point and defect spectra and $\mathcal{R}$ denotes the collection of all commuting $n$-tuples of Riesz operators which commute with $a_1, \ldots, a_n$.

Analogous results are obtained for the Harte spectrum, the Taylor spectrum, the Slodkowski spectra $\sigma_{\pi,k} \cup \sigma_{\delta,l}$ and their split versions. We show that necessary and sufficient for a commuting $n$-tuple $\mathbf{a} = (a_1, \ldots, a_n)$ to be Taylor-Browder is that $\mathbf{a} = \mathbf{c} + \mathbf{s}$ where $\mathbf{c} = (c_1, \ldots, c_n)$ is a commuting tuple of compact operators, $\mathbf{s} = (s_1, \ldots, s_n)$ is Taylor-invertible and $c_i s_j = s_j c_i$ for all $i, j$.

Multivariable analogues of the notion of a *pole* and a *Riesz point* for an operator are introduced for commuting tuples $\mathbf{a} = (a_1, \ldots, a_n)$. We use poles to investigate a several variable version of N. Dunford's minimal equation theorem and Riesz points are used to characterise commuting tuples of Riesz operators. Applications to a multivariable Weyl's Theorem are considered.

## REFERENCES

[1] R.E. Curto and A.T. Dash, Browder Spectral Systems. *Proc. Amer. Math. Soc.*, 103(2):407-413, 1988.
[2] D. Kitson, Ascent and descent for sets of operators. *Studia Math.*, to appear.
[3] V. Kordula, V. Müller and V. Rakočević, On the Semi-Browder Spectrum. *Studia Math.*, 123(1):1-13, 1997.

## Polynomials on Riesz Spaces

John Loane
*johnloane@yahoo.com*

This is an abstract of the PhD thesis *Polynomials on Riesz Spaces* written by John Loane under the supervision of Dr. Ray Ryan at the Department of Mathematics, National University of Ireland, Galway and submitted in December 2007.

Mathematicians have been exploring the concept of polynomial and holomorphic mappings in infinite dimensions since the late 1800's. From the beginning the importance of representing these functions locally by monomial expansions was noted. Recently Matos studied the classes of homogeneous polynomials on a Banach space with unconditional basis that have pointwise unconditionally convergent monomial expansions relative to this basis. More recently still Grecu and Ryan noted that these polynomials coincide with the polynomials that are regular with respect to the Banach lattice structure of the domain.

In this thesis we investigate polynomial mappings on Riesz spaces. This is a natural first step towards building up an understanding of polynomials on Banach lattices and thus eventually gaining an insight into holomorphic functions.

We begin in Chapter 1 with some definitions. A polynomial is defined to be positive if the corresponding symmetric multilinear mappings are positive. We discuss monotonicity for positive homogeneous polynomials and then give a characterization of positivity of homogeneous polynomials in terms of forward differences.

In Chapter 2 we show that, as in the linear case positive multilinear and positive homogeneous polynomial mappings are completely determined by their action on the positive cone of the domain and furthermore additive mappings on the positive cone extend to the whole space. We conclude by proving formulas for the positive part, the negative part and the absolute value of a polynomial mapping.

In Chapter 3 we prove extension theorems for positive and regular polynomial mappings. We consider the Aron-Berner extension for homogeneous polynomials on Riesz spaces.

In Chapter 4 we first review the Fremlin tensor product for Riesz spaces and then consider a symmetric Fremlin tensor product. We discuss symmetric $k$-morphisms and define the concept of polymorphism. We give several characterizations of $k$-morphisms in terms of these polymorphisms. Finally we consider orthosymmetric multilinear mappings.

## References

[1] C. D. Aliprantis and O. Burkinshaw, *Positive Operators,* Pure and Applied Mathematics, 119. Academic Press, Inc., Orlando, FL, 1985. xvi+367 pp. ISBN: 0-12-050260-7.

[2] G. Buskes and A. van Rooij, *Bounded variation and tensor products of Banach lattices,* Positivity and its applications (Nijmegen, 2001). Positivity 7 (2003), no. 1-2, 47–59.

[3] D. H. Fremlin, *Tensor products of Archimedean vector lattices,* Amer. J. Math. 94, 1972, 777–798.

[4] B. C. Grecu and R. A. Ryan, *Polynomials on Banach spaces with unconditional bases,* Proc. Amer. Math. Soc. 133 (2005), no. 4, 1083–1091.

## A Theoretical Study of Spin Filtering and its Application to Polarizing Antiprotons

DOMHNAILL O'BRIEN
*donie@maths.tcd.ie*

This is an abstract of the PhD thesis *"A theoretical study of spin filtering and its application to polarizing antiprotons"* written by Domhnaill O'Brien under the supervision of Dr. Nigel Buttimore at the School of Mathematics, Trinity College Dublin and submitted in June 2008.

There has been much recent research into possible methods of polarizing an antiproton beam, the most promising being spin filtering, the theoretical understanding of which is currently incomplete. The method of polarization buildup by spin filtering requires many of the beam particles to remain within the beam after repeated interaction with an internal target in a storage ring. Hence small scattering angles, where we show that electromagnetic effects dominate hadronic effects, are important. All spin-averaged and spin-dependent electromagnetic cross-sections and spin observables for elastic spin 1/2 - spin 1/2 scattering, for both point-like particles and non-point-like particles with internal structure defined by electromagnetic form factors, are derived to first order in QED. Particular attention is paid to spin transfer and depolarization cross-sections in antiproton-proton, antiproton-electron and positron-electron scattering, in the low $|t|$ region of momentum transfer. A thorough mathematical treatment of spin filtering is then presented, identifying the key physical processes involved and highlighting the dynamical properties of the physical system. We present and solve sets of differential equations which describe the buildup of polarization by spin filtering in many different scenarios of interest. The advantages of using a lepton target are outlined, and finally a proposal to polarize antiprotons by spin filtering off an opposing polarized electron beam is investigated.

# DEPARTMENTAL NEWS

This section contains important news (such as permanent appointments, retirements, prizes awarded, etc.) as supplied by the Mathematics Departments of Universities in Ireland.

The Editor does not take any responsibility for the accuracy of the information provided.

## University College Cork

Newly appointed to UCC were: Dr Christian Ewald, Senior Lecturer in Financial Mathematics, since September 2008; Dr Edward Lee, one-year research contract, since September 2008; Dr Jesse Ratzkin, one-year research contract, since September 2008.

On the other hand, Professor Brian Twomey retired in May 2008 after many years of service. His retirement was marked by a well-attended conference on Complex Function Theory held at UCC in April and supported, in part, by the IMS.

## Trinity College Dublin

Dr Rupert Levene was appointed to a J. L. Synge Instructorship at TCD for 2008–10. Dr Derek Kitson was appointed to a temporary lectureship. Pietro Giudice, Anton Ilderton, Osvaldo Santillán and Ryo Suzuki were appointed as postdocs.

Dr Donal O'Donovan was elected Head of the School of Mathematics, following the completion of Professor Samson Shatashvili's term as Head. Dr Richard Timoney was promoted to Associate Professor while Drs Michael Peardon, John Stalker and Dmitri Zaitsev became Senior Lecturers.

Profesor Adrian Constantin resigned the Erasmus Smith's Chair of Mathematics at TCD (which he held from 2004) to take up a Chair in Vienna.

## University College Dublin

Professor Tom Laffey (UCD) and Mr Gordon Lessels (UL) were each presented with a clock by the IMS in November, on the occasion of the presentation of the Fergus Gaines' cup, in recognition of their long and continued work with the Irish Mathematical Olympiad team.

## National University of Ireland Galway

Dr Aisling McCluskey, Mathematics Department, NUI Galway, won a President's Award for Excellence in Teaching.

## University of Limerick

New postdoctoral researchers in MACSI, UL are: Marguerite Robinson, who completed her PhD at UL; Jonathan Ward, who completed his PhD at the University of Bristol; Joanna Mason, who completed her PhD at the University of Bristol.

# Optimisation Problems for the Determinant of a Sum of $3 \times 3$ Matrices

FINBARR HOLLAND

Abstract. Given a pair of positive definite $3 \times 3$ matrices $A, B$, the maximum and minimum values of $\det(U^*AU + V^*BV)$ are determined when $U, V$ vary within the collection of unitary $3 \times 3$ matrices.

## 1. Introduction

Let $m, n$ be a pair of natural numbers. Suppose $A_1, A_2, \dots, A_n$ are $m \times m$ Hermitian positive definite matrices. What are the maximum and minimum values of the expression

$$\det\Big(\sum_{i=1}^{n} U_i^* A_i U_i\Big)$$

as $U_1, U_2, \dots, U_n$ range over the group $G_m$ of $m \times m$ unitary matrices? The case $m = 2$ of this arose in the context of an interesting maximum-likelihood problem which is discussed in [3], and the minimum value was determined there when the given matrices were real and symmetric, and the $U$s members of the subgroup of $G_2$ of orthogonal matrices.

In this note we address the above problem only in the case $m = 3$, and resolve it when $n = 2$. However, the methods used here don't appear to generalise to the case of general $m$, even when $n = 2$. Accordingly, a different strategy has been devised to deal with this more general case, which will be the subject of another paper. However, at the time of writing, the general case of arbitrary $m, n$ remains open.

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2000 *Mathematics Subject Classification.* Primary 15A45.

*Key words and phrases.* Positive definite matrices, unitary matrices, doubly-stochastic matrices, extreme points, rearrangement inequality.

## 2. Statement of the Main Result

**Theorem 1.** *Let $S$ and $T$ be two $3 \times 3$ positive definite matrices with spectra $\sigma(S) = \{s_1, s_2, s_3\}$ and $\sigma(T) = \{t_1, t_2, t_3\}$, respectively, where $s_1 \geq s_2 \geq s_3 > 0$ and $t_1 \geq t_2 \geq t_3 > 0$. Then*

$$\min\{\det(S + U^*TU) : U \in G_3\} = \prod_{i=1}^{3}(s_i + t_i),$$

*and*

$$\max\{\det(S + U^*TU) : U \in G_3\} = \prod_{i=1}^{3}(s_i + t_{4-i}).$$

## 3. Two Preparatory Lemmas

**Lemma 1.** *Let $A = [a_{ij}]$ be a $3 \times 3$ matrix. Let*

$$M = \begin{bmatrix} x + a_{11} & a_{12} & a_{13} \\ a_{21} & y + a_{22} & a_{23} \\ a_{31} & a_{32} & z + a_{33} \end{bmatrix}.$$

*Then*
$\det M = xyz + yza_{11} + zxa_{22} + xya_{33} + xA_{11} + yA_{22} + zA_{33} + \det A.$

*Proof.* Here and later, we use the customary notation $A_{ij}$ for the cofactor of the typical element $a_{ij}$, so that, in particular, $A_{11}$, $A_{22}$, $A_{33}$ are the principal minors of $A$ of order $2 \times 2$. Expanding by elements of the first row,

$$\begin{aligned}
\det M &= (x + a_{11})[(y + a_{22})(z + a_{33}) - a_{23}a_{32}] \\
&\quad - a_{12}[a_{21}(z + a_{33}) - a_{31}a_{23}] + a_{13}[a_{21}a_{32} - a_{31}(y + a_{22})] \\
&= (x + a_{11})(y + a_{22})(z + a_{33}) - [xa_{23}a_{32} + ya_{13}a_{31} + za_{12}a_{21}] \\
&\quad - a_{11}a_{23}a_{32} - a_{12}[a_{21}a_{33} - a_{31}a_{23}] + a_{13}[a_{21}a_{32} - a_{31}a_{22}] \\
&= (x + a_{11})(y + a_{22})(z + a_{33}) - a_{11}a_{23}a_{32} \\
&\quad - [xa_{23}a_{32} + ya_{13}a_{31} + za_{12}a_{21}] - a_{12}A_{12} + a_{13}A_{13} \\
&= xyz + xya_{33} + yza_{22} + zxa_{11} + x[a_{22}a_{33} - a_{23}a_{32}] \\
&\quad + y[a_{11}a_{33} - a_{13}a_{31}] + z[a_{11}a_{22} - a_{12}a_{21}] \\
&\quad\quad + a_{11}A_{11} - a_{12}A_{12} + a_{13}A_{13} \\
&= xyz + xya_{33} + yza_{22} + zxa_{11} + xA_{11} + yA_{22} + zA_{33} + \det A.
\end{aligned}$$

□

We wish to exploit this result when $A = U^*TU$, where $T$ is a diagonal matrix with positive diagonal elements $p \geq q \geq r > 0$, and $U = [u_{ij}]$ is unitary. A calculation shows that

$$a_{ij} = p\overline{u_{i1}}u_{j1} + q\overline{u_{i2}}u_{j2} + r\overline{u_{i3}}u_{j3},\ i, j = 1, 2, 3.$$

In particular,

$$a_{ii} = p|u_{i1}|^2 + q|u_{i2}|^2 + r|u_{i3}|^2,\ i = 1, 2, 3.$$

In addition, $A$ is invertible and $A^{-1} = U^*T^{-1}U = (\det A)^{-1}[A_{ij}]^t$, whence

$$\frac{A_{ii}}{pqr} = p^{-1}|u_{i1}|^2 + q^{-1}|u_{i2}|^2 + r^{-1}|u_{i3}|^2,\ i = 1, 2, 3,$$

or

$$A_{ii} = qr|u_{i1}|^2 + rp|u_{i2}|^2 + pq|u_{i3}|^2,\ i = 1, 2, 3.$$

Observe too that

$$\sum_{i=1}^{3} |u_{ij}|^2 = \sum_{j=1}^{3} |u_{ij}|^2 = 1,\ i, j = 1, 2, 3,$$

and so the matrix $[|u_{ij}|^2]$ is doubly-stochastic. With this in mind we prove a rearrangement inequality.

**Lemma 2.** *Let $[p_{ij}]$ stand for an arbitrary $n \times n$ doubly-stochastic matrix. Let $a, b$ be two real $n \times 1$ vectors whose entries are in decreasing order. Then*

$$\sum_{i=1}^{n} a_i b_{n-i+1} \leq \sum_{i,j=1}^{n} a_i b_j p_{ij} \leq \sum_{i=1}^{n} a_i b_i.$$

*Proof.* Consider the function $f$ defined on the convex set $\mathcal{P}$ of all $n \times n$ doubly-stochastic matrices $P = [p_{ij}]$ by

$$f(P) = \sum_{i,j=1}^{n} a_i b_j p_{ij},\quad P \in \mathcal{P}.$$

Clearly, $f$ is linear in $P$, and so convex on $\mathcal{P}$. Hence it attains its maximum and minimum at an extreme point of $\mathcal{P}$. But, by Birkhoff's theorem [1], the set of extreme points of the latter consists of the set of permutation matrices $\{\pi(I) = [\delta_{i\pi(j)}] : \pi \in S_n\}$, where $S_n$

denotes the group of permutations of $\{1,2,\ldots,n\}$. Hence

$$\begin{aligned}\min\{f(P):P\in\mathcal{P}\} &= \min\{f(\pi(I)):\pi\in S_n\}\\ &= \min\{\sum_{i,j=1}^{n} a_i b_j \delta_{i\pi(j)}:\pi\in S_n\}\\ &= \min\{\sum_{j=1}^{n} a_{\pi(j)} b_j:\pi\in S_n\}\\ &= \sum_{j=1}^{n} a_j b_{n-j+1},\end{aligned}$$

by the elementary rearrangement inequality, since $a,b$ are similarly ordered [2]. This argument establishes that

$$\sum_{i=1}^{n} a_i b_{n-i+1} \le \sum_{i,j=1}^{n} a_i b_j p_{ij},$$

with equality when $p_{ij}=\delta_{i(n-j+1)},\ i,j=1,2,\ldots,n,$.

The maximum can be handled in the same way. □

## 4. Proof of the Main Result

Define $F$ on the group $G_3$ of $3\times 3$ unitary matrices by

$$F(U)=\det(S+U^*TU),\ U\in G_3.$$

In the first place, there are matrices $V,W\in G_3$ such that

$$S=V\begin{bmatrix} s_1 & 0 & 0\\ 0 & s_2 & 0\\ 0 & 0 & s_3\end{bmatrix}V^*\equiv V\Delta V^*,$$

and

$$T=W\begin{bmatrix} t_1 & 0 & 0\\ 0 & t_2 & 0\\ 0 & 0 & t_3\end{bmatrix}W^*\equiv W\Lambda W^*,$$

say. Hence

$$F(WUV^*)=\det(\Delta+U^*\Lambda U),$$

whence it's enough to deal with the case where $S=\Delta, T=\Lambda$. This being so, we can appeal to Lemma 1, taking

$$A=U^*\Delta U=\Big[\sum_{k=1}^{3} t_k\overline{u_{ik}}u_{jk}\Big],$$

and obtain that

$$\begin{aligned}
\det(\Delta + U^*\Lambda U) &= \det(\Delta + A) \\
&= s_1 s_2 s_3 + s_1 s_2 s_3 \sum_{k=1}^{3} s_k^{-1} a_{kk} + \sum_{k=1}^{3} s_k A_{kk} + \det A \\
&= s_1 s_2 s_3 + s_1 s_2 s_3 \sum_{i=1}^{3} s_i^{-1} \sum_{j=1}^{3} t_j |u_{ij}|^2 \\
&\qquad + t_1 t_2 t_3 \sum_{i=1}^{3} s_i \sum_{j=1}^{3} t_j^{-1} |u_{ij}|^2 + t_1 t_2 t_3 \\
&= s_1 s_2 s_3 + s_1 s_2 s_3 \sum_{i,j=1}^{3} s_i^{-1} t_j |u_{ij}|^2 \\
&\qquad + t_1 t_2 t_3 \sum_{i,j=1}^{3} s_i t_j^{-1} |u_{ij}|^2 + t_1 t_2 t_3 \\
&\geq s_1 s_2 s_3 + s_1 s_2 s_3 \sum_{i=1}^{3} s_i^{-1} t_i + t_1 t_2 t_3 \sum_{i=1}^{3} s_i t_i^{-1} + t_1 t_2 t_3,
\end{aligned}$$

by Lemma 2, since $s_1, s_2, s_3$, and $t_1^{-1}, t_2^{-1}, t_3^{-1}$ are oppositely ordered. It follows that

$$\begin{aligned}
\det(\Delta + U^*\Lambda U) &\geq s_1 s_2 s_3 + t_1 s_2 s_3 + t_2 s_1 s_3 + t_3 s_1 s_2 \\
&\qquad + s_1 t_2 t_2 + t_2 s_1 s_3 + t_3 s_1 s_2 + t_1 t_2 t_3 \\
&= (s_1 + t_1)(s_2 + t_2)(s_3 + t_3),
\end{aligned}$$

with equality when $U = I$, the identity matrix. Hence

$$\min\{F(U) : U \in G_3\} = \prod_{i=1}^{3} (s_i + t_i).$$

Arguing in a similar manner, it can be seen that

$$\max\{F(U) : U \in G_3\} = \prod_{i=1}^{3} (s_i + t_{4-i}).$$

This completes the proof of Theorem 1.

## REFERENCES

[1] R. BHATIA, *Matrix Analysis*, Springer-Verlag, 1997.
[2] G. H. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, Cambridge University Press, 1934.
[3] F. HOLLAND AND K. ROY CHOUDHREY, Likelihood ratio tests for equality of shape under varying degrees of orientation invariance, *J. Multivariate Analysis* **99** (2008), 1772–1792.

Finbarr Holland,
Mathematics Department,
University College,
Cork, Ireland,
*f.holland@ucc.ie*

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# Hermitian Morita Theory: a Matrix Approach

DAVID W. LEWIS AND THOMAS UNGER

ABSTRACT. In this note an explicit matrix description of hermitian Morita theory is presented.

## 1. INTRODUCTION

Let $K$ be a field of characteristic different from two and let $A$ be a central simple $K$-algebra equipped with an involution $*$. By a well-known theorem of Wedderburn, $A$ is of the form $M_n(D)$, a full matrix algebra over a division $K$-algebra $D$. Furthermore, there exists an involution— on $D$ of the same kind as $*$ such that $*$ and—have the same restriction to $K$. Then $*$ is the adjoint involution $\mathrm{ad}_{h_0}$ of some nonsingular $\varepsilon_0$-hermitian form $h_0$ over $(D, -)$,

$$h_0 : D^n \times D^n \longrightarrow D,$$

with $\varepsilon_0 = \pm 1$. Thus

$$X^* = \mathrm{ad}_{h_0}(X) = S\overline{X}^t S^{-1}, \quad \forall X \in M_n(D),$$

where $S \in GL_n(D)$ is the matrix of $h_0$, so that $\overline{S}^t = \varepsilon_0 S$.

Let $\mathrm{Gr}_\varepsilon(A, *)$ and $W_\varepsilon(A, *)$ denote the Grothendieck group and Witt group of $\varepsilon$-hermitian forms over $(A, *)$, respectively. Hermitian Morita theory furnishes us with isomorphisms

$$\mathrm{Gr}_\varepsilon(A, *) \cong \mathrm{Gr}_{\varepsilon_0\varepsilon}(D, -) \quad \text{and} \quad W_\varepsilon(A, *) \cong W_{\varepsilon_0\varepsilon}(D, -).$$

These isomorphisms are the result of the following equivalences of categories

$$\left\{\begin{matrix}\varepsilon\text{-hermitian}\\ \text{forms over}\\ (M_n(D), *)\end{matrix}\right\} \xleftrightarrow{\text{scaling}} \left\{\begin{matrix}\varepsilon_0\varepsilon\text{-hermitian}\\ \text{forms over}\\ (M_n(D), -^t)\end{matrix}\right\} \xleftrightarrow[\text{equivalence}]{\text{Morita}} \left\{\begin{matrix}\varepsilon_0\varepsilon\text{-hermitian}\\ \text{forms over}\\ (D, -)\end{matrix}\right\}$$

(all forms are assumed to be nonsingular) which respect isometries, orthogonal sums and hyperbolic forms.

In this note we describe these correspondences explicitly. In particular we give a matrix description of Morita equivalence which does not seem to be generally known. Other explicit descriptions can be found in [3, 4, 5]. The subject is often treated in a more abstract manner, such as in [1] and [2, Chap. I, §9].

## 2. SCALING

Let $M$ be a right $M_n(D)$-module and let $h : M \times M \longrightarrow M_n(D)$ be an $\varepsilon$-hermitian form with respect to $*$, i.e.

$$h(y,x) = \varepsilon h(x,y)^* = \varepsilon S\overline{h(x,y)}^t S^{-1}.$$

**Proposition 2.1.** *The form*

$$S^{-1}h : M \times M \longrightarrow M_n(D),\ (x,y) \longmapsto S^{-1}h(x,y)$$

*is $\varepsilon_0\varepsilon$-hermitian over $(M_n(D), \overline{\phantom{x}}^t)$.*

*Proof.* Sesquilinearity of $S^{-1}h$ with respect to $\overline{\phantom{x}}^t$ follows easily from sesquilinearity of $h$ with respect to $*$:

$$\begin{aligned}(S^{-1}h)(x\alpha, y) &= S^{-1}h(x\alpha,y) = S^{-1}\alpha^* h(x,y)\\ &= S^{-1}S\overline{\alpha}^t S^{-1}h(x,y) = \overline{\alpha}^t S^{-1}h(x,y)\end{aligned}$$

for any $\alpha \in M_n(D)$ and any $x, y \in M$.

Furthermore, using the fact that $\overline{S}^t = \varepsilon_0 S$, we get

$$\begin{aligned}(S^{-1}h)(y,x) &= S^{-1}h(y,x)\\ &= S^{-1}\varepsilon S\overline{h(x,y)}^t S^{-1}\\ &= \varepsilon\overline{h(x,y)}^t S^{-1}\\ &= \varepsilon\varepsilon_0\overline{h(x,y)}^t\overline{(S^{-1})}^t\\ &= \varepsilon\varepsilon_0\overline{(S^{-1}h)(x,y)}^t\end{aligned}$$

for any $x, y \in M$. ∎

**Remark 2.2.** By the first part of the proof, scaling of a sesquilinear form $h$ (rather than an $\varepsilon$-hermitian form $h$) with respect to $*$ results in a sesquilinear form $S^{-1}h$ with respect to $\overline{\phantom{x}}^t$.

**Remark 2.3.** The matrix $S$ is not determined uniquely, but only up to scalar multiplication by $\lambda \in K$, since $\lambda S$ and $S$ give the same involution $\mathrm{ad}_{h_0}$. Hence the scaling correspondence is not canonical.

## 3. MORITA EQUIVALENCE

Every module over $M_n(D) \cong \mathrm{End}_D(D^n)$ is a direct sum of simple modules, namely copies of $D^n$. Let $(D^n)^k$ be such a module. We identify $(D^n)^k$ with $D^{k\times n}$, the $k \times n$-matrices over $D$. We view each row of a $k\times n$-matrix over $D$ as an element of $D^n$. Note that $M_n(D)$ acts on $D^{k\times n}$ on the right.

Now let
$$h : D^{k\times n} \times D^{k\times n} \longrightarrow M_n(D)$$
be an $\varepsilon$-hermitian form over $(M_n(D), -^t)$.

**Proposition 3.1.** *There exists an $\varepsilon$-hermitian $k \times k$-matrix $B \in M_k(D)$ such that*
$$h(x,y) = \overline{x}^t By,\ \forall x,y \in D^{k\times n}. \tag{1}$$

*Proof.* Let $B = (b_{ij})$. We will determine the entries $b_{ij}$. Let $e_{ij} \in D^{k\times n}$, $e'_{ij} \in D^{n\times k}$ and $E_{ij} \in M_n(D)$ respectively denote the $k \times n$-matrix, the $n\times k$-matrix and the $n\times n$-matrix with 1 in the $(i,j)$-th position and zeroes everywhere else. One can easily verify that
$$e_{if}E_{f\ell} = e_{i\ell}, \tag{2}$$
where $1 \le i \le k$ and $1 \le f,\ell \le n$. Also note that if $C \in M_n(D)$, then computing the product $E_{ij}C$ picks the $j$-th row of $C$ and puts it in row $i$ while making all other entries zero. Similarly, computing the product $CE_{ij}$ picks the $i$-th column of $C$ and puts it in column $j$ while making all other entries zero. The matrices $e_{ij}$ and $e'_{ij}$ behave in a similar fashion.

The matrices $\{e_{ij} \mid 1 \le i \le k,\ 1 \le j \le n\}$ generate $D^{k\times n}$ as a right $M_n(D)$-module. Thus it suffices to compute $h(e_{if}, e_{jg})$ where $1 \le i,j \le k$ and $1 \le f,g \le n$. Let us first compute $h(e_{ii}, e_{jj})$:
$$\begin{aligned} h(e_{ii}, e_{jj}) &= h(e_{ii}E_{ii}, e_{jj}E_{jj}) \\ &= E_{ii}h(e_{ii}, e_{jj})E_{jj} \\ &= m_{ij}E_{ij}, \end{aligned}$$
where $m_{ij}$ is the $(i,j)$-th entry of $h(e_{ii}, e_{jj}) \in M_n(D)$. In other words, the matrix $h(e_{ii}, e_{jj})$ has only one non-zero entry, namely $m_{ij}$ in position $(i,j)$.

Next, let us compute $h(e_{if}, e_{jg})$. We will use the fact that

$$e_{if} = e_{ii}E_{if},$$

where $1 \leq i \leq k$ and $1 \leq f \leq n$, which follows from (2). We get

$$\begin{aligned} h(e_{if}, e_{jg}) &= h(e_{ii}E_{if}, e_{jj}E_{jg}) \\ &= E_{fi}h(e_{ii}, e_{jj})E_{jg} \\ &= \big(h(e_{ii}, e_{jj})\big)_{ij}E_{fg} \\ &= m_{ij}E_{fg}. \end{aligned}$$

Let $b_{ij} = m_{ij}$ where $1 \leq i, j \leq k$. We have

$$\begin{aligned} \overline{e_{if}}^t Be_{jg} &= e'_{fi}Be_{jg} \\ &= b_{ij}E_{fg} \\ &= m_{ij}E_{fg}. \end{aligned}$$

Therefore, $h(e_{if}, e_{jg}) = \overline{e_{if}}^t Be_{jg}$ where $1 \leq i, j \leq k$ and $1 \leq f, g \leq n$, which establishes (1).

Finally,

$$m_{ji}E_{ji} = h(e_{jj}, e_{ii}) = \varepsilon\overline{h(e_{ii}, e_{jj})}^t = \varepsilon\overline{m_{ij}}E_{ji}, \text{ for } 1 \leq i, j \leq k,$$

which implies $m_{ji} = \varepsilon\overline{m_{ij}}$, for $1 \leq i, j \leq k$. In other words, $\overline{m_{ji}} = \varepsilon m_{ij}$, for $1 \leq i, j \leq k$, so that $\overline{B}^t = \varepsilon B$, which finishes the proof. ∎

So, given an $\varepsilon$-hermitian form $h$ over $(M_n(D), -^t)$, we have obtained an $\varepsilon$-hermitian form over $(D, -)$ with matrix $B$ as in Proposition 3.1. Conversely, given an $\varepsilon$-hermitian form

$$\varphi : D^k \times D^k \longrightarrow D,$$

represented by the matrix $B$ (i.e., $B = \big(\varphi(e_i, e_j)\big)$ for a $D$-basis $\{e_i\}$ of $D^k$), we define

$$h : D^{k\times n} \times D^{k\times n} \longrightarrow M_n(D)$$

by

$$h(x, y) := \overline{x}^t By, \ \forall x, y \in D^{k\times n},$$

which gives an $\varepsilon$-hermitian form over $(M_n(D), -^t)$.

**Remark 3.2.** The correspondence $h \leftrightarrow \varphi$ already works for forms that are just sesquilinear, without assuming any hermitian symmetry. Since scaling also preserves sesquilinearity, as remarked earlier, we conclude that the category equivalences of §1 already hold for sesquilinear forms over $(M_n(D), *)$, $(M_n(D), -^t)$ and $(D, -)$, respectively.

## References


[1] A. Fröhlich and A. M. McEvett, Forms over rings with involution, *J. Algebra* **12** (1969), 79–104.

[2] M.-A. Knus, *Quadratic and Hermitian forms over rings*, Grundlehren der Mathematischen Wissenschaften **294**, Springer-Verlag, Berlin, 1991.

[3] D. W. Lewis, Forms over real algebras and the multisignature of a manifold, *Advances in Math.* **23** (1977), no. 3, 272–284.

[4] C. Riehm, Effective equivalence of orthogonal representations of finite groups, *J. Algebra* **196** (1997), no. 1, 196–210.

[5] C. Riehm, Orthogonal, symplectic and unitary representations of finite groups, *Trans. Amer. Math. Soc.* **353** (2001), no. 12, 4687–4727.



David Lewis and Thomas Unger,
School of Mathematical Sciences,
University College Dublin,
Belfield, Dublin 4, Ireland,
*david.lewis@ucd.ie*, *thomas.unger@ucd.ie*


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# Recent Trends on Order Bounded Disjointness Preserving Operators [1]

KARIM BOULABIAR

## 1. Introduction

Disjointness preserving operators have been introduced in some form or other in the forty's. Indeed, linear multiplicative operations in the work [55] by Vulikh are disjointness preserving operators in disguise. However, only during the last decades have they made a formal entry into the development of vector lattices. A first systematic study of disjointness preserving operators goes back to the pioneering note [4] by Abramovich, Veksler, and Koldunov published in the end of the seventies. From then on, the interest in disjointness preserving operators has steadily grown and a series of works devoted to the subject appeared in the literature. In this regard, spectral properties of order bounded disjointness preserving operators were considered in great details in [8, 30]. On the other hand, invertible disjointness preserving operators occupied a prominent role in a vast literature, such as [7, 20, 33, 35] and mainly the remarkable memoir [3] by Abramovich and Kitover. One of the external reasons for the continuing interest in disjointness preserving operators is the fact that precisely the order bounded disjointness preserving operators allow multiplicative representations as weighted composition operators and, more generally, polar decompositions [2, 19, 28, 39]. They thus found applications in the theory of singular and integral equation, dynamical system, and differential equations with delayed time [38, 46, 51].

The present survey on order bounded disjointness preserving operators has two main objectives. First, convince the young researchers in vector lattices that disjointness preserving operators constitute an honorable research activity. Secondly, inform the experts about

[1] Most of the content of this survey was presented by the author in the Instructional Workshop organized by Professor Anthony Wickstead in the summer of 2008 at Queen's University Belfast.

results on the subject obtained recently by the author. In fact, this is a study of modest length—selectivity is a must—with the choices of illustrations being left, for good or ill, to personal taste and prejudices. In this prospect, no attention has been paid to aspects of order bounded disjointness preserving operators evoked in the preceding paragraph, except of their representations as weighted composition operators.

The exposition is divided in six sections. This introduction is followed by a section dealing with what is considered today as the most important result in the theory of order bounded disjointness preserving operators, namely, Meyer's Theorem [43]. In the third section, order bounded disjointness preserving operators on certain spaces of continuous functions are characterized. In this prospect, some results are presumably known. However, the author has not been able to locate precise references for them. For this reason, complete proofs will be given. The fourth section contains extensions of results in Section 3 to the more general setting of functions algebra in the sense of Birkhoff and Pierce [15]. Section 5 gives a look at disjointness preserving operators from a different 'global' point of view. Indeed, the lattice structure of certain sets of order bounded operators preserving disjointness are investigated and facts on the so-called orthomorphisms are extended to such sets. Some results in this section are new and they will be proved completely. The last section concerns algebraic (in the sense of Kaplansky [37]) order bounded disjointness preserving operators. To be a little more precise, we focus on order bounded disjointness preserving operators that satisfy a nontrivial polynomial equation. At last, we point out that this survey contains some open problems. The hope of the author is that the reader will find them rather worthy of interest.

The books [1, 6, 41, 45, 59] on the theory of vector lattices and operators between them are used in this survey as the sources of unexplained terminology and notation.

## 2. Meyer's Theorem

Unless otherwise stated, $L$ and $M$ stand throughout for Archimedean vector lattices. A (linear) operator $T$ from $L$ into $M$ is said to be *disjointness preserving* (or to *preserve disjointness*) if

$$f, g \in L \text{ and } |f| \wedge |g| = 0 \text{ in } L \quad \text{imply} \quad |T(f)| \wedge |T(g)| = 0 \text{ in } M.$$

It is not hard to see that the operator $T$ from $L$ into $M$ preserves disjointness if and only if $|T(f)| \wedge |T(g)| = 0$ for all $f, g \in L$ with $f \wedge g = 0$. Also, $|T(f)| = |T(|f|)|$ for all $f \in L$ is a necessary and sufficient condition for the operator $T$ from $L$ into $M$ in order to be disjointness preserving. Obviously, the operator $T$ from $L$ into $M$ is a lattice homomorphism if and only if $T$ is a positive disjointness preserving operator. As mentioned in the title, this survey will deal with disjointness preserving operators that are order bounded. It should be pointed out here that a disjointness preserving operator need not be order bounded. An example in this direction is given next.

*Example* 2.1. A real-valued function $f$ defined on the real interval $[0, \infty)$ is said to be essentially polynomial if there exist a real polynomial $p_f$ and a real number $\lambda_f$ with $f(x) = p_f(x)$ for all $x \in [\lambda_f, \infty)$. The set $L$ of all essentially polynomial functions on $[0, \infty)$ is an Archimedean vector lattice with respect to the usual pointwise operations and ordering. The operator $T$ from $L$ into the vector lattice $\mathbb{R}$ of all real numbers defined by $Tf = p_f(0)$ for all $f \in L$ preserves disjointness but is not order bounded.

At this point, recall that the set $\mathcal{L}(L, M)$ of all operators from $L$ into $M$ is an Archimedean ordered vector space with respect to the usual pointwise operation and ordering. The set $\mathcal{L}_b(L, M)$ of all order bounded operators from $L$ into $M$ is an ordered vector subspace of $\mathcal{L}(L, M)$. It is well-known that $\mathcal{L}(L, M)$ is not a vector lattice, in general. In this regard, even an order bounded operator from $L$ into $M$ need not have an absolute value nor in $\mathcal{L}(L, M)$ neither in $\mathcal{L}_b(L, M)$. In spite of that, $\mathcal{L}_b(L, M)$ is a Dedekind complete lattice-subspace [1] of $\mathcal{L}(L, M)$ as soon as $M$ in addition Dedekind complete. This leads us to the what is considered today as the fundamental theorem of order bounded disjointness preserving operators, namely, the following very famous Meyer's Theorem.

**Theorem 2.2.** *Let $T$ be an order bounded disjointness preserving operator from $L$ into $M$. Then there exist unique lattice homomorphisms $T^+, T^-$ from $L$ into $M$ such that $T = T^+ - T^-$ and*

$$T^+(f) = (T(f))^+, T^-(f) = (T(f))^- \quad \text{for all } f \in L^+.$$

*In particular, $T$ has an absolute value $|T|$ in $\mathcal{L}(L, M)$ and $|T| = T^+ + T^-$. Moreover, $|T|$ is a lattice homomorphism from $L$ into $M$*

*such that*

$$|T|(|f|) = |T(f)| = ||T|(f)| \quad \textit{for all } f \in L.$$

The first proof of the Theorem 2.2 was given in [43] by Meyer himself. Meyer's proof is not constructive, that is, it is based upon the Zorn's Lemma (i.e., the Axiom of Choice). Later, two Zorn's Lemma free proofs of Theorem 2.2 were provided by Bernau in [12] and de Pagter in [47], respectively.

Meyer's Theorem is in many ways the starting point of our investigation of order bounded disjointness preserving operators. Let us collect some consequences. First of all, it is readily verified that the kernel $\ker T$ of a lattice homomorphism $T$ from $L$ into $M$ is an order ideal of $L$. Theorem 2.2 yields directly that if $T$ is an order bounded disjointness preserving operator from $L$ into $M$ then $\ker T = \ker |T|$. So, the kernel of any order bounded disjointness preserving linear operator is again an order ideal. However, contrary to lattice homomorphisms, the range $\operatorname{Im} T$ of the order bounded disjointness preserving linear operator $T$ from $L$ into $M$ need not be a vector sublattice of $M$. To see this, consider the order bounded disjointness preserving operator from the Archimedean vector lattice $C([0,1])$ of all real-valued continuous functions on the real interval $[0,1]$ into itself defined by $(T(f))(x) = xf(x)$ for all $f \in C([0,1])$ and $x \in [0,1]$. Another nice application of Meyer's Theorem was obtained by Huijsmans and Wickstead in [34]. That is, if $T$ is a bijective order bounded disjointness preserving operator from $L$ into $M$, then the inverse $T^{-1}$ is an order bounded disjointness preserving operator from $M$ into $L$. Moreover, the equality $|T|^{-1} = |T^{-1}|$ holds in $\mathcal{L}(M,L)$. In [8], Arendt proved that if $L$ and $M$ are Banach lattices then an operator $T$ from $L$ into $M$ is an order bounded disjointness preserving operator if and only if $|T(f)| \leq |T(g)|$ holds in $M$ whenever $|f| \leq |g|$ holds in $L$. Relying on Meyer's Theorem, [33] Huijsmans and de Pagter extended the characterization obtained by Arendt to arbitrary Archimedean vector lattices. This result is interesting in part because it gives one equivalent condition to both order boundedness and disjointness preservation. Since we are evoking Arendt and his work [8], we point out that he called order bounded disjointness preserving operators shortly *Lamperti operators*.

Now, we shift our emphasis from the general case to the particular setting of band preserving operators. Let $T$ be an operator on $L$ (i.e., from $L$ into $L$) and recall that a nonvoid subset $D$ of $L$ is said to be $T$-*invariant* whenever $T$ maps $D$ into $D$. An operator $T$ on $L$ is said to be *band preserving* if every band of $L$ is $T$-invariant. Hence, the operator $T$ on $L$ is band preserving if and only if $|T(f)| \wedge |g| = 0$ for all $f, g \in L$ with $|f| \wedge |g| = 0$. Obviously, if any band preserving operator on $L$ preserves disjointness. This implication is not reversible, of course. On the other hand, the following example (obtained by Meyer in [44]) shows that a band preserving operators need not be order bounded.

*Example* 2.3. The set $PL([0,1))$ of all piecewise linear functions on $[0,1)$ is an Archimedean vector lattice with respect to the usual pointwise operations and ordering. Notice here that $f \in PL([0,1))$ if and only if there exists a partition $0 = x_0 < x_1 < ... < x_{n-1} < x_n = 1$ of $[0,1)$ such that $f$ is linear on $[x_{i-1}, x_i)$ for each $i \in \{1, ..., n\}$. The band preserving operator $T$ on $PL([0,1))$ defined by

$$T(f)(x) = f'_r(x) \quad \text{for all } f \in PL([0,1)) \text{ and } x \in [0,1),$$

where $f'_r$ indicates the right derivative of $f$, is not order bounded.

Only order bounded band preserving operators will be considered in this study. In this prospect, an order bounded band preserving operator $T$ on $L$ is called an *orthomorphism* on $L$. Obviously, the identity operator $I_L$ on $L$ is an orthomorphism on $L$. Moreover, the set $\mathrm{Orth}(L)$ of all orthomorphisms on $L$ is an ordered vector subspace of $\mathcal{L}(L) = \mathcal{L}(L, L)$. Actually, $\mathrm{Orth}(L)$ is much more than a simple ordered vector subspace of $\mathcal{L}(L)$. Indeed, Bigard and Keimel in [14] and, independently, Conrad and Diem in [24] proved that $\mathrm{Orth}(L)$ is a generalized vector sublattice (in the sense of [5]) of $\mathcal{L}(L)$ with the lattice operations given pointwise, meaning that, if $S, T \in \mathrm{Orth}(L)$ then $(S \vee T)(f) = S(f) \vee T(f)$ and $(S \wedge T)(f) = S(f) \wedge T(f)$ for all $f \in L^+$. In particular, if $T \in \mathrm{Orth}(L)$ then the absolute value $|T|$ exists and $|T|(|f|) = |T(f)| = ||T|(f)|$ for all $f \in L$. The latter can be obtained alternatively from Theorem 2.2 since $T$ in particular preserves disjointness.

## 3. Concrete Situations

By and large, the notation and terminology of the great text [27] by Gillman and Jerison will be used in this section unless it conflicts with the by now standard notation used by workers in vector lattices. In particular, $\mathbb{R}^X$ will indicate the universally complete [6] vector lattice of all real-valued functions on a nonvoid set $X$ under the usual pointwise addition, scalar multiplication, and ordering. Moreover, the constant function on $X$ whose constant value is the real number $r$ is denoted by $\mathbf{r}_X$. Furthermore, if $X$ is a topological space then $C(X)$ denoted the (relatively) uniformly complete [41] vector sublattice of $\mathbb{R}^X$ of all continuous functions on $X$. The main objective of this section is to characterize order bounded disjointness preserving operators on $C(X)$-type vector lattices under suitable restrictions on $X$.

A topological space is called a *Tychonoff space* if it is a subspace of a compact Hausdorff space. In [54], Tychonoff himself proved that the topological space $X$ is Tychonoff if and only if $X$ is Hausdorff and *completely regular*, that is, whenever $F$ is a closed set in $X$ and $x_F \in X$ with $x_F \notin F$, there exists $f \in C(X)$ such that $f(x_F) = 0$ and $f(x) = 1$ for all $x \in F$. It was known to both Stone [53] and Čech [25] that for each topological space $X$, there is a Tychonoff space $X^*$ such that $C(X)$ and $C(X^*)$ are isomorphic as vector lattices. In fact, $X^*$ is obtained by first identifying those points which cannot be separated by continuous functions, inducing the functions of $C(X)$ on $X^*$ in the obvious manner, and then furnishing $X^*$ with the weakest topology in which these functions are continuous (see Theorem 3.9 in [27] for more details). This observation eliminates any reason for considering vector lattices of real-valued continuous functions on other than Tychonoff spaces. Therefore, it will be assumed henceforth that $X$ is a Tychonoff space unless the contrary is stated explicitly.

Hewitt's great paper [32] built on the aforementioned works of Stone and Čech and laid the foundation for the study of the interplay between $C(X)$ and $X$. In today's terminology the Tychonoff space $X$ is said to be *realcompact* if there is no strictly large Tychonoff space $Y$ such that $X$ is dense in $Y$ and every $f \in C(X)$ has an extension in $C(Y)$. Actually, Hewitt in [32] used the terminology *$Q$-space* instead of realcompact space and proved that $X$ is realcompact if and only if it is homeomorphic to a closed subspace of a product of real lines equipped with the usual product topology. This characterization is often used as a definition of realcompact spaces. Later, Shirota [52] showed that $X$ is realcompact if and only if to each algebra homomorphism $\varphi$ from $C(X)$ onto the real field $\mathbb{R}$ there corresponds a point $x$ of $X$ such that $\varphi(f) = f(x)$ for all $f \in C(X)$. This remarkable necessary and sufficient condition for a Tychonoff space to be realcompact was obtained very recently by Ercan and Önal [26] *via* an elementary approach. These observations will be used next to obtain an alternative characterization of realcompact spaces which is a little more fit for our study.

**Lemma 3.1.** *Let $X$ be a Tychonoff space $X$. Then the following are equivalent.*

(i) *$X$ is realcompact*

(ii) *To each lattice homomorphism $\varphi$ from $C(X)$ onto $\mathbb{R}$ with $\varphi(\mathbf{1}_X) = 1$ there corresponds a point $x$ of $X$ such that $\varphi(f) = f(x)$ for all $f \in C(X)$.*

*Proof.* By the above Shirota's result, it suffices to prove that if $\varphi$ is a mapping of $C(X)$ to $\mathbb{R}$ with $\varphi(\mathbf{1}_X) = 1$, then $\varphi$ is a lattice homomorphism if and only if $\varphi$ is an algebra homomorphism. So, let $\varphi$ be such a mapping and assume $\varphi$ to be an algebra homomorphism. If $f \in C(X)$ then

$$0 \leq \left(\varphi\left(|f|^{1/2}\right)\right)^2 = \varphi(|f|) = \left(\varphi\left(f^2\right)\right)^{1/2} = \left((\varphi f)^2\right)^{1/2} = |\varphi(f)|.$$

It follows that $\varphi$ is a lattice homomorphism. Conversely, suppose that $\varphi$ is a lattice homomorphism and let $f, h \in C(X)$ such that $\varphi(h) = 0$. For every $\varepsilon \in (0, \infty)$, the inequalities $\mathbf{0}_X \leq |fh| \leq \varepsilon\left|f^2h\right| + \varepsilon^{-1}|h|$ hold in $C(X)$. Thus,

$$0 \leq |\varphi(fh)| \leq \varepsilon\varphi\left(\left|f^2h\right|\right) + \varepsilon^{-1}\varphi(|h|) = \varepsilon\varphi\left(\left|f^2h\right|\right).$$

As $\varepsilon$ is arbitrary in $(0, \infty)$, we get $\varphi(fh) = 0$. We derive that if $g \in C(X)$ then $\varphi(g - \varphi(g)\mathbf{1}_X) = 0$. Therefore,

$$\varphi(fg) = \varphi(fg) - \varphi((g - \varphi(g)\mathbf{1}_X)f) = \varphi(f)\varphi(g).$$

So, $\varphi$ is an algebra homomorphism and we are done. □

From now on, $Y$ stands for an arbitrary topological space. The *cozero-set* of a function $w \in C(Y)$ is the set

$$\operatorname{coz}(w) = \{y \in Y : w(y) \neq 0\}$$

Now, let $w \in C(Y)$ and $\tau$ be a function of $Y$ to $X$ which is continuous on $\operatorname{coz}(w)$. It is readily verified that the mapping $T$ from $C(X)$ into $C(Y)$ defined by

$$T(f)(y) = w(y)f(\tau(y)) \quad \text{for all } f \in C(X) \text{ and } y \in Y$$

is an order bounded disjointness preserving operator. Such a mapping is usually called a *weighted composition operator*. Next, we discuss the question whether this implication is reversible. Surprisingly, this question has an affirmative answer if $X$ in addition is realcompact.

**Theorem 3.2.** *Let $T$ be a mapping of $C(X)$ to $C(Y)$ and assume $X$ to be realcompact. Then the following are equivalent.*

(i) *$T$ is an order bounded disjointness preserving operator.*
(ii) *There exists $w \in C(Y)$ and a function $\tau$ of $Y$ into $X$ such that $\tau$ is continuous on $\operatorname{coz}(w)$ and*

$$T(f)(y) = w(y) f(\tau(y)) \quad \text{for all } f \in C(X) \text{ and } y \in Y.$$

*Proof.* Only necessity will be proved. Let $T$ be an order bounded disjointness preserving operator from $C(X)$ into $C(Y)$. Hence $T^+$ is a lattice homomorphism from $C(X)$ into $C(Y)$ (see Theorem 2.2). Put $w_+ = T^+(\mathbf{1}_X)$ and define for each $f \in C(X)$ the function $S(f) \in C(\operatorname{coz}(w_+))$ by

$$S(f)(y) = \frac{T^+(f)(y)}{w_+(y)} \quad \text{for all } y \in \operatorname{coz}(w_+).$$

The mapping $S$ thus defined from $C(X)$ into $C(\operatorname{coz}(w_+))$ is linear, obviously. Moreover, if $y \in \operatorname{coz}(w_+)$ then

$$\begin{aligned}(\delta_y \circ S)|f| = S(|f|)(y) &= \frac{T^+(|f|)(y)}{w_+(y)} \\ &= \frac{|T^+(f)(y)|}{w_+(y)} = |S(f)(y)| = |(\delta_y \circ S) f|.\end{aligned}$$

Hence, $\delta_y \circ S$ is a lattice homomorphism from $C(X)$ onto $\mathbb{R}$ with $(\delta_y \circ S)(\mathbf{1}_X) = 1$. Lemma 3.1 yields that a point $x_y$ of $X$ can be found so that

$$S(f)(y) = (\delta_y \circ S)(f) = f(x_y) \quad \text{for all } f \in C(X).$$

Let $\tau_+$ be the mapping of $\operatorname{coz}(w_+)$ to $X$ defined by $\tau_+(y) = x_y$ for all $y \in \operatorname{coz}(w_+)$. By Theorem 3.8 in [27], $\tau_+$ is continuous. We get also

$$T^+(f)(y) = w_+(y)(f \circ \tau_+)(y) \quad \text{for all } f \in C(X), y \in \operatorname{coz}(w_+).$$

On the other hand, let $f \in C(X)$ and $y \in Y$ such that $T^+(\mathbf{1}_X)(y) = 0$. Also, let $\varepsilon \in (0, \infty)$ and observe that

$$\mathbf{0}_X \leq |f| \leq |f| \wedge \varepsilon \mathbf{1}_X + \varepsilon^{-1} f^2.$$

Therefore,

$$\begin{aligned}0 &\leq \left|T^{+}(f)(y)\right| \\ &\leq \left|T^{+}(f)(y)\right| \wedge \varepsilon T^{+}(\mathbf{1}_X)(y) + \varepsilon^{-1}T^{+}\left(f^2\right)(y) \\ &= \varepsilon^{-1}T^{+}\left(f^2\right)(y).\end{aligned}$$

So, $(T^{+}(f))(y) = 0$ because $\varepsilon$ is arbitrary in $(0, \infty)$. Now, choose an arbitrary extension of $\tau_+$ to $Y$ and denote such an extension again by $\tau_+$. We derive directly that

$$\left(T^{+}(f)\right)(y) = w_+(y)\, f(\tau_+(y)) \quad \text{for all } f \in C(X) \text{ and } y \in Y.$$

Analogously, if $w_- = T^{-}(\mathbf{1}_X)$ then we may find a function $\tau_-$ of $Y$ into $X$ such that $\tau_-$ is continuous on $\operatorname{coz}(w_-)$ and

$$T^{-}(f)(y) = w_-(y)\, f(\tau_-(y)) \quad \text{for all } f \in C(X) \text{ and } y \in Y.$$

Observe now that

$$w_+ \wedge w_- = T^{+}(\mathbf{1}_X) \wedge T^{-}(\mathbf{1}_X) = (T(\mathbf{1}_X))^{+} \wedge (T(\mathbf{1}_X))^{-} = \mathbf{0}_X$$

(where we use once more Theorem 2.2). But then

$$\operatorname{coz}(w_+) \cap \operatorname{coz}(w_-) = \emptyset \quad \text{and} \quad \operatorname{coz}(w_+) \cup \operatorname{coz}(w_-) = \operatorname{coz}(w),$$

where $w = w_+ - w_- \in C(Y)$. Choose a function $\tau$ of $Y$ to $X$ so that

$$\tau(y) = \tau_+(y) \text{ if } y \in \operatorname{coz}(w_+) \text{ and } \tau(y) = \tau_-(y) \text{ if } y \in \operatorname{coz}(w_-).$$

Since $T = T^{+} - T^{-}$, we derive that

$$T(f)(y) = w(y)\, f(\tau(y)) \quad \text{for all } f \in C(X) \text{ and } y \in Y$$

and we are done. □

Next, we shall say a few words to see that the condition of realcompactness imposed on the Tychonoff space $X$ in Theorem 3.2 is close to being the best possible for order bounded disjointness preserving operators from $C(X)$ into $C(Y)$ to be automatically weighted composition operators. Indeed, assume that $\mathbb{R}^{C(X)}$ is endowed with the usual product topology. The mapping $\pi$ of $X$ to $\mathbb{R}^{C(X)}$ defined by $\pi(x)(f) = f(x)$ for all $x \in X$ and $f \in C(X)$sends $X$ homeomorphically to $\pi(X) = \{\pi(x) : x \in X\}$ (see [27]). Let $\upsilon X$ denote the closure of $\pi(X)$ in $\mathbb{R}^{C(X)}$. Hence, $\upsilon X$ is the unique (up to a homeomorphism leaving $X$ pointwise fixed) realcompact topological space such that $X$ is dense in $\upsilon X$ and every function $f \in C(X)$ has a unique extension $f^{\upsilon} \in C(\upsilon X)$. The realcompact space $\upsilon X$ is referred to as the *realcompactification* of $X$ (see again [27]). Moreover, the mapping $\upsilon$ of $C(X)$ to $C(\upsilon X)$ defined by $\upsilon(f) = f^{\upsilon}$ for all $f \in C(X)$ is a lattice isomorphism as it was observed by Shirota in [52] and Henriksen [31].

At this point, assume that $X$ and $Y$ are both locally compact (and then Tychonoff). Recall that the set $C_0(X)$ ($C_\infty(X)$ in the book [27]) of all real-valued continuous functions on $X$ vanishing at infinity is a vector sublattice of $\mathbb{R}^X$. Meyer-Nieberg proved in [45] that a mapping $T$ of $C_0(X)$ to $C_0(Y)$ is a lattice homomorphism if and only if there exist a positive (real-valued) function $w$ on $Y$ which is continuous on $\operatorname{coz}(w)$ and a continuous function $\tau$ of $\operatorname{coz}(w)$ to $X$ such that, if $f \in C_0(X)$, then $T(f)(y) = w(y) f(\tau(y))$ if $y \in \operatorname{coz}(w)$ and $T(f)(y) = 0$ of $y \notin \operatorname{coz}(w)$. In view of Theorem 2.2, a same argument as previously used in the end of the proof of Theorem 3.2 leads straightforwardly to the following result.

**Theorem 3.3.** *Assume that $X$ and $Y$ are locally compact and let $T$ be a mapping of $C_0(X)$ to $C_0(Y)$. Then the following are equivalent.*

(i) *$T$ is an order bounded disjointness preserving operator.*
(ii) *There exist a real-valued function $w$ on $Y$ which is continuous on $\operatorname{coz}(w)$ and a continuous function $\tau$ of $\operatorname{coz}(w)$ to $X$ such that, if $f \in C_0(X)$, then*

$$T(f)(y) = \begin{cases} w(y) f(\tau(y)) & \text{if} \quad y \in \operatorname{coz}(w) \\ \\ 0 \quad \text{if} \quad y \notin \operatorname{coz}(w). \end{cases}$$

At last, it should be pointed out that Theorem 3.3 was obtained in an alternative way by Jeang and Wong in [36]. Also, notice that Theorem 3.2 and Theorem 3.3 have the same compact version. Such a version has been obtained earlier by Arendt in [8] (see [35] by Jarosz for a different approach). In this regard, one might hope that Theorem 3.3 can be obtained from its compact version by extending the order bounded disjointness preserving operator $T$ from $C_0(X)$ into $C_0(Y)$ to an order bounded disjointness preserving operator $T^{\alpha}$ from $C(\alpha X)$ into $C(\alpha Y)$, where $\alpha X$ denotes the one-point compactification of $X$ (see [27]). However, Jeang and Wong provided in [36] the following example of an order bounded disjointness preserving operator $T$ from $C_0(X)$ into $C_0(Y)$ which does not have any such extension.

*Example* 3.4. Let $X = [0, \infty)$ and $Y = \mathbb{R}$ with the usual topology and define $w, \tau$ from $\mathbb{R}$ into $\mathbb{R}$ by

$$w(y) = \begin{cases} 1 \quad \text{if} \quad y > 2 \\ \\ y - 1 \quad \text{if} \quad 0 \leq y \leq 2 \\ \\ -1 \quad \text{if} \quad y < 0 \end{cases} \quad \text{and } \tau(y) = \begin{cases} y \quad \text{if} \quad y \geq 0 \\ \\ -y \quad \text{if} \quad y < 0. \end{cases}$$

Let $T$ be the mapping from $C_0(X)$ to $C_0(Y)$ defined by $T(f)(y) = w(y \; \mathrm{f}\tau(y))$ for all $f \in C(X)$ and $y \in Y$. Clearly, $T$ is an order bounded disjointness preserving operator from $C_0(X)$ into $C_0(Y)$. But no extension $T^a$ from $C(\alpha X)$ into $C(\alpha Y)$ of $T$ can be an order bounded disjointness preserving operator.

## 4. A Multiplicative Aspect

In this section, we show how can results in Section 3 be extended to the more general setting of function algebras in the sense of Birkhoff and Pierce [15]. A vector lattice $L$ which is simultaneously an associative algebra such that $fg \in L^+$ for all $f, g \in L^+$ is called a *lattice ordered algebras* (briefly, an *$\ell$-algebra*). In [15], Birkhoff and Pierce called the $\ell$-algebra $L$ a *function algebra* (shortly, an *$f$-algebra*) if $(fh) \wedge g = (hf) \wedge g = 0$ for all $f, g, h \in L^+$ with $f \wedge g = 0$. One of the most classical examples of $f$-algebras is $\mathbb{R}^X$ for any nonvoid set $X$. Moreover, if $X$ is a topological space, then $C(X)$ is a uniformly complete $f$-subalgebra of $\mathbb{R}^X$. In this space, we focus only on Archimedean $f$-algebras. The Archimedean $f$-algebra $L$ with a multiplicative unity is *semiprime*, meaning that, 0 is the only nilpotent element of $L$. Orthomorphisms on an Archimedean vector lattice $L$ is an important example of $f$-algebras. Indeed, the Archimedean vector lattice $\mathrm{Orth}(L)$ is $f$-algebra with respect to the composition and the identity operator $I_L$ on $L$ is a multiplicative unity in $\mathrm{Orth}(L)$. On the other hand, if $L$ is an Archimedean semiprime $f$-algebra, then $L$ can be embedded in $\mathrm{Orth}(L)$ as an $f$-subalgebra. Below, we shall identify $L$ with an $f$-subalgebra of $\mathrm{Orth}(L)$ without further ado. The reader is referred to the surveys [22, 23] for more information on $f$-algebras. Also, Chapter 20 in [60] by Zaanen presents an excellent study of $f$-algebras based upon the Ph.D. thesis [48] of de Pagter. Next, we shall describe another important instance of $f$-algebras, namely, the $f$-algebra of all extended orthomorphisms on an Archimedean vector lattice.

Let $L$ be an Archimedean vector lattice. Luxemburg and Schep in [40] defined an order bounded operator $T$ from an order dense order ideal $D_T$ of $L$ into $L$ to be an *extended orthomorphism* of $L$ if $|f| \wedge |g| = 0$ in $D_T$ implies $|Tf| \wedge |g| = 0$ in $L$. Of course, an extended orthomorphism $T$ of $L$ is an orthomorphism of $L$ if $D_T = L$. A natural equivalence relation can be introduced in the set of all extended orthomorphisms of $L$ as follows. Two extended orthomorphisms of $L$ are equivalent whenever they agree on an order dense order ideal in $L$ or, equivalently, they are equal on the intersection of their domains. Notice that the intersection of two order dense order ideals in $L$ is obviously again an order dense order ideal in $L$. The set of all equivalence classes of extended orthomorphisms of $L$ is denoted by $\mathrm{Orth}^{\infty}(L)$. With respect to the pointwise addition, scalar multiplication, and ordering, $\mathrm{Orth}^{\infty}(L)$ is an Archimedean vector lattice. The lattice operations in the vector lattice $\mathrm{Orth}^{\infty}(L)$ are given pointwise. It turns out that the vector lattice $\mathrm{Orth}^{\infty}(L)$ is an $f$-algebra under the composition as multiplication. Moreover, since extended orthomorphisms are order continuous, the set $\mathrm{Orth}(L)$ of all orthomorphisms of $L$ can be embedded naturally in $\mathrm{Orth}^{\infty}(L)$ as an $f$-subalgebra. Obviously, the identity operator $I_L$ of $L$ serves as a multiplicative unity in $\mathrm{Orth}^{\infty}(L)$. All these facts can be found in the fundamental papers [40] by Luxemburg and Schep and [49] by de Pagter.

As previously pointed out, this section gives a look at concrete situations presented in Section 3 from a 'purely algebraic' point of view. In this prospect, some extra observations are needed. Let $X$ be a realcompact space and $Y$ be a Tychonoff space. It is shown in Theorem 3.2 that if $T$ is an order bounded disjointness preserving operator from $C(X)$ into $C(Y)$ then there exist $w \in C(Y)$ and a function $\tau$ of $Y$ into $X$ which is continuous on $\mathrm{coz}(w)$ such that $T(f)(y) = w(y) f(\tau(y))$ for all $f \in C(X)$ and $y \in Y$. The observation to make here is that if $f$ is in $C(X)$ then the real-valued function $S(f)$ defined on $Y$ by $S(f)(y) = f(\tau(y))$ for all $y \in Y$ need not be in $C(Y)$. However, it is readily verified that $S(f)$ is continuous on some dense open set in $Y$. In other words, $S(f)$ belongs to $\mathrm{Orth}^{\infty}(C(Y))$. Indeed, $\mathrm{Orth}^{\infty}(C(Y))$ is essentially the algebra of all continuous functions defined on some dense open set of $Y$ (see [29] by Hager, [49] by de Pagter, and [57] by Wickstead). Accordingly, the mapping $S$ defined from $C(X)$ into $\mathbb{R}^Y$ by $S(f) = f \circ \tau$ for all $f \in C(X)$ is actually a lattice and algebra homomorphism from

$C(X)$ into $\mathrm{Orth}^{\infty}(C(Y))$. Summarizing, Theorem 3.2 can be stated algebraically as follows. A mapping $T$ from $C(X)$ into $C(Y)$ is an order bounded disjointness preserving operator if and only if there exist $w \in C(Y)$ and a lattice and algebra homomorphism $S$ from $C(X)$ into $\mathrm{Orth}^{\infty}(C(Y))$ such that $T(f)(y) = w(y)(S(f))(y)$ for all $f \in C(X)$ and $y \in Y$. It seems to be natural therefore to ask whether this 'algebraic' version of Theorem 3.2 can be extended to the more general setting of $f$-algebras. From now on, *$A$ and $B$ stand for Archimedean semiprime $f$-algebras with $B$ uniformly complete.*

**Theorem 4.1.** *Assume $A$ to have a multiplicative unity and let $T$ be a mapping from $A$ into $B$. Then the following are equivalent.*

(i) *$T$ is an order bounded disjointness preserving operator.*
(ii) *There exist $w \in B$ and a lattice and algebra homomorphism from $A$ into $\mathrm{Orth}^{\infty}(B)$ such that $T(f) = wS(f)$ for all $f \in A$.*

Next, we focus on Theorem 3.3 in which we have seen that if $X$ and $Y$ are locally compact spaces and $T$ is an order bounded disjointness preserving operator from $C_0(X)$ into $C_0(Y)$, then there exist a real-valued function $w$ on $Y$ which is continuous on $\mathrm{coz}(w)$ and a continuous function $\tau$ of $\mathrm{coz}(w)$ to $X$ such that, if $f \in C_0(X)$, then $T(f)(y) = w(y)f(\tau(y))$ if $y \in \mathrm{coz}(w)$, and $T(f)(y) = 0$ if $y \notin \mathrm{coz}(w)$. In [17], it shown that $C_b(Y)$ and $\mathrm{Orth}(C_0(Y))$ are isomorphic as $f$-algebras. Also, the $f$-algebras $\mathrm{Orth}^{\infty}(C_b(Y))$ and $\mathrm{Orth}^{\infty}(C(Y))$ are isomorphic (see again [29, 49, 57]). Hence, the above result can be stated as follows. A mapping $T$ from $C_0(X)$ into $C_0(Y)$ is an order bounded disjointness preserving operator if and only if there exist $w \in \mathrm{Orth}^{\infty}(C_b(Y))$ and a lattice and algebra homomorphism $S$ from $C(X)$ into $\mathrm{Orth}^{\infty}(C_b(Y))$ such that $T(f)(y) = w(y)(S(f))(y)$ for all $f \in C_0(X)$ and $y \in Y$. This result holds in $f$-algebras as we shall see next. First of all, the $f$-algebra $A$ is said to be $n^{\text{th}}$-*root closed* for some nonzero natural number $n$ if for every $g \in A^+$ there exists $f \in A^+$ such that $f^n = g$ (such an $f$ is unique since $A$ is assumed to be semiprime). The proof of the following theorem can be found in the recent survey [23].

**Theorem 4.2.** *Assume that $A$ is $n^{\text{th}}$-root closed for some positive integer and let $T$ be a mapping from $A$ into $B$. Then the following are equivalent.*

(i) *$T$ is an order bounded disjointness preserving operator.*

(ii) *There exist* $w \in \mathrm{Orth}^{\infty}(\mathrm{Orth}(B))$ *and a lattice and algebra homomorphism* $S$ *from* $A$ *into* $\mathrm{Orth}^{\infty}(\mathrm{Orth}(B))$ *such that* $T(f) = wS(f)$ *for all* $f \in A$.

The proof of Theorem 4.2 presented in [23] is based upon a beautiful theorem by Hart [30], namely, if $L$ and $M$ are Archimedean vector lattices and $T$ is an order bounded disjointness preserving operator $T$ from $L$ into $M$, then there exists a lattice and algebra homomorphism $\widetilde{T}$ from $\mathrm{Orth}(L)$ into $\mathrm{Orth}(\mathcal{R}(T(L)))$ (where $\mathcal{R}(T(L))$ is the vector sublattice of $M$ generated by $T(L)$) such that $\widetilde{T}(S)(T(f)) = T(S(f))$ for all $T \in \mathrm{Orth}(L)$ and $f \in L$.We end this section with an example (see [23]) showing that the condition imposed on $A$ in Theorem 4.2 cannot be deleted.

*Example* 4.3. Let $A$ be the $f$-algebra of the piecewise polynomial functions on $[0,1]$ that are 0 at 0. Then the real-valued lattice homomorphism $T$ on $A$ that assigns to a function its right derivative at 0 is not representable as in the main theorem above. Indeed, denote the identity function on $[0,1]$ by $f$. Suppose that $T$ has a representation as above with $S$ a lattice and algebra homomorphism from $A$ onto $\mathbb{R}$ and $\alpha$ a nonzero real number such that $T = \alpha S$. Then $S(f) \neq 0$, hence $S(f^2) \neq 0$, but $T(f^2) = 0$, a contradiction.

## 5. A Global Point of View

In this section, we look at disjointness preserving operators from a certain 'global' point of view. Indeed, we focus on the lattice structure of certain sets of operators preserving disjointness rather that the behavior of the disjointness preserving operators themselves. We start our investigation by introducing the notion of disjointness preserving sets. A nonvoid subset $\mathcal{D}$ of $\mathcal{L}_b(L,M)$ is called a *disjointness preserving set* in $\mathcal{L}_b(L,M)$ if $|S(f)| \wedge |T(g)| = 0$ for all $S,T \in \mathcal{D}$ and $f,g \in L$ with $|f| \wedge |g| = 0$. Several elementary properties follow straightforwardly from the definition. Let us single out a few as particularly worthy. For instance, an order bounded operator $T$ from $L$ into $M$ preserve disjointness if and only if $\{T\}$ is a disjointness preserving set in $\mathcal{L}_b(L,M)$. Therefore, any element in a disjointness preserving set in $\mathcal{L}_b(L,M)$ is an order bounded disjointness preserving operator. Moreover, the non-void subset $\mathcal{D}$ of $\mathcal{L}_b(L,M)$ is a disjointness preserving set in $\mathcal{L}_b(L,M)$ if and only if each pair $\{S,T\}$ of elements of $\mathcal{D}$ is a disjointness preserving set in $\mathcal{L}_b(L,M)$.

Thus, if $\mathcal{D}$ is a disjointness preserving set in $\mathcal{L}_b(L, M)$ then so is any nonvoid subset of $\mathcal{D}$. However, the next property of disjointness preserving sets is not so visible at first sight. Indeed, it turns out that each pair in a disjointness preserving set in $\mathcal{L}_b(L, M)$ has a supremum and an infimum in $\mathcal{L}(L, M)$. More details are given in the following.

**Lemma 5.1.** *Let $\mathcal{D}$ be a disjointness preserving set in $\mathcal{L}_b(L, M)$. Then each pair $\{S, T\}$ of elements of $\mathcal{D}$ has a supremum $S \vee T$ and a infimum $S \wedge T$ in $\mathcal{L}(L, M)$ such that*

$$(S \vee T)(f) = S(f) \vee T(f) \text{ and } (S \wedge T)(f) = S(f) \wedge T(f)$$

*for all $f \in L^+$.*

*Proof.* Let $S, T \in \mathcal{D}$ and $f, g \in L$ such that $|f| \wedge |g| = 0$. Since $\mathcal{D}$ is a disjointness preserving set in $\mathcal{L}_b(L, M)$, the sets $\{S(f), T(f)\}$ and $\{S(g), T(g)\}$ are disjoint. Hence,

$$0 \leq |S(f) - T(f)| \wedge |S(g) - T(g)| = 0$$

So, the difference $S - T$ is an order bounded disjointness preserving operator from $L$ into $M$. By Theorem 2.2, $S - T$ has an absolute value $|S - T|$ in the ordered vector space $\mathcal{L}(L, M)$ such that

$$|S - T|(f) = |S(f) - T(f)| \quad \text{for all } f \in L^+.$$

This yields quickly that the pair $\{S, T\}$ has a least upper bound $S \vee T$ and a great lower bound $S \wedge T$ in $\mathcal{L}(L, M)$ given by

$$S \vee T = \frac{1}{2}(S + T + |S - T|) \quad \text{and} \quad S \wedge T = \frac{1}{2}(S + T - |S - T|).$$

Now, we prove that these supremum and infimum are given pointwise. On the other hand, if $f \in L^+$ then

$$\begin{aligned}(S \vee T)(f) &= \left(\frac{1}{2}(S + T + |S - T|)\right)(f) \\ &= \frac{1}{2}(S(f) + T(f) + |S - T|(f)) \\ &= \frac{1}{2}(S(f) + T(f) + |S(f) - T(f)|) = S(f) \vee T(f).\end{aligned}$$

The formula

$$(S \wedge T)(f) = S(f) \wedge T(f) \quad \text{for all } f \in L^+$$

is obtained in the same way completing the proof of the lemma. □

Now, let $S, T$ be order bounded disjointness preserving operators from $L$ into $M$. A short's moment thought (see the first lines of the previous proof) reveals that if $\{S, T\}$ is a disjointness preserving set in $\mathcal{L}_b(L, M)$ then the sum $S+T$ preserves disjointness. The question wether this implication is reversible is discussed next.

**Lemma 5.2.** *Let $S, T$ be order bounded disjointness preserving operators from $L$ into $M$. Then the following are equivalent.*

(i) *$S+T$ preserves disjointness*

(ii) *The pair $\{S, T\}$ is a disjointness preserving set in $\mathcal{L}_b(L, M)$.*

*Proof.* Only necessity is proved. So, assume $S+T$ to be disjointness preserving. By Theorem 2.2, the absolute value $|S+T|$ exists in the ordered vector space $\mathcal{L}(L, M)$. Furthermore, if $f \in L^+$ then

$$0 \leq |(|S|-|T|)(f)| = |S(f)|-|T(f)| \leq |S(f)+T(f)| = |S+T|(f).$$

It follows readily that $|S|-|T|$ preserves disjointness. So, the absolute value $||S|-|T||$ of $|S|-|T|$ exists in $\mathcal{L}(L, M)$ and if $f \in L^+$ then

$$||S|-|T||(f) = ||S|(f)-|T|(f)| = ||S(f)|-|T(f)||.$$

As in the proof of Lemma 5.1, the pair $\{|S|, |T|\}$ has a supremum $|S| \vee |T|$ and a infimum $|S| \wedge |T|$ in $\mathcal{L}(L, M)$. Moreover, if $f \in L^+$ then

$$(|S| \vee |T|)(f) = |S(f)| \vee |T(f)| \text{ and } (|S| \wedge |T|)(f) = |S(f)| \wedge |T(f)|.$$

At this point, let $f, g \in L$ such that $|f| \wedge |g| = 0$. Since $|S|$ and $|T|$ are lattice homomorphisms from $L$ into $M$, we can write

$$\begin{aligned}
(|S| \vee |T|)(|f|) + (|S| \vee |T|)(|g|) &= (|S| \vee |T|)(|f| + |g|) \\
&= (|S| \vee |T|)(|f| \vee |g|) \\
&= (|S|(|f| \vee |g|)) \vee (|T|(|f| \vee |g|)) \\
&= |S|f|| \vee |S|g|| \vee |T|f|| \vee |T|g|| \\
&= |S|f|| \vee |T|f|| \vee |S|g|| \vee |T|g|| \\
&= (|S| \vee |T|)(|f|) \vee (|S| \vee |T|)(|g|).
\end{aligned}$$

Thus,

$$\begin{aligned}
0 &\leq |S(f)| \wedge |T(g)| \\
&= ||S|(f)| \wedge ||T|(g)| = |S(f)| \wedge |T(g)| \\
&\leq (|S| \vee |T|)(|f|) \wedge (|S| \vee |T|)(|g|) = 0.
\end{aligned}$$

We derive that $\{S, T\}$ is a disjointness preserving set and we are done. $\square$

In [13], Bernau, Huijsmans, and de Pagter studied sums of order bounded disjointness preserving operators and gave various properties of such sums. More recently in [50], de Pagter in collaboration with Schep furnished several necessary and sufficient conditions for a sum of two order bounded disjointness preserving operators in order to be again preserving disjointness. Lemma 5.2 above can be seen as a modest contribution to this study.

Now, recall that the set $\operatorname{Orth}(M)$ of all orthomorphisms on $M$ is a generalized vector sublattice of $\mathcal{L}(M)$ the lattice operations of which are given pointwise. Actually, this nice property of $\operatorname{Orth}(M)$ has something to do with the fact that $\operatorname{Orth}(M)$ is a particular disjointness preserving set in $\mathcal{L}_b(M)$. Let us say some additional words in order to explain our point of view. Assume that $\mathcal{D}$ is a disjointness preserving set in $\mathcal{L}_b(M)$ that contains $\operatorname{Orth}(M)$. Hence, $I_M \in \mathcal{D}$ and $\{I_M, T\}$ is a disjointness preserving set in $\mathcal{L}_b(M)$ for all $T \in \mathcal{D}$. It follows that $T$ is an orthomorphism on $M$, that is, $\mathcal{D} = \operatorname{Orth}(M)$. In other words, $\operatorname{Orth}(M)$ is a maximal element in the set of all disjointness preserving sets in $\mathcal{L}_b(M)$ with respect to the inclusion ordering. Surprisingly, it turns out that any maximal disjointness preserving set in $\mathcal{L}_b(L, M)$ is a generalized vector sublattice of $\mathcal{L}(L, M)$. To see this, let us define a disjointness preserving set $\mathcal{M}$ in $\mathcal{L}_b(L, M)$ to be *maximal* if there is no strictly large disjointness preserving set in $\mathcal{L}_b(L, M)$. We are in position now to prove the central theorem of this section.

**Theorem 5.3.** *Let $\mathcal{M}$ be a maximal disjointness preserving set in $\mathcal{L}_b(L, M)$. Then $\mathcal{M}$ is a generalized vector sublattice of $\mathcal{L}(L, M)$. Moreover, if $S, T \in \mathcal{M}$ then*

$$(S \vee T)(f) = S(f) \vee T(f) \text{ and } (S \wedge T)(f) = S(f) \wedge T(f)$$

*for all $f \in L^+$.*

*Proof.* Let $S, T \in \mathcal{M}$ and $a$ be a real number. Since $\{S, T\}$ is a disjointness preserving set in $\mathcal{L}_b(L, M)$, Lemma 5.2 yields that the sum $S+T$ preserves disjointness. Now, let $R \in \mathcal{M}$ and $f, g \in L$ such that $|f| \wedge |g| = 0$. Hence, $\{R, S, T\}$ is a disjointness preserving set in $\mathcal{L}_b(L, M)$ so

$$|R(f)| \wedge |S(g)| = |R(f)| \wedge |T(g)| = 0$$

We derive that

$$0 \leq |R(f)| \wedge |(S+T)(g)| \leq |R(f)| \wedge (|S(g)| + |T(g)|) = 0.$$

Thus, $\{R, S+T\}$ is a disjointness preserving set in $\mathcal{L}_b(L, M)$. It follows easily that $\mathcal{M} \cup \{S+T\}$ is again a disjointness preserving set in $\mathcal{L}_b(L, M)$. Since $\mathcal{M}$ is maximal, $S+T \in \mathcal{M}$. Analogously, it is readily checked that $\mathcal{M} \cup \{aT\}$ is a disjointness preserving set in $\mathcal{L}_b(L, M)$ and then, by maximality, $aT \in \mathcal{M}$. This implies that $\mathcal{M}$ is a vector subspace of $\mathcal{L}(L, M)$.

At this point, let $T \in \mathcal{M}$. Since $T$ is an order bounded disjointness preserving operator from $L$ into $M$, the absolute value $|T|$ of $T$ in $\mathcal{L}(L, M)$ exists (where we use Theorem 2.2). Let $S \in \mathcal{M}$ and $f, g \in L$ such that $|f| \wedge |g| = 0$. Theorem 2.2 together with the fact that $\mathcal{M}$ is a disjointness preserving set in $\mathcal{L}_b(L, M)$ leads to

$$|R(f)| \wedge ||T|(g)| = |R(f)| \wedge |T(g)| = 0.$$

We derive that $\mathcal{M} \cup \{|T|\}$ is a disjointness preserving set in $\mathcal{L}_b(L, M)$. Since $\mathcal{M}$ is maximal as a disjointness preserving set in $\mathcal{L}_b(L, M)$, we get $\mathcal{M} = \mathcal{M} \cup \{|T|\}$ so $|T| \in \mathcal{M}$. It follows that $\mathcal{M}$ is a generalized vector sublattice of $\mathcal{L}(L, M)$ and we are done. □

Alternative aspects of maximal disjointness preserving sets can be found in the recent works [10, 11]. For instance, this concept is used in [11] to give an elementary proof of the existence of the modulus of complex order bounded disjointness preserving operators between two arbitrary complex vector lattices. This fact was first proved via Zorn's Lemma by Meyer in [42] for uniformly complete complex vector lattices. More recently, Grobler and Huijsmans obtained the Meyer result constructively [28]. In [11], the result was proved without assuming the complex vector lattices under consideration to be uniformly complete. Another property of maximal disjointness preserving set in $\mathcal{L}_b(L, M)$ is discussed next.

Wickstead proved in [56] if $L$ in addition is Dedekind complete, then $\mathrm{Orth}(L)$ is a band of the vector lattice $\mathcal{L}_b(L)$. This result is extended in what follows to more general setting of maximal disjointness preserving sets. We should recall here that if $M$ is Dedekind complete, then the ordered vector space $\mathcal{L}_b(L, M)$ is a Dedekind complete vector lattice. The following proposition follows from Proposition 2.2 in [10] .

**Proposition 5.4.** *If $M$ is Dedekind complete, then any maximal disjointness preserving set in $\mathcal{L}_b(L, M)$ is a band of $\mathcal{L}_b(L, M)$.*

Now, assume that $L$ has a strong order unit $e > 0$ (that is, the inequality $|f| \leq a|e|$ holds for all $f \in L$ and some real number $a$) and let $\mathcal{M}$ be a order ideal of $\mathcal{L}_b(L, M)$. Define the positive operator $\Pi_e$ from $\mathcal{M}$ into $M$ by

$$\Pi_e(T) = T(e) \quad \text{for all } T \in \mathcal{M}.$$

The following result is a direct inference of Theorem 3.3 in [10].

**Theorem 5.5.** *Assume $L$ to have a strong order unit $e$ and $M$ to be Dedekind complete. If $\mathcal{M}$ is an order ideal of $\mathcal{L}_b(L, M)$ then the following are equivalent.*

(i) *$\mathcal{M}$ is a maximal disjointness preserving set in $\mathcal{L}_b(L, M)$.*
(ii) *$\Pi_e$ is a lattice isomorphism.*
(iii) *$\Pi_e$ is bijective.*

In particular, if $L$ has a strong order unit $e$ and $M$ is Dedekind complete, then $\mathcal{L}_b(L, M)$ has a unique (up to a lattice isomorphism) maximal disjointness preserving set, which is a vector lattice copy of $M$. This fact turns out to be an extension of a classical fact due to Zaanen in [58], namely, if $M$ is a Dedekind complete vector lattice with a strong order unit, then $\mathrm{Orth}(M)$ and $M$ are isomorphic as vector lattices. Actually, Zaanen proved a stronger result, viz., $\mathrm{Orth}(M)$ and $M$ are isomorphic as vector lattices as soon as $M$ is uniformly complete and has a strong order unit. It seems to be natural therefore to ask the following question.

*Problem* 5.6. Do the equivalences in Theorem 5.5 hold if $M$ is only uniformly complete?

The next paragraph deals with maximal disjointness preserving sets on certain $C(X)$-spaces. First of all, let $X$ and $Y$ be topological spaces and let $\tau$ be a function of $Y$ to $X$. For every $w \in C(Y)$, let $C_{\omega,\tau}$ indicate the mapping from $C(X)$ into $\mathbb{R}^Y$ defined by

$$C_{w,\tau}(f)(y) = w(f) f(\tau(y)) \quad \text{for all } f \in C(X) \text{ and } y \in Y.$$

Moreover, put

$$\Omega_\tau = \{w \in C(Y) : C_{\omega,\tau}(f) \in C(Y) \quad \text{for all } f \in C(X)\}$$

and

$$O_\tau = \cup_{w \in \Omega_\tau} \mathrm{coz}(w).$$

Obviously, $O_\tau$ is an open set in $Y$. In [9], Benamor observes that $\tau$ is continuous on $O_\tau$ and defines $\tau$ to be *maximal* if there is no large open set in $Y$ on which $\tau$ is continuous. A sleight modification of the proof of Corollary 1 in [9] yields directly to the following characterization of maximal disjointness preserving sets in $\mathcal{L}_b\left(C\left(X\right), C\left(Y\right)\right)$ when $X$ in addition is compact.

**Theorem 5.7.** *Assume that $X$ is a compact space and let $\mathcal{M}$ be a non-void set of $\mathcal{L}_b\left(C\left(X\right), C\left(Y\right)\right)$. Then the following are equivalent.*

(i) *$\mathcal{M}$ is a maximal disjointness preserving set.*
(ii) *There exists a maximal function $\tau$ from $Y$ into $X$ such that $\mathcal{M} = \{C_{w,\tau} : w \in \Omega_\tau\}$.*

At last, a careful examination of Theorem 5.7 leads naturally to the following open problem.

*Problem* 5.8. Let $T$ be a lattice homomorphism from $L$ into $M$ and put

$$\mathcal{D}\left(T\right) = \left\{S \circ T : T \in \operatorname{Orth}\left(M\right)\right\}.$$

It is not hard to see that $\mathcal{D}$ is a disjointness preserving set in $\mathcal{L}_b(L, M)$. Under what conditions is $\mathcal{D}$ maximal? Conversely, if such conditions are satisfied and $\mathcal{D}$ is an arbitrary maximal disjointness preserving set in $\mathcal{L}_b\left(L, M\right)$. Does there exist a lattice homomorphism $T$ from $L$ into $M$ such that $\mathcal{D} = \mathcal{D}\left(T\right)$?

## 6. Algebraic Disjointness Preserving Operators

Consider a square matrix $T$ for which on every row there is at most one nonzero entry. Let $n$ be the degree of its minimal polynomial and let $m$ be its valuation, that is, the multiplicity of 0 as a root of that minimal polynomial. Then $T^{n!}$ is diagonal, when restricted to the range of $T^m$. The latter looks surprising and one suspects that the result is known, but we have not been able to locate a reference for it. In this section we offer a wide ranging generalization of this matrix result. The condition above simply states that the matrix represents an operator that preserves disjointness in the pointwise ordering. The question arises naturally, as to whether general operators on vector lattices that preserve disjointness behave in a similar fashion. For obvious reasons, when leaving the domain of finite dimensional vector spaces, some form or another of continuity is reasonably imposed on the operators considered. Thus we consider order bounded disjointness preserving operators on Archimedean vector lattices. Fortunately, there is a concept of diagonal in vector lattices such that, surprisingly, order bounded disjointness preserving operators that satisfy a polynomial equation do behave like the matrix case. Indeed, the algebraic orthomorphisms serve the role of diagonals. This brings us to the main topic of the present section, algebraic order bounded disjointness preserving operators. Let us recall some of the relevant notions.

First, we recall the reader that $L$ is an Archimedean vector lattice. As usual, $\mathbb{R}[X]$ indicates the ring of all polynomials with coefficients in the real field $\mathbb{R}$. An operator $T$ on $L$ is said to be *algebraic* if $\Pi(T) = 0$ for some nonzero polynomial $\Pi \in \mathbb{R}[X]$. Hence, $T \in \mathcal{L}(L)$ is algebraic if and only if the ring ideal

$$\mathcal{I}(T) = \{\Pi \in \mathbb{R}[X] : \Pi(T) = 0\}$$

is not equal to $\{0\}$. Let $T \in \mathcal{L}(T)$ be an algebraic operator. Since the ring $\mathbb{R}[X]$ is principal, the ring ideal $\mathcal{I}(T)$ is generated by a unique monic polynomial $\Pi_T$, usually called the *minimal polynomial* of $T$. In particular, if $\Pi \in \mathbb{R}[X]$ then $\Pi(T) = 0$ if and only if $\Pi_T$ divides $\Pi$. The notion of algebraic operators has been introduced by Kaplansky [37] for operators on Banach spaces.

Now, we say that an operator $T$ on $L$ is *strongly diagonal* if there exist pairwise disjoint components $P_1, P_2, ..., P_m$ of the identity operator $I_I$ and real numbers $\alpha_1, \alpha_2, ..., \alpha_n$ such that

$$T = \alpha_1 P_1 + \alpha_2 P_2 + \cdots + \alpha_m P_m.$$

Recall here that by a *component* of $I_L$ we mean a positive orthomorphism $P$ such that the equality $P \wedge (I_L - P) = 0$ holds in the vector lattice $\mathrm{Orth}(L)$ of all orthomorphisms on $L$. Strongly diagonal operators are usually called *$I_L$-step functions* [6, 59]. Clearly, a strongly diagonal operator on $L$ is an orthomorphism. It turns out that the converse holds if the orthomorphism under consideration is algebraic (see Theorem 3.3 in [21]).

**Proposition 6.1.** *Let $T$ an operator on $L$. Then the following are equivalent.*

(i) *$T$ is a strongly diagonal*
(ii) *$T$ is an algebraic orthomorphism.*

The proof of the main result of this section is based upon the classical Kakutani–Bohnenblust–Krein representation theorem [1] and the representation of order bounded disjointness preserving operators on $C(X)$-spaces as weighted composition operators (see Section 3). However, the proof also uses the following lemma dealing with the existence of invariant principal order ideals, which is of independent interest (see Lemma 5.2 in [21]). First, recall that if $T \in \mathcal{L}(L)$ then a subset $D$ of $L$ is *$T$-invariant* if $T$ sends $D$ to $D$. Also, recall that an order ideal of $L$ which is generated by one element is said to be *principal.*

**Lemma 6.2.** *Let $T$ be an algebraic order bounded disjointness preserving operator on $L$. For every $f \in L$ there exists a $T$-invariant principal order ideal of $L$ containing $f$.*

We are in position now to state the central result of this section. The details can be found in Theorem 5.3 in [21].

**Theorem 6.3.** *Let $T$ be an order bounded disjointness preserving operator on $L$. Then the following are equivalent.*

(i) *$T$ is algebraic.*

(ii) *There exist natural numbers $m$ and $n$ with $n > m$ such that the restriction of $T^{n!}$ to the vector sublattice of $L$ generated by the range $T^m(L)$ of $T^m$ is strongly diagonal.*

*Furthermore, when $T$ is algebraic, $n$ (respectively, $m$) can be chosen as the the degree (respectively, the valuation) of the minimal polynomial of $T$.*

Once we observe that $|T^n| = |T|^n$ for all natural number $n$ and each order bounded disjointness preserving operator $T$ on $L$, it follows quickly from Theorem 6.3 that the absolute value of an algebraic order bounded disjointness preserving operator is algebraic as well. This seems far from obvious without the representation in Theorem 6.3 and contrasts with the fact that the absolute value of a finite rank operator need not be a finite rank operator (see [1]). On the other hand, in the above theorem we really need both $n$ and $m$, that is to say, it is possible that $T^n$ is not an orthomorphism on $L$ for any $n$. The following simple example illustrates that fact, whereas special cases where one can take $m = 0$ will be discussed next.

*Example* 6.4. Let $L$ be the Archimedean vector lattice $\mathbb{R}^2$ with coordinatewise addition, scalar multiplication, and ordering and define

$T \in \mathcal{L}(L)$ by $T(x,y) = (x,x)$ for all $(x,y) \in \mathbb{R}^2$. Clearly, $T$ is an order bounded disjointness preserving which is not an orthomorphism. Now, observe that $T^n = T$ for all $n \in \mathbb{N}$.

In [18], it is shown that if $T$ is an algebraic operator on a vector space, then $T$ is injective if and only if $T$ is surjective. Combining this fact with Theorem 6.3 we get easily the following.

**Corollary 6.5.** *Let $T$ be a surjective (or injective) order bounded disjointness preserving operator from $L$ into $M$. Then the following are equivalent.*

(i) *$T$ is algebraic.*
(ii) *There exists a natural number $n$ such that $T^{n!}$ is a strongly diagonal.*

*Furthermore, $n$ can be chosen to be the degree the minimal polynomial of $T$.*

At this point, we turn our attention to locally algebraic disjointness preserving operators. First of all, recall that the operator $T$ on the vector space $L$ is said to be *locally algebraic* if for every $f \in L$ there exists a nonzero polynomial $\Pi \in \mathbb{R}[X]$ (depending on $f$) such that $\Pi(T)f = 0$ (see again [37] by Kaplansky). Obviously, any algebraic operator is locally algebraic. Next, we present a characterization of locally algebraic orthomorphism (see [21] for the proof).

**Proposition 6.6.** *Let $T$ an orthomorphism on $L$. Then the following are equivalent.*

(i) *$T$ is locally algebraic.*
(ii) *The restriction of $T$ to each principal band of $L$ is algebraic (or strongly diagonal).*

On the other hand, recently in [21], the notion of Kaplansky complete vector lattice was introduced as follows. The vector lattice $L$ is said to be *Kaplansky complete* if for every infinite countable subset $D$ of $L$ there exists $f \in L$ and an infinite subset $G$ of $D$ such that $f \wedge g = 0$ for all $g \in G$. For instance, Banach lattices and vector lattices with weak order units are Kaplansky complete. This concept turns out to be crucial for understanding when every locally algebraic orthomorphism is strongly diagonal. For details, we refer to [21].

**Theorem 6.7.** *The following are equivalent.*

(i) *$L$ is Kaplansky complete.*
(ii) *Every locally algebraic orthomorphism on $L$ is algebraic (and then a strongly diagonal operator).*

As a consequence, we conclude that, for $\sigma$-Dedekind complete vector lattices, the condition that every orthomorphism is strongly diagonal is very strong as we can see next (see [21] for the proof).

**Corollary 6.8.** *If $L$ is $\sigma$-Dedekind complete, then the following are equivalent.*

(i) *Every orthomorphism on $L$ is a strongly diagonal operator.*
(ii) *$L$ is finite dimensional.*

The condition of Dedekind $\sigma$-completeness is not superfluous as it can be seen *via* the next Zaanen's example [58].

*Example* 6.9. Let $L$ be the Archimedean vector lattice of all real-valued continuous functions on $[0,1]$ which are piecewise linear. So, $L$ is not $\sigma$-Dedekind complete and $\operatorname{Orth}(L) = \{aI_L : a \in \mathbb{R}\}$.

Finally, it seems to be natural now to ask for the corresponding versions of Proposition 6.6 and Theorem 6.7 for order bounded disjointness preserving operators on the vector lattice $L$.

*Problem* 6.10. As for locally algebraic orthomorphisms, can a characterization of locally algebraic order bounded disjointness preserving operators be obtained in terms of algebraic operators?

*Problem* 6.11. What could be a necessary and sufficient condition on $L$ for locally algebraic order bounded disjointness preserving operators on $L$ to be algebraic?

## References

[1] Y. A. Abramovich, C. D. Aliprantis, *An Invitation to Operator Theory*, Grad. Stud. Math., vol. 50, Amer. Math. Soc., Providence, RI, 2002.

[2] Y. A. Abramovich, E. L. Arensen and A. K. Kitover, *Banach $C(K)$-modules and Operators Preserving Disjointness*, Pitman Research Notes in Mathematics Series, 277, Longman Scientific & Technical, Harlow, 1992.

[3] Y. A. Abramovich, and A. K. Kitover, Inverses of disjointness preserving operators, *Memoirs Amer. Math. Soc.*, 143 (2000), no 679.

[4] Y. A. Abramovich, A. I. Veksler, and A. V. Koldunov, Operators preserving disjointness, *Dokl. Akad. Nauk.*, 248 (1979), 1033-1036.

[5] Y. A. Abramovich and A. W. Wickstead, Recent results on the order structure of compact operators, *Irish Math. Soc. Bull.,* 32 (1994), 32–45.

[6] C. D. Aliprantis, O. Burkinshaw, *Positive Operators*, Academic Press, Orlando, 1985.

[7] J. Araujo, E. Beckenstein and L. Narici, Biseparating maps and homeomorphic real-compactifications, *J. Math. Ana. Appl.*, 12 (1995), 258-265.

[8] W. Arendt, Spectral properties of Lamperti operators, *Indiana Univ. Math. J.*, 32 (1983), 199-215.

[9] F. Benamor, Riesz spaces of order bounded disjointness preserving operators, *Comment. Math. Univ. Carolinae*, 48 (2007), 607-622.

[10] F. Benamor and K. Boulabiar, Maximal ideals of disjointness preserving operators, *J. Math. Anal. Appl.*, 322 (2006), 599-609.

[11] F. Benamor and K. Boulabiar, On the modulus of disjointness preserving operators on complex vector lattices, *Algebra Univ.*, 54 (2005), 185-193.

[12] S. J. Bernau, Orthomorphisms of Archimedean vector lattices, *Math. Proc. Cambridge Philos. Soc.*, 89 (1981), 119-128.

[13] S. J. Bernau, C. B. Huijsmans, and B. de Pagter, Sums of lattice homomorphisms, *Proc. Amer. Math. Soc.*, 115 (1992), 151-156.

[14] A. Bigard and K. Keimel, Sur les endomorphismes conservant les polaires d'un groupe réticulé archimédien, *Bull. Soc. Math. France*, 97 (1969), 381-398.

[15] G. Birkhoff and R. S. Pierce, Lattice-ordered rings, *An. Acad. Brasil. Ciènc.*, 28 (1956), 41-69.

[16] K. Boulabiar, Order bounded separating linear maps on $\Phi$-algebras, *Houston J. Math.*, 30 (2004), 1143-1155.

[17] K. Boulabiar and G. Buskes, A note on bijective disjointness preserving operators, in: Positivity IV - Theory and Applications, pp. 29-33, Technische Universität Dresden, 2006.

[18] K. Boulabiar and G. Buskes, After the determinants are down: a criterion for invertibility, *Amer. Math. Monthly*, 119 (2003), 737-741.

[19] K. Boulabiar and G. Buskes, Polar decomposition of order bounded disjointness preserving operators, *Proc. Amer. Math. Soc.*, 132 (2004), 799-806.

[20] K. Boulabiar, G. Buskes, and M. Henriksen, A Generalization of a Theorem on Biseparating Maps, *J. Math. Ana. Appl.*, 280 (2003), 334-339.

[21] K. Boulabiar, G. Buskes, and G. Sirotkin, Algebraic order bounded disjointness preserving operators and strongly diagonal operators, *Integral Equa. Opera. Theory*, 54 (2006), 9-31.

[22] K. Boulabiar, G. Buskes, and A. Triki, Recent trends and advances in certain lattice-ordered algebras, *Contemporary Math.*, 328 (2003), 99-133.

[23] K. Boulabiar, G. Buskes, and A. Triki, Results in $f$-algebras, in: Positivity (Boulabiar et al. Eds), Trends in Mathematics, pp 73-96, Birkhäuser, Basel-Boston-Berlin, 2007.

[24] P. F. Conrad and J. E. Diem, The ring of polar preserving endomorphisms on an Abelian lattice-ordered group, *Illinois J. Math.*, 15 (1971), 222-240.

[25] E. Čech, On bicompact spaces, *Ann. Math.*, 38 (1937), 823-844.

[26] Z. Ercan and S. Önall, A remark on the homomorphism on $C(X)$, *Proc. Amer. Math. Soc.*, 133 (2005), 3609-3611.

[27] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Springer Verlag, Berlin-Heidelberg-New York, 1976.

[28] J. J. Grobler and C. B. Huijsmans, Disjointness preserving operators on complex Riesz spaces, *Positivity*, 1 (1997), 155-164.

[29] A. W. Hager, Isomorphism with a $C(Y)$ of the maximal ring of quotients of $C(X)$, *Fund. Math.*, 66 (1969), 7-13.

[30] D. R. Hart, Some properties of disjointness preserving operators, *Indag. Math.*, 88 (1985), 183-197.

[31] M. Henriksen, On the equivalence on the ring, lattice, and semigroup of continuous functions, *Proc. Amer. Math. Soc.*, 7 (1956), 959-960.

[32] E. Hewitt, Rings of real-valued continuous functions I, *Trans. Amer. Math. Soc.*, 64 (1948), 54-99.

[33] C. B. Huijsmans and B. de Pagter, Invertible disjointness preserving operators, *Proc. Edinburgh Math. Soc.*, 37 (1993), 125-132.

[34] C. B. Huijsmans and A. W. Wickstead, The inverse of band preserving and disjointness preserving operators, *Indag. Math.*, 3 (1992), 179-183.

[35] K. Jarosz, *Automatic continuity of separating linear isomorphisms*, Bull. Canadian Math. Soc. 33 (1990), 139-144.

[36] J. S. Jeang and N. C. Wong, Weighted composition operators of $C_0(X)$'s, *J. Math. Anal. Appl.*, 201 (1996), 981-993.

[37] I. Kaplansky, *Infinite Abelian Groups*, University of Michigan Press, Ann Arbor, 1954.

[38] Y. I. Karlovich and V. T. Kravchenko, Singular integral equations with non Carleman shift on anopen contour, *Diff. Equat.*, 17 (1981), 2212-2223.

[39] S. S. Kukatuladze, Support set for sublinear operators, *Soviet Math. Dokl.*, 21 (1976), 1428-1431.

[40] W. A. J. Luxemburg and A. R. Schep, A Radon-Nikodym type theorem for positive operators and a dual, *Indag. Math.*, 40 (1978), 357-375.

[41] W. A. J. Luxemburg and A. C. Zaanen, *Riesz spaces* I, North-Holland, Amsterdam-London, 1971.

[42] M. Meyer, Les homomorphismes d'espaces vectoriels réticulés complexes, *C. R. Acad. Sci. Paris Serie I*, 292 (1981), 793-796.

[43] M. Meyer, Le stabilateur d'un espace vectoriel réticulé, *C.R. Acad. Sci. Paris, Serie I*, 283, (1976), 249-250.

[44] M. Meyer, Quelques propriétés des homomorphismes d'espaces vectoriels réticulés, Equipe d'Analyse Paris VI, preprint 131, 1979.

[45] P. Meyer-Nieberg, *Banach Lattices*, Springer, Berlin-Heidelberg-New York, 1991.

[46] A. D. Myshkis, Certain problems of the theory of differential equations with deviation argument, *Uspechi Mat. Nauk.*, 32 (1974), 173-202.

[47] B. de Pagter, A note on disjointness preserving operators, *Proc. Amer. Math. Soc.*, 90 (1984), 543-549.

[48] B. de Pagter, $f$-Algebras and Orthomorphisms, Thesis, Leiden, 1981.

[49] B. de Pagter, The space of extended orthomorphisms on a vector lattice, *Pacific J. Math.*, 112 (1984), 193-210.

[50] B. de Pagter and A. R. Schep, Band decomposition for disjointness preserving operators, *Positivity*, 4 (2000), 259-288.

[51] R. J. Sakel and C. R. Sell, A spectral theory for linear differential systems, *J. Diff. Equat.*, 27 (1978), 320-328.

[52] T. Shirota, A class of topplogical spaces, *Osaka Math. J.*, 4 (1952), 23-40.

[53] M. H. Stone, Applications of the theory of Boolean rings to general topology, *Trans. Amer. Math. Soc.*, 41 (1937), 275-481.

[54] A. Tychonoff, Über die topologische Erweiterung von Raümen, *Math. Ann.*, 102 (1930), 544-561.

[55] B. Z. Vulikh, On linear multiplicative operations, *Dokl. Akad. Nauk.*, 41 (1943), 148-151.

[56] A. W. Wickstead, Representation and duality of multiplication operators on Archimedean Riesz spaces, *Compositio Math.*, 35 (1977), 225-238.

[57] A. W. Wickstead, The injective hull of an Archimedean $f$-algebra, *Compositio Math.*, 62 (1987), 329-342.

[58] A.C. Zaanen, Examples of orthomorphisms, *J. Approx. Theory*, 13 (1975) 192-204.

[59] A. C. Zaanen, *Introduction to Operator Theory in Riesz Spaces*, Springer Verlag, Berlin-Heidelberg-New York, 1997.

[60] A. C. Zaanen, *Riesz Spaces* II, North-Holland, Amsterdam-London, 1983.



Karim Boulabiar,
Institut Préparatoire aux Etudes Scientifiques et Techniques,
Université du 7 Novembre à Carthage,
BP 51, 2070-La Marsa, Tunisia
*karim.boulabiar@ipest.rnu.tn*


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# Minimizing Oblique Errors for Robust Estimating

DIARMUID O'DRISCOLL, DONALD E. RAMIREZ,
AND REBECCA SCHMITZ

Abstract. The slope of the best fit line from minimizing the sum of the squared oblique errors is shown to be the root of a polynomial of degree four. We introduce a median estimator for the slope and, using a case study, we show that the median estimator is robust.

## 1. Introduction

With ordinary least squares (OLS) regression, we have data

$$\{(x_1, Y_1|X = x_1), \ldots, (x_n, Y_n|X = x_n)\}$$

and we minimize the sum of the squared *vertical errors* to find the *best-fit* line $y = h(x) = \beta_0 + \beta_1 x$. With OLS it is assumed that the independent or causal variable is measured without error.

J. L. Gill [2] states that "some regression prediction or estimation must be made in a direction opposite to the natural causality of one variable by another." This is found from the inverse function $h^{-1}(y_0) = x_0 = y_0/\beta_1 - \beta_0/\beta_1$. He adds "Geometric mean regression could be more valid than either direct or inverse regression if both variables are subject to substantial measurement error."

For inverse prediction we will want both $h(x)$ and $h^{-1}(y)$ to model the data. To accomplish this, we try to determine a fit so that the squared vertical and the squared horizontal errors will both be small. The vertical errors are the squared distances from $(x, y)$ to $(x, h(x))$ and the horizontal errors are the squared distances from $(x, y)$ to $(h^{-1}(y), y)$. As a compromise, we will consider the errors at the median or midpoint to the predicted vertical and predicted horizontal values. All of the estimated regression models we consider (including the geometric mean and perpendicular methods) are contained in the parametrization (with $0 \leq \lambda \leq 1$) of the line from $(x, h(x))$

to $(h^{-1}(y), y)$. For the squared vertical errors, set $\lambda = 1$ and correspondingly, for the horizontal errors, set $\lambda = 0$. Our Maple codes and the data set for our case study can be found here:

`people.virginia.edu/~der/pdf/oblique_errors`

Our paper first introduces the Oblique Error Method in Section 2. In Section 3, we show how the Geometric Mean and Perpendicular Methods are included in our parametrization. In Section 4, we include a weighted regression procedure and Section 5 contains a small case study showing the robustness of the proposed median slope estimator.

## 2. Minimizing Squared Oblique Errors

From the data point $(x_i, y_i)$ to the fitted line $y = h(x) = \beta_0 + \beta_1 x$ the vertical length is $a_i = |y_i - \beta_0 - \beta_1 x_i|$, the horizontal length is $b_i = |x_i - (y_i - \beta_0)/\beta_1| = |(\beta_1 x_i - y_i + \beta_0)/\beta_1| = |a_i/\beta_1|$ and the perpendicular length is $h_i = a_i/\sqrt{1+\beta_i^2}$. With standard notation,

$$S_{xx} = \sum_{i=1}^{n}(x_i - \overline{x})^2, S_{yy} = \sum_{i=1}^{n}(y_i - \overline{y})^2, S_{xy} = \sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})$$

with the correlation $\rho = S_{xy}/\sqrt{S_{xx}S_{yy}}$. A basic fact is $-1 \le \rho \le 1$ or equivalently $0 \le S_{xy}^2 \le S_{xx}S_{yy}$.

For the oblique length from $(x_i, y_i)$ to $(h^{-1}(y_i) + \lambda(x_i - h^{-1}(y_i)), y_i + \lambda(h(x_i) - y_i))$, the horizontal length is $(1-\lambda)b_i = (1-\lambda)a_i/\beta_1$ and the vertical length is $\lambda a_i$. Since $SSE_h(\beta_0, \beta_1, \lambda) = \left(\sum_{i=1}^n a_i^2\right)/\beta_1^2$ and $SSE_v(\beta_0, \beta_1, \lambda) = \sum_{i=1}^n a_i^2$, we have

$$\begin{aligned} SSE_o(\beta_0, \beta_1, \lambda) &= (1-\lambda)^2 SSE_h + \lambda^2 SSE_v \\ &= \sum_{i=1}^{n}\left\{\frac{(1-\lambda)^2 a_i^2}{\beta_1^2} + \lambda^2 a_i^2\right\} = \frac{(1-\lambda)^2 + \lambda^2\beta_1^2}{\beta_1^2}\sum_{i=1}^{n} a_i^2. \end{aligned}$$

Setting $\partial SSE_o/\partial\beta_0 = 0$, then $\beta_0 = \overline{y} - \beta_1\overline{x}$ and

$$\begin{aligned} \sum_{i=1}^{n} a_i^2 &= \sum_{i=1}^{n}\left\{(y_i - \overline{y}) - \beta_1(x_i - \overline{x})\right\}^2 \\ &= S_{yy} - 2\beta_1 S_{xy} + \beta_1^2 S_{xx}. \end{aligned}$$

Hence

$$SSE_o = ((1-\lambda)^2\beta_1^{-2} + \lambda^2)\left(S_{yy} - 2\beta_1 S_{xy} + \beta_1^2 S_{xx}\right) \tag{1}$$

with

$$\frac{\partial SSE_o}{\partial \beta_1} = -2(1-\lambda)^2\beta_1^{-3}S_{yy}+2(1-\lambda)^2\beta_1^{-2}S_{xy}-2\lambda^2 S_{xy}+2\lambda^2\beta_1 S_{xx}.$$

Thus the oblique estimator is a root of the fourth degree polynomial in $\beta_1$, namely

$$P_4(\beta_1) = \lambda^2\sqrt{\frac{S_{xx}}{S_{yy}}}\beta_1^4 - \lambda^2\rho\beta_1^3 + (1-\lambda)^2\rho\beta_1 - (1-\lambda)^2\sqrt{\frac{S_{yy}}{S_{xx}}}. \quad (2)$$

We claim that $P_4(\beta_1)$ has exactly two real roots, one positive and one negative. By inspection, since the leading coefficient of $P_4(\beta_1)$ is positive and the constant coefficient is negative, $P_4(\beta_1)$ necessarily has at least one positive and one negative root. That these are the only real roots will be important in establishing the global minimum value for $SSE_o$.

The Complete Discrimination System $\{D_1, \ldots, D_n\}$ of Yang [4] is a set of explicit expressions that determine the number (and multiplicity) of roots of a polynomial. In the case of a fourth degree polynomial, the polynomial has exactly two real roots, each with multiplicity one, provided $D_4 < 0$; where $D_4 = 256a_0^3a_4^3+...+144a_0^2a_2a_4a_3^2$. The expression for $D_4$ has 16 terms involving the five coefficients $\{a_0, \ldots, a_4\}$ of the polynomial and it is of order 6.

For the polynomial $P_4(\beta_1)$ (with some manipulations),

$$\begin{aligned} D_4 &= \lambda^6(1-\lambda)^6(-256+192\rho^2+6\rho^4+4\rho^6) \\ &\quad -27\lambda^4(1-\lambda)^4\rho^4\left(\frac{S_{xx}}{S_{yy}}(1-\lambda)^4+\lambda^4\frac{S_{yy}}{S_{xx}}\right). \end{aligned}$$

Since $|\rho| \le 1$, it follows that $D_4 < 0$. And thus $P_4(\beta_1)$ has exactly one positive and one negative root.

Evaluating $\partial SSE_o/\partial\beta_1$ at $\beta_1 = S_{xy}/S_{xx}$ and using the inequality $0 \le S_{xy}^2 \le S_{xx}S_{yy}$ and the equality $S_{xx}S_{yy} - S_{xy}^2 = (1-\rho^2)S_{xx}S_{yy}$,

$$\begin{aligned} \frac{\partial SSE_o}{\partial \beta_1} &= \frac{-2(1-\lambda)^2}{\beta_1^2}\left\{\frac{S_{yy}}{S_{xy}/S_{xx}} - S_{xy}\right\} + 2\lambda^2\left\{-S_{xy} + \frac{S_{xy}}{S_{xx}}S_{xx}\right\} \\ &= \frac{-2(1-\lambda)^2}{\beta_1^2}\frac{1}{S_{xy}}S_{xx}S_{yy}(1-\rho^2) \end{aligned}$$

which has the sign of $-S_{xy}$. Similarly evaluating $\partial SSE_o/\partial\beta_1$ at $\beta_1 = S_{yy}/S_{xy}$

$$\frac{\partial SSE_o}{\partial\beta_1} = 2\lambda^2 \frac{1}{S_{xy}} S_{yy} S_{xx}(1-\rho^2)$$

which has the sign of $S_{xy}$.

We use the Intermediate Value Theorem to assert that (1) If $S_{xy} > 0$, then $0 < S_{xy}/S_{xx} \le \beta_1 \le S_{yy}/S_{xy}$; (2) If $S_{xy} < 0$, then $S_{yy}/S_{xy} \le \beta_1 \le S_{xy}/S_{xx} < 0$; and (3) If $S_{xy} = 0$, $\beta_1 = \pm\left(((1-\lambda)^2 S_{yy})/(\lambda^2 S_{xx})\right)^{1/4}$.

The Second Derivative Test assures that a root of $P_4(\beta_1)$ is a local minimum of $SSE_o$ by

$$\frac{\partial^2 SSE_o}{\partial\beta_1^2} = \frac{6(1-\lambda)^2 S_{yy}}{\beta_1^4} - \frac{4(1-\lambda)^2 S_{xy}}{\beta_1^3} + 2\lambda^2 S_{xx}$$

$$= \frac{2(1-\lambda)^2}{\beta_1^4}\left[3S_{yy} - 2\beta_1 S_{xy}\right] + 2\lambda^2 S_{xx},$$

with $3S_{yy} - 2\beta_1 S_{xy} = 3S_{yy} - 2|\beta_1 S_{xy}| \ge 3S_{yy} - 2S_{yy} = S_{yy} > 0$.

Suppose $S_{xy} > 0$. Note from Equation (1) that $SSE_o(|\beta_1|) < SSE_o(-|\beta_1|)$. Let $\beta_1^+$ be the positive root of $P_4(\beta_1)$ and let $\beta_1^-$ be the negative root of $P_4(\beta_1)$. Then $SSE_o(\beta_1^+) \le SSE_o(|\beta_1^-|) < SSE_o(\beta_1^-)$. This assures that the positive root gives the global minimum for $SSE_0(\beta_1)$. A similar result holds when $S_{xy} < 0$.

## 3. Minimizing Squared Perpendicular and Squared Geometric Mean Errors

The perpendicular error model dates back to Adcock [1] who introduced it as a procedure for fitting a straight line model to data with error measured in both the $x$ and $y$ directions.

For squared perpendicular errors we minimize $SSE_p(\beta_0, \beta_1) = \sum_{i=1}^n a_i^2/(1+\beta_1^2)$ with solutions $\beta_0^p = \overline{y} - \beta_1^p \overline{x}$ and

$$\beta_1^p = \frac{(S_{yy} - S_{xx}) \pm \sqrt{(S_{yy} - S_{xx})^2 + 4S_{xy}^2}}{2S_{xy}}, \tag{3}$$

(provided $S_{xy} \ne 0$).

Note with $S_{xy} \ne 0$ and $S_{xx} = S_{yy}$, then $\beta_1^p = \pm 1$ showing that under standardization this method is functionally independent of the correlation between $x$ and $y$!

For squared geometric mean errors, we minimize $SSE_g(\beta_0, \beta_1) = \sum_{i=1}^n \left(\sqrt{|a_i b_i|}\right)^2 = \sum_{i=1}^n a_i^2/|\beta_1|$ with solutions $\beta_0^g = \overline{y} - \beta_1^g \overline{x}$ and $\beta_1^g = \pm\sqrt{S_{yy}/S_{xx}}$. Note that $\beta_1^g$ is always functionally independent of the correlation between $x$ and $y$ and also under standardization $b_1^g = \pm 1$ as in the perpendicular model.

The solutions to the above equations for both $\beta_1^p$ and $\beta_1^g$ are also roots of $P_4(\beta_1)$ for particular values of $\lambda$ which can be seen from the geometry of the model. See [3] and [2] for applications of the perpendicular and geometric mean estimators.

## 4. Minimizing Squared Weighted Average Errors

If the user wishes to incorporate the effect of different variances in $x$ and $y$, this can be achieved by using a weighed average of the squared vertical and squared horizontal errors with $(0 \leq \alpha \leq 1)$ and $SSE_w = \alpha SSE_v + (1-\alpha) SSE_h$. A typical value for $\alpha$ might be $\alpha = \sigma_y^2/(\sigma_x^2 + \sigma_y^2)$ to standardize the data. Recall from Section 2 that $SSE_o = \lambda^2 SSE_v + (1-\lambda)^2 SSE_h$. On setting $(1-\lambda)^2/\lambda^2 = (1-\alpha)/\alpha$, we get the quadratic equation $(2\alpha - 1)\lambda^2 - 2\alpha\lambda + \alpha = 0$, which has root

$$\lambda = \begin{cases} \dfrac{\alpha - \sqrt{\alpha(1-\alpha)}}{(2\alpha - 1)} & \alpha \neq \dfrac{1}{2} \\ \dfrac{1}{2} & \alpha = \dfrac{1}{2}. \end{cases} \tag{4}$$

## 5. Case Study

In this section, we introduce the median estimator $\beta_1^m$ using $P_4(\beta_1)$ with $\lambda = 1/2$. Our small case study reveals the desirable robustness inherent in the median estimator. The data set is from [2] with $n = 40$. The case study shows that the perpendicular estimator is highly influenced by outliers in the data, with the vertical and horizontal estimators also being significantly influenced by outliers. The geometric mean estimator, as expected, is more robust; and our median estimator, introduced in this paper, being the most robust in this case study. For the Weighted Average procedure, $\alpha = S_{yy}/(S_{yy} + S_{xx}) = 0.671$ which from Equation 4 yields $\lambda = 0.588$.

The first table below gives the values for the slope $\beta_1$, $y$ -intercept $\beta_0$, $\lambda$, and $SSE$. To study the effect of outliers, we pick a row from the data set and perturb the values by some factor.

The second table contains the basic values and, in addition, the square of the shifts in the slope and $y$-intercept caused by perturbing the $x$-data by a factor of 7.5 for the data point for case $k = 5$. Note that the median estimator has the smallest squared shift distance. The third table shows similar values by perturbing the $y$-data by a factor of 0.5 for case $k = 5$. Note that the perpendicular model has been greatly influenced by this one outlier.

| | Vert | Horiz | Perp | Geom | Median | Wt Avg |
|---|---|---|---|---|---|---|
| $\beta_1$ | 1.28 | 1.59 | 1.48 | 1.43 | 1.38 | 1.35 |
| $\beta_0$ | 136 | 104 | 115 | 121 | 126 | 130 |
| $\lambda$ | 1.00 | 0.00 | 0.312 | 0.412 | 0.500 | 0.588 |
| $SSE$ | 12565 | 6163 | 4330 | 4494 | 4908 | 5581 |

Table 1. Gill Data for Vertical (Vert), Horizontal (Horiz), Perpendicular (Perp), Geometric Mean (Geom), Median and Weighted Average (Wt Avg) Procedures

| | Vert | Horiz | Perp | Geom | Median |
|---|---|---|---|---|---|
| $\beta_1$ | 0.0937 | 2.33 | 0.118 | 0.467 | 0.654 |
| $\beta_0$ | 259 | $-4.39$ | 256 | 215 | 193 |
| $SSE$ | 62007 | 284364 | 61327 | 95987 | 129360 |
| $(\beta_1^* - \beta_1)^2$ | 1.41 | 0.541 | 1.87 | 0.923 | 0.531 |
| $(\beta_0^* - \beta_0)^2$ | 15040 | 11723 | 19855 | 8818 | 4506 |

Table 2. Gill Data perturbed with $x^*[5] = 7.5\ x[5]$

| | Vert | Horiz | Perp | Geom | Median |
|---|---|---|---|---|---|
| $\beta_1$ | 0.875 | 1.99 | 1.51 | 1.32 | 1.23 |
| $\beta_0$ | 174 | 57.2 | 107 | 127 | 137 |
| $SSE$ | 30977 | 17770 | 13339 | 13841 | 14521 |
| $(\beta_1^* - \beta_1)^2$ | 0.165 | 0.161 | 0.000717 | 0.0116 | 0.0228 |
| $(\beta_0^* - \beta_0)^2$ | 1410 | 4875 | 3446 | 2789 | 2312 |

Table 3. Gill Data perturbed with $y^*[4] = 0.5\ y[4]$

We replicated the above perturbation procedure for each of the $n = 40$ cases and record in Table 4 and Table 5 the average squared change in slope and the average squared change in the $y$-intercept denoted $\{E(\beta_1^* - \beta_1)^2, E(\beta_0^* - \beta_0)^2\}$ by perturbing the original $x$-data and $y$-data values by a factor of $\{7.5, 0.5\}$ respectively. Table 6 records the average squared changes where the data has been jointly perturbed for $(x[k], y[k])$ by the factors $\{7.5, 0.5\}$ respectively.

| | Vert | Horiz | Perp | Geom | Median |
|---|---|---|---|---|---|
| $E(\beta_1^* - \beta_1)^2$ | 1.41 | 4.01 | 1.87 | 1.07 | 0.656 |
| $E(\beta_0^* - \beta_0)^2$ | 14966 | 57192 | 19820 | 10358 | 5649 |

Table 4. Gill Data perturbed with $x^*[k] = 7.5\ x[k]$

| | Vert | Horiz | Perp | Geom | Median |
|---|---|---|---|---|---|
| $E(\beta_1^* - \beta_1)^2$ | 0.0163 | 0.201 | 0.0968 | 0.0333 | 0.0143 |
| $E(\beta_0^* - \beta_0)^2$ | 165 | 2509 | 1276 | 488 | 229 |

Table 5. Gill Data perturbed with $y^*[k] = 0.5\ y[k]$

| | Vert | Horiz | Perp | Geom | Median |
|---|---|---|---|---|---|
| $E(\beta_1^* - \beta_1)^2$ | 1.90 | 20.6 | 2.59 | 0.975 | 0.487 |
| $E(\beta_0^* - \beta_0)^2$ | 20175 | 258272 | 27880 | 8611 | 3364 |

Table 6. Gill Data perturbed with $\{x^*[k] = 7.5\ x[k], y^*[k] = 0.5\ y[k]\}$

The results in Table 4 with an outlier in the $x$-data show the sensitivity with the vertical, horizontal and perpendicular procedures. The results in Table 5 with an outlier in the $y$-data show the sensitivity with the horizontal and perpendicular procedures. Table 6, with $(x, y)$ both perturbed, shows the robustness of the geometric and median procedures with the median estimators uniformly superior to the geometric estimators in this small case study. These preliminary results commend the method for further investigation.

## References


[1] R. J. Adcock, A problem in least-squares, *The Analyst*, **5** (1878), 53–54.
[2] J. L. Gill, Biases in regression when prediction is inverse to causation, *American Society of Animal Science* **64** (1987), 594–600.
[3] L. Leng, T. Zhang, L. Kleinman, and W. Zhu, Ordinary least square regression, orthogonal regression, geometric mean regression and their applications in aerosol science, *Journal of Physics: Conference Series* **78** (2007), 012084–012088.
[4] L. Yang, Recent advances on determining the number of real roots of parametric polynomials, *J. Symbolic Computation* **28** (1999), 225–242.



Diarmuid O'Driscoll,
Department of Mathematics and Computer Studies,
Mary Immaculate College,
Limerick, Ireland,
*diarmuid.odriscoll@mic.ul.ie*

*Donald E. Ramirez,*
*Department of Mathematics,*
*University of Virginia,*
*Charlottesville, VA 22904, USA*
*der@virginia.edu*

*Rebecca Schmitz,*
*ITCEP,*
*University of Minnesota,*
*Minneapolis, MN 55455, USA*
*schm2676@umn.edu*


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