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EDITORIAL

This summer issue of the Bulletin *once again contains a mix of contributions of varying kind—which is the preferred format. However, it appears to be difficult to attract research notes of a good quality for the* Bulletin*. Many of the submissions have to be rejected outright. Maybe members of the Society can consider to submit well-written, short article (of up to 10 pages length, say) that highlight their research and can also encourage their colleagues and collaborators at other universities to do the same? This would be much appreciated, since the only way to attract good-quality papers is to publish some.*

The section "Announcements of Conferences" will be closed due to too little material provided by organisers; in future, please refer to the IMS website at

`http://www.maths.tcd.ie/pub/ims/Calendar-ie/`

for up-to-date and quickly accessible information. Only the annual IMS meeting, and maybe events very closely connected with it, will be announced in the future, always in the winter issue for the following year.

However, I would like to resurrect the "Departmental News", which were stopped some time ago, again because of too little material. A reminder to provide the Editor with pertinent material (new appointments, prizes, etc.) shall be sent out via `mathdep` *later this year, to be published in Volume 62.*

—MM

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Applying for I.M.S. Membership

1. The Irish Mathematical Society has reciprocity agreements with the American Mathematical Society, the Irish Mathematics Teachers Association, the New Zealand Mathematical Society and the Real Sociedad Matemática Española.

2. The current subscription fees (as from 1 January 2002) are given below:

Institutional member	130 euro
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4. Any member with a bank account in the Irish Republic may pay his or her subscription by a bank standing order using the form supplied by the Society.

5. Any ordinary member who has reached the age of 65 years and has been a fully paid up member for the previous five years may pay at the student membership rate of subscription.

6. Subscriptions normally fall due on 1 February each year.

7. Cheques should be made payable to the Irish Mathematical Society. If a Eurocheque is used then the card number should be written on the back of the cheque.

8. Any application for membership must be presented to the Committee of the I.M.S. before it can be accepted. This Committee meets twice each year.

9. Please send the completed application form with one year's subscription to:

   The Treasurer, I.M.S.
   Department of Mathematics
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Information from the European Mathematical Society

The European Mathematical Society is increasing its activities and its membership. We are working harder than ever to make sure that mathematics is represented properly when funding decisions are taken at a European level, and this is beginning to bear fruit. An example is the recent call by the European Science Foundation for proposals for research conferences in mathematics

`http://www.esf.org/index.php?id=4602`

Also, we now have 56 national member societies from all over Europe, which brings huge opportunities for collaborative work of all kinds.

We would like to increase our individual membership, which now comes with free access to Zentralblatt

`http://www.zentralblatt-math.org/portal/en/`

as well as our superb Newsletter

`http://www.ems-ph.org/journals/journal.php?jrn=news`

and many other benefits, as you can see from our new web site

`http://www.euro-math-soc.eu/`

Membership is not expensive, and joining is easy: you can do it either through the Irish Mathematical Society or on the EMS web page.

Ari Laptev, President
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IRISH MATHEMATICAL SOCIETY
President's Report 2007

Thanks: The first item to be noted in my report is to thank Maurice O'Reilly for the extraordinary job he did as President in 2005 and 2006. Two other notable Officers of the IMS whose term of office ended in 2007 are David Wraith (Treasurer) and Ann O'Shea (Secretary), they have held these positions over the last four to six years and the IMS owes a large debt of gratitude to them. Finally the Society would like to thank David Armitage, who is retiring. David has been a long-standing committee member of the IMS and served as President of the Society too.

Future fees: The Society has decided to raise its fees for 2009 for normal members from €20 to €25 with commensurate changes in other membership categories.

Membership communication: To improve communications with our members, we will be including an insert in the next Bulletin for members to update their details including e-mail addresses.

Foyles' Discount: Foyles' Bookshop in London offers IMS members a 10% discount on all orders. To avail of this offer (negotiated by David Wraith) members will need to set up an account with Foyles, see:

`http://www.maths.tcd.ie/pub/ims/Books/foylesform.pdf`

For security reasons, on your application you will need to enter your IMS membership number, the latter can be obtained by e-mailing the IMS Treasurer.

Links: The IMS has established links (via Maurice O'Reilly) with the Société Mathématique de France, these links take the form of exchanging information regarding our respective societies' activities in the others' publications and it is hoped there will be SMF involvement at the BMC in 2009 in Galway. The Society has set up a reciprocity agreement with the New Zealand Mathematical Society (`http://www.waikato.ac.nz/NZMS/NZMS.html`), which will start in 2008. In Ireland, the Society has established links with the newly founded Irish Applied Mathematics Teachers Association (`http://www.iamta.ie/`).

Website: The IMS website (`http://www.maths.tcd.ie/pub/ims/`) continues to be maintained by Richard Timoney and in 2007 he added the IMS Diary page. This page lists conferences that are being held in Ireland and is designed for conference organisers to avoid clashes.

SFI Mathematics Initiative: I twice circulated (via MATHDEP) mathematicians in Ireland asking for their views and comments on this year's SFI Mathematics Initiative. I received about 30 replies mostly from people who did not apply this year under the initiative. Consequently, I met with Dr Gary Crawley in July to discuss the opinions expressed, and his responses are on the IMS website

`http://www.maths.tcd.ie/pub/ims/business/SFIMaths2.pdf`

Annual conferences: Dr Crawley opened and addressed the IMS September conference in UCD in September 2007. The conference was well attended and featured a diverse range of talks over the two days. The IMS expressed its thanks to the five main organizers Christopher Boyd, Sean Dineen, Michael Mackey, Rhona Preston and myself for their work in this regard.

The Society also held a joint symposium with DIAS in December, which covered a range of topics in mathematics and physics over two days. The IMS would like to thank DIAS for the financial support they provided for this symposium.

Fergus Gaines' Cup: The Irish Mathematical Society awards the Fergus Gaines' Cup annually to the best performer in the Irish Mathematical Olympiad. The cup was awarded on 16th November 2006 to Galin Ganchev in St Patrick's College, Dublin and on 15th November 2007 to Stephen Dolan again in St Patrick's College.

Teaching: A special committee (chaired by Tom Carroll) produced a discussion document on service teaching of mathematics, which can be found on the IMS website. This document was developed by the standing committee on teaching and educational matters (chaired by Ann O'Shea) and may be viewed here:

`http://www.maths.tcd.ie/pub/ims/business/2007-09-CSTM2.pdf`

Brendan Guilfoyle joined both these committees and also kindly agreed to act as the Public Relations Officer of the Society.

Institute of Technology members: I have particularly been trying to recruit members from the IoT sector in 2007 and have met with some limited success. In order to attract and maintain such members it is important that the Society changes from its university focus to a more inclusive one. This would take the form of supporting IoT sector conferences financially and having IoT speakers at IMS conferences. This policy is being implemented as we now have a good representation of the IoT sector on the Committee of the Society. We are also including talks on educational matters and of more general interest at our September conference (however, more needs to be done). To address the last point I set up a sub-committee on IoT membership with Jim Cruickshank as Chairperson. He has peopled this sub-committee with four IoT members (so far) and their remit is to:

1. report back on what the IMS is expected/can do for IoT members and
2. consider the timing of our 'September' conference.

The report will be prepared for the Committee meeting in September 2008.

Conference support: The number of conference organisers applying for support from the IMS for conferences has dropped. The reasons for this may be two-fold: The IMS is not a cash-rich Society and so our average grant is only about €250 to €350, secondly conference organisers had to write a short report on the conference if they had received support. There is little the Society can do about the general level of funding, but the second condition has now been eliminated and instead organisers will just be asked to acknowledge that we helped to fund the conference and perhaps to distribute some membership forms at it.

Future meetings: Future planned meetings of the IMS are as follows:

- September 1-2, 2008, Cork IoT: IMS Conference and AGM
- December 2008, DIAS (Dublin): Joint DIAS/IMS Symposium
- April 6-9 2009, NUI Galway: Joint meeting of the British Mathematical Colloquium and the IMS
- December 2009, DIAS (Dublin): Joint DIAS/IMS Symposium and IMS AGM

- September 2010, Dublin IoT: IMS Conference and AGM
- September 2011: A non-Dublin venue is being sought for the IMS Conference and AGM
- September 2012: Tallaght IoT? IMS Conference and AGM

Russell Higgs
President of the Irish Mathematical Society
15th December 2007

Algebraic Properties of the Index of Invariant Subspaces of Operators on Banach Spaces

GEORGE CHAILOS

Abstract. For an operator S on a Banach space X, let $Lat\,(S,X)$ be the collection of all its invariant subspaces. We consider the *index* function on $Lat\,(S,X)$ and establish various algebraic properties of it. Amongst others we show that if S is a bounded below operator, then

$$ind\,M + ind\,N \geq ind(M \cap N) + ind(M \vee N).$$

If, in addition, $ind\,M = ind\,N = 1$ and $M \cap N \neq \{0\}$ then $ind(M \vee N) = 1$.

1. Introduction

If S is an operator on a Banach space X, then a closed subspace M of X is called invariant for S if $SM \subset M$. The collection of invariant subspaces of an operator S is denoted by $Lat\,(S,X)$. It forms a complete lattice with respect to intersections and closed spans. One of the important notions in the general theory of operators, such as bounded below operators, is the index of an element in $Lat\,(S,X)$, which is defined as follows. (This definition is taken from [1].)

Definition 1.1. The map

$$ind : Lat\,(S,X) \longrightarrow \{0\} \cup \mathbb{N} \cup \{\infty\}$$

is defined as $ind\,M = dim\,(M/SM)$ and $ind\,M = 0$ if and only if $M = \{0\}$. We say that M has index n if $ind\,M = n$.

The index function plays an essential role in the study of invariant subspaces of Banach spaces. (For example, see an extensive study in [5] of index 1 invariant subspaces in Banach spaces of analytic

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functions.) In this article we give various algebraic properties of the index function. Amongst others, and as a corollary to our main result, we show that if $M, N \in Lat(S, X)$, $ind\, M = ind\, N = 1$ and $M \cap N \neq \{0\}$ then $ind(M \vee N) = 1$, where $M \vee N$ denotes the closed span of M and N. (Equivalently, $M \vee N$ is the closure of $M + N$). This result, but in not such a general setting as the one presented here, was proved by Richter ([5], Corollary 3.12), using operator theoretical tools and results from analysis. Here we prove it using only algebraic tools and a rather standard result from functional analysis.

2. Algebraic Properties of the Index Function

Theorem 2.1. *Let* $\mathbf{R}$ *be a commutative ring with identity and let* A, A', B' *be free unitary* $\mathbf{R}$*-modules such that* A' *and* B' *are free submodules of* A. *Then*

$$rank(A/A') + rank(A/B') = rank(A/(A' \cap B')) + rank(A/(A' + B')).$$

Proof. Consider the following sequence

$$0 \longrightarrow A/(A' \cap B') \xrightarrow{f} A/A' \oplus A/B' \xrightarrow{g} A/(A' + B') \longrightarrow 0,$$

where $f([y]) = ([y], [y])$, $g([x], [y]) = [x - y]$ and $[\cdot]$ denotes the equivalence class in the appropriate quotient module. We claim that the sequence above is exact.

To prove the claim we first show that f and g are well-defined homomorphisms. Letting $[y] \in A/(A' \cap B')$ and $x \in A' \cap B'$, we obtain that $f([y + x]) = ([y + x], [y + x]) = ([y], [y])$. Hence, f is well defined. Moreover, f is a homomorphism, since

$$\begin{aligned} f([y] + [z]) &= ([y] + [z], [y] + [z]) = ([y], [y]) + ([z], [z]) \\ f(r[y]) &= (r[y], r[y]) = r([y], [y]),\ r \in \mathbf{R}. \end{aligned}$$

Similarly, if $([x], [y]) \in A/A' \oplus A/B'$, and $x_1 \in A'$, $x_2 \in B'$, then

$$\begin{aligned} g([x + x_1], [y + y_1]) &= [(x + x_1) - (y + y_1)] \\ &= [(x - y) + (x_1 - y_1)] = [x - y], \end{aligned}$$

since $x_1 - y_1 \in A' + B'$. Thus, g is well defined.

Moreover, g is a homomorphism, since

$$\begin{aligned} g(([x],[y]) + ([x'],[y'])) &= g([x]+[x'],[y]+[y']) \\ &= g([x+x'],[y+y']) \\ &= [(x+x')-(y+y')] = [x-y+x'-y'] \\ &= [x-y]+[x'-y'] \end{aligned}$$

and $\quad g(r([x],[y])) = g([rx],[ry]) = [rx-ry] = r[x-y],\ r \in \mathbf{R}.$

It remains to show that $\ker g = \operatorname{im} f$. For this let $([x],[y]) \in A/A' \oplus A/B'$ be such that $g([x],][y]) = 0$. Then $[x-y]=0$, and thus $x-y \in A'+B'$. This implies that $x+A' = y+B'$, i.e., $[x]_{A/A'} = [y]_{A/B'}$ wherefore $([x]_{A/A'},[y]_{A/B'}) \in \operatorname{im} f$, and hence $\ker g \subset \operatorname{im} f$.

Conversely, if $([x],[y]) \in \operatorname{im} f$ then $x+A' = y+B'$ and hence $x+A'+B' = y+A'+B'$. It follows that $g([x],[y]) = [x-y] = 0$ so that $\operatorname{im} f \subset \ker g$. The proof of the claim is complete.

Since $A/(A'+B')$ is a free module, it is in particular projective, and hence the above exact sequence splits (see [4]). Therefore

$$A/A' \oplus A/B' = A/(A'\cap B') \oplus A/(A'+B').$$

This immediately implies that

$$rank(A/A') + rank(A/B') = rank(A/(A'\cap B')) + rank(A/(A'+B'))$$

concluding the proof of the theorem. □

As every vector space is free over its ground field, the following is an immediate consequence of the above theorem.

Corollary 2.2. *If X is a Banach space and S an operator on X, for all $M,N \in Lat(S,X)$*

$$ind\, M + ind\, N = ind(M\cap N) + ind(M+N).$$

In then case when S is a bounded below operator, like the shift operator on Banach spaces of analytic functions, the following holds.

Lemma 2.3. *Suppose $M,N \in Lat(S,X)$, where S is a bounded below operator on a Banach space X. Then*

$$ind(M\vee N) \le ind(M+N) \le ind\, M + ind\, N.$$

Proof. If either $ind\, M$ or $ind\, N$ is infinite, then there is nothing to prove. So we may assume that $ind\, M < \infty$ and $ind\, N < \infty$. Thus there are finite-dimensional subspaces M_1 and N_1 of M and N, respectively, such that $M = SM + M_1, \quad N = SN + N_1$, where $dim\, M_1 = ind\, M$ and $dim\, N_1 = ind\, N$. We find that

$$\begin{aligned} M + N &= SM + M_1 + SN + N_1 \\ &= S(M + N) + M_1 + N_1 \\ &\subseteq S(M \vee N) + (M_1 + N_1) \\ &\subseteq M \vee N. \end{aligned}$$

Since S is a bounded below operator, its range is closed (see, e.g., [2], Proposition 6.4, chapter VII), and hence the second to last expression is the sum of a closed and a finite-dimensional subspace, hence it is closed. Since $M + N$ is dense in $M \vee N$ we obtain that the last inclusion in above is actually an equality. From this it follows that

$$ind(M \vee N) \leq dim(M_1 + N_1) = ind(M + N) \leq ind\, M + ind\, N.$$

□

The next theorem, which is our main result, follows immediately from Corollary 2.2 and Lemma 2.3.

Theorem 2.4. *If X is a Banach space and S a bounded below operator on X then, for all $M, N \in Lat(S, X)$,*

$$ind\, M + ind\, N \geq ind(M \cap N) + ind(M \vee N).$$

Corollary 2.5. *Suppose that $M_1, M_2 \in Lat(S, X)$ are such that $ind\, M_1 = ind\, M_2 = 1$, where S, X are as in the previous theorem. If $M_1 \cap M_2 \neq \{0\}$ then $ind\,(M_1 \vee M_2) = 1$.*

Proof. If $M_1 \cap M_2 \neq \{0\}$ then $ind(M_1 \cap M_2) \geq 1$. As $ind(M_1 \vee M_2) \geq 1$, Theorem 2.4 implies that $ind\,(M_1 \vee M_2) = 1$. □

Example 2.6. In [5], Proposition (2.16 b), Richter considered the case where S is the shift operator on any Banach space $\mathcal{B}$ of analytic functions on an open and connected subset of the complex plane. He showed that if $m \geq 2$ and there is a space in $Lat(S, \mathcal{B})$ of index m, and furthermore if $n_1, n_2 \in \mathbb{N} \cup \{\infty\}$, $n_1 + n_2 = m$, then there are invariant subspaces N_1, N_2 such that $ind\, N_i = n_i$, $i = 1, 2$ and $ind(N_1 \vee N_2) = indN_1 + indN_2$. In these cases, Theorem 2.4 implies that $ind(N_1 \cap N_2) = 0$ and hence $N_1 \cap N_2 = \{0\}$. Thus, $N_1 \vee N_2 =$

$N_1 \oplus N_2$. (For example, it is well known ([3], Corollary 6.5) that when S is the shift operator on a weighted Bergman space on the unit disk, then for all $1 \leq m \leq \infty$ there are invariant subspaces of index m.)

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Lipschitz Character of Solutions to the Inner Obstacle Problems

SŁAWOMIR JAGODZIŃSKI, ANNA OLEK AND KUBA SZCZEPANIAK

Abstract. In our paper we consider the inner problem with $l \in \mathbb{N}$ impediments from below, the inner problem with $m \in \mathbb{N}$ impediments from above and the double inner problem with $l + m$ impediments. Assuming the Lipschitz character of the obstacles we show that the corresponding solutions are also Lipschitz. We extend here the result given in [SV], where the author considered the inner obstacle problem with a single impediment from below. Our work is based on the ideas introduced by J. Jordanov from 1982 who investigated $H^{1,p}(\Omega)$ regularity of solutions to inner obstacle problems.

1. Introduction

In many physical processes "obstacles" appear in a natural way having strong influence on the character of the examined problem. A simple example of such a situation is the study of contrast between a vibrating membrane and a vibrating membrane set between obstacles.

In the 1970's there was considerable interest in the analysis of obstacle problems. This was connected with the development of research on variational inequalities and has been studied by many authors (see [BC], [BS], [T] and references therein). The majority of results concentrated on, natural from a mathematical point of view, problems of existence and uniqueness of the solutions. However, in case of variational inequalities corresponding to obstacle problems additional questions regarding, e.g., the coincidence set (cf. [DS1], [DS2]) or regularity of the solutions (cf. [BS]) can be posed. These problems seem to be interesting due to possible applications.

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Key words and phrases. Inner obstacle problems, Lipschitz continuity, variational inequalities.

The fundamental result where regularity of solutions with regard to regularity of obstacles is studied in case of global obstacle problems can be found in [KS].

Recently the interest in the analysis of the obstacle problems has increased. This is due to appearance of works on the inner problems (see [BSz], [JOS], [Ro] and references therein). Among other things examination of regularity of solutions to the inner obstacle problems is a matter of significant importance.

A study of the Lipschitz character of the solutions to the obstacle problems was initiated in [SV]. The authors showed that the solution of the global problem and the inner problem with one obstacle from below is Lipschitz continuous assuming that the impediments are Lipschitz. Later on the papers [Ch1], [Ch2] appeared where one can find theorems concerning the Lipschitz continuity of the solutions to the global inverse and double global problems.

In our work we aim at transferring results concerning Lipschitz character of solutions of global obstacle problems to the case of inner ones. It is worth mentioning that the construction presented in our paper enables us to identify each inner problem with the corresponding global one. This fact makes it possible to carry out the complete analysis of the inner problems with the help of methods available for the global ones.

We offer a comprehensive study of the Lipschitz regularity of solutions of the inner obstacle problems. The present paper is a part of the research program on free boundary problems.

2. Notation and Basic Definitions

Throughout the paper we assume that $\Omega \subset \mathbb{R}^n$ is an open, bounded set with the smooth boundary $\partial\Omega$. The functions $a_{ij} : \bar{\Omega} \to \mathbb{R}$ for $1 \leq i, j \leq n$ belong to $C^1(\Omega)$ and satisfy the ellipticity condition, i.e., there exist $\gamma, \mu > 0$ such that

$$\mu|\xi|^2 \geq a_{ij}(x)\xi_i\xi_j \geq \gamma|\xi|^2 \quad \text{for } x \in \Omega \text{ and } \xi \in \mathbb{R}^n, \tag{1}$$

where the summation convention is adopted. We also introduce the second order elliptic operator

$$L = -\partial_{x_i}\big(a_{ij}(x)\partial_{x_j}\big). \tag{2}$$

Remark 2.1. The operator L defined by (2) considered as the mapping $L : H_0^1(\Omega) \to H^{-1}(\Omega)$ defines (see [KS]) a bilinear, continuous and coercive form on $H_0^1(\Omega)$ as follows:

$$a(u,v) = \langle Lu, v\rangle = \int_\Omega a_{ij}(x) u_{x_i}(x) v_{x_j}(x)\,dx \quad (u,v\in H_0^1(\Omega)). \quad (3)$$

Now we pass to the precise definitions of fundamental concepts of this work. Let us consider $l\in\mathbb{N}$ functions $\Psi_i\in H^1(E_i)$ where $E_i\subset\Omega$ are compact sets such that ∂E_i is smooth, $E_i\cap E_j=\emptyset$ for $i,j=1,\dots,l$ and $i\neq j$. Next we take $m\in\mathbb{N}$ functions $\Phi_i\in H^1(F_i)$ where $F_i\subset\Omega$ are compact sets such that ∂F_i is smooth, $F_i\cap F_j=\emptyset$ for $i,j=1,\dots,m$ and $i\neq j$. Moreover, we assume that:

$$\Psi_i \le \Phi_j \quad \text{on } E_i\cap F_j \text{ for } i=1,\dots,l \text{ and } j=1,\dots,m. \quad (4)$$

We denote by K_l, K^m and K_l^m the following admissible sets:

$$\begin{aligned} K_l &= \{v\in H_0^1(\Omega) \,:\, v\ge\Psi_i \text{ on } E_i \text{ for } 1\le i\le l\}, & (5)\\ K^m &= \{v\in H_0^1(\Omega) \,:\, v\le\Phi_i \text{ on } F_i \text{ for } 1\le i\le m\}, & (6)\\ K_l^m &= \{v\in H_0^1(\Omega) \,:\, v\ge\Psi_i \text{ on } E_i \ \wedge v\le\Phi_j \text{ on } F_j, & (7)\\ &\qquad\qquad \text{for } 1\le i\le l, 1\le j\le m\}. \end{aligned}$$

Definition 2.2. Let l be a fixed natural number. For the form defined by (3) and $f\in H^{-1}(\Omega)$ the problem:
Find $u_l\in K_l$ such that

$$a(u_l, v-u_l)\ge\langle f, v-u_l\rangle \quad \text{for any } v\in K_l, \quad (8)$$

where K_l is defined by (5) is called an $l-$inner obstacle problem with the impediments Ψ_i $(i=1,\dots,l)$.

We shall use the notation $l-IP$ to denote the $l-$inner obstacle problem.

Definition 2.3. Let m be a fixed natural number. For the form defined by (3) and $f\in H^{-1}(\Omega)$ the problem:
Find $u^m\in K^m$ such that

$$a(u^m, v-u^m)\ge\langle f, v-u^m\rangle \quad \text{for any } v\in K^m, \quad (9)$$

where K^m is defined by (6) is called an $m-$inner inverse obstacle problem with the impediments Φ_i $(i=1,\dots,m)$.

We shall use the notation $m-IIP$ to denote the $m-$inner inverse obstacle problem.

Definition 2.4. Let l, m be fixed natural numbers. For the form defined by (3) and $f\in H^{-1}(\Omega)$ the problem:

Find $u_l^m \in K_l^m$ such that

$$a(u_l^m, v - u_l^m) \geq \langle f, v - u_l^m \rangle \quad \text{for any } v \in K_l^m, \tag{10}$$

where K_l^m is defined by (7) is called an $l, m-$double inner obstacle problem with the impediments Ψ_i $(i = 1, \dots, l)$ and Φ_j $(j = 1, \dots, m)$.

We shall use the notation $l, m - DIP$ to denote the $l, m-$double inner obstacle problem.

Remark 2.5. If we put $l = 1$, take $E_1 = \Omega$ and assume that $\Psi_1 = \Psi$ satisfies $\Psi\big|_{\partial\Omega} \leq 0$ then Definition 2.2 in fact is identical with the definition of the global obstacle problem with the impediment Ψ. The admissible set $\tilde{K}_1$ will be defined by

$$\tilde{K}_1 = \{v \in H_0^1(\Omega) \ : \ v \geq \Psi \text{ on } \Omega\}. \tag{11}$$

We shall use the notation GP to denote the global obstacle problem.

Remark 2.6. If we put $m = 1$, take $F_1 = \Omega$ and assume that $\Phi_1 = \Phi$ satisfies $\Phi\big|_{\partial\Omega} \geq 0$ then Definition 2.3 gives the definition of the inverse global obstacle problem with the impediment Φ.The admissible set $\tilde{K}^1$ will be defined by

$$\tilde{K}^1 = \{v \in H_0^1(\Omega) \ : \ v \leq \Phi_i \text{ on } \Omega\}. \tag{12}$$

We shall use the notation GIP to denote the inverse global obstacle problem.

Remark 2.7. If we put $l = m = 1$, take $E_1 = F_1 = \Omega$ and assume that $\Psi_1 = \Psi$ and $\Phi_1 = \Phi$ are such that $\Psi\big|_{\partial\Omega} \leq 0$, $\Phi\big|_{\partial\Omega} \geq 0$ and $\Psi \geq \Phi$ then Definition 2.4 gives the definition of the double global obstacle problem with the impediments Ψ and Φ. The admissible set $\tilde{K}_1^1$ will be defined by

$$\tilde{K}_1^1 = \{v \in H_0^1(\Omega) \ : \ \Psi \leq v \leq \Phi \text{ on } \Omega\}. \tag{13}$$

We shall use the notation DGP to denote the double global obstacle problem.

The existence and uniqueness theorems for GP, GIP, DGP can be found in [KS].

3. LIPSCHITZ REGULARITY

In this section we present the main results of our paper, i.e., the Lipschitz regularity of the solutions in the case of the inner obstacle problems. Since our approach is based on identification of the inner problem with the corresponding global one, we now recall the following lemma (see [SV], [Ch1], [Ch2]).

Lemma 3.1. *The solutions: $\tilde{u}_1$ — of GP with the impediment Ψ and $f = 0$, $\tilde{u}^1$ — of GIP with the impediment Φ and $f = 0$, $\tilde{u}_1^1$ — of DGP with the impediments Ψ, Φ and $f = 0$ are Lipschitz ($\tilde{u}_1$, $\tilde{u}^1$, $\tilde{u}_1^1 \in H^{1,\infty}(\Omega)$) provided Ψ, $\Phi \in H^{1,\infty}(\Omega)$.*

We start with presenting the result for $l - IP$ (for $l \geq 1$). The following theorem is a generalisation of the one included in [SV] where $l = 1$.

Theorem 3.2. *If $\Psi_i \in H^{1,\infty}(E_i)$ then there exists a unique solution u_l to $l - IP$ with the impediments Ψ_i ($i = 1, \ldots, l$). Moreover, if $f = 0$ this solution is Lipschitz continuous.*

Proof. Let us construct for each $i = 1, \ldots, l$ the functions $\tilde{\Psi}_i : \Omega \to \mathbb{R}$ in the following way

$$\tilde{\Psi}_i = \begin{cases} w_i & \text{in } \Omega \backslash E_i \\ \Psi_i & \text{in } E_i, \end{cases} \tag{14}$$

where $w_i \in H^1(\Omega \backslash E_i)$ solves the following problem

$$\begin{cases} L w_i = 0 & \text{in } \Omega \backslash E_i \\ w_i = \Psi_i & \text{in } \partial E_i \\ w_i = 0 & \text{in } \partial\Omega. \end{cases} \tag{15}$$

Next we put

$$\Psi = \max\{\tilde{\Psi}_1, \ldots, \tilde{\Psi}_l\}.$$

It is well known (see [KS]) that in order to show existence and uniqueness of the solution u_l of $l - IP$ it is enough to show non-emptiness of the set K_l. It is easy to see that $\Psi \in K_l$ which yields the desired conclusion.

On the other hand, we observe that $\tilde{\Psi}_i$ ($i = 1, \ldots, l$) and consequently Ψ are Lipschitz continuous. Then we notice that u_l satisfies $L u_l \geq 0$ in Ω. Indeed, for arbitrary $v \in H_0^1(\Omega)$ such that $v \geq 0$ in Ω we have $u_l + v \in K_l$. Thus we can write

$$a(u_l, u_l + v - u_l) \geq 0.$$

The above inequality gives that $a(u_l, v) \geq 0$ for arbitrary $v \in H_0^1(\Omega)$ such that $v \geq 0$ in Ω.

We have $\tilde{\tilde{\Psi}}_i = \Psi_i$ in E_i, $L\tilde{\Psi}_i = 0$ in $\Omega \backslash E_i$, $\tilde{\Psi}_i\big|_{\partial\Omega} = 0$ and $Lu_l \geq 0$ in $\Omega \backslash E_i$. Therefore using the maximum principle we derive that $u_l \geq \tilde{\Psi}_i$ in Ω.

Moreover, we have that $u_l \geq \Psi$ in Ω. It is true due to the fact that $u_l \geq \tilde{\Psi}_i$ in Ω for $i = 1, \ldots, l$. This together with $u_l \in H_0^1(\Omega)$ implies that $u_l \in \tilde{K}_1$ where $\tilde{K}_1$ is defined in (11).

Let us denote by $\tilde{u}_1$ the solution to GP with the impediment Ψ and f being equal to zero. Since $u_l \in \tilde{K}_1$ we can state that

$$a(\tilde{u}_1, u_l - \tilde{u}_1) \geq 0.$$

On the other hand, $\tilde{K}_1 \subset K_l$ since if $v \in \tilde{K}_1$ then $v \in H_0^1(\Omega)$ and $v \geq \Psi \geq \Psi_i$ on E_i for $i = 1, \ldots, l$. Hence we can write

$$a(u_l, \tilde{u}_1 - u_l) \geq 0,$$

because $u_l \in K_l$ solves the variational inequality (8) with $f = 0$. Having added the last two inequalities we shall obtain (using the coercivity of the form $a(\cdot, \cdot)$ in $H_0^1(\Omega)$) that there exists $\nu > 0$ such that

$$\nu \parallel \tilde{u}_1 - u_l \parallel^2 \leq a(\tilde{u}_1 - u_l, \tilde{u}_1 - u_l) \leq 0,$$

which implies that $\tilde{u}_1 = u_l$ in Ω. Lipschitz continuity of $\tilde{u}_1$ (see Lemma 3.1) completes the proof. □

The result similar to the one presented in Theorem 3.2 can be obtained for $m - IIP$ $(m \geq 1)$.

Theorem 3.3. *If $\Phi_i \in H^{1,\infty}(F_i)$ then there exists a unique solution u^m to $m - IIP$ with the impediments Φ_i $(i = 1, \ldots, m)$. Moreover, if $f = 0$ this solution is Lipschitz continuous.*

The proof of Theorem 3.3 is almost identical to the previous one, so we omit it.

In the third theorem we consider $1, 1 - DIP$. In this case we extend the result given in [SV] by adding one impediments from above.

Theorem 3.4. *If the functions $\Psi_1 \in H^{1,\infty}(E_1)$, $\Phi_1 \in H^{1,\infty}(F_1)$ satisfy*

$$\Psi_1 \leq \Phi_1 \quad \text{in } E_1 \cap F_1, \tag{16}$$

then there exists a unique solution u_1^1 *to* $1,1-DIP$ *with the impediments* Ψ_1, Φ_1. *Moreover, if* $f=0$ *this solution is Lipschitz continuous.*

Proof. Firstly, applying the ideas presented in the proof of Theorem 3.2 (see (14)) we construct the Lipschitz extensions of Ψ_1 and Φ_1 onto the whole domain Ω. In order to reduce the complexity of notations those extensions we still call Ψ_1 and Φ_1. Then we observe that $\max\{\Psi_1,0\}+\min\{\Phi_1,0\}\in K_1^1$ which gives us existence of the unique solution $u_1^1\in K_1^1$ to the $1,1-DIP$.

In the proof we shall construct two Lipschitz functions Ψ and Φ such that the solution $\tilde{u}_1^1$ of DGP with the impediments Ψ and Φ and f being equal to zero will coincide with u_1^1 – the solution of $1,1-DIP$.

Let us consider the coincidence set $I[u_1^1]$ for $1,1-DIP$. Obviously it is contained in $E_1\cup F_1$ (as (16) holds). We denote by $I_F[u_1^1]$ that part of $I[u_1^1]$ which is contained in $F_1\cap\overline{(\Omega\backslash E_1)}$ where $u_1^1=\Phi_1$ and by $I_E[u_1^1]$ that part of $I[u_1^1]$ which is contained in $E_1\cap\overline{(\Omega\backslash F_1)}$ where $u_1^1=\Psi_1$. Now we define

$$\tilde{\Psi}_1=\begin{cases}\Psi_1 & \text{in } E_1\\ \min\{\Psi_1,\Phi_1\} & \text{in } \Omega\backslash E_1,\end{cases} \tag{17}$$

$$\tilde{\Phi}_1=\begin{cases}\Phi_1 & \text{in } F_1\\ \max\{\Psi_1,\Phi_1\} & \text{in } \Omega\backslash F_1.\end{cases} \tag{18}$$

Both functions $\tilde{\Psi}_1$ and $\tilde{\Phi}_1$ are continuous. Moreover, they are both Lipschitz.

Now let us take a Lipschitz function $\xi\in H_0^{1,\infty}(\Omega)$ such that $\xi|_{\partial(\Omega\backslash E_1)}=0$ and $\xi<0$ in $\Omega\backslash E_1$. Next we consider the Lipschitz function $\delta\in H^{1,\infty}(\Omega)$ where we put $\delta=\tilde{\Psi}_1+\xi$. We know that

$$u_1^1=\Phi_1=\tilde{\Phi}_1 \quad \text{in } \mathrm{I_F[u_1^1]}. \tag{19}$$

It also satisfies

$$u_1^1>\delta=\tilde{\Psi}_1+\xi \quad \text{in } \mathrm{I_F[u_1^1]} \tag{20}$$

since $\tilde{\Psi}_1+\xi=\min\{\Psi_1,\Phi_1\}+\xi<\tilde{\Phi}_1$ in $I_F[u_1^1]$. From the continuity of u_1^1, ξ and $\tilde{\Psi}_1$ we state that there exists a neighbourhood O_F of $I_F[u_1^1]$ where the inequality (20) holds.

Now we choose a set D_F with the smooth boundary in the following way:

$$I_F[u_1^1]\subset D_F\subset\bar{D}_F\subset O_F\cap\overline{(\Omega\backslash E_1)}.$$

Let ψ be the solution of the problem:

$$\begin{cases} L\psi = 0 & \text{in } (\Omega\backslash E_1)\backslash \bar{D}_F \\ \psi = \delta & \text{in } \partial((\Omega\backslash E_1)\backslash \bar{D}_F). \end{cases} \tag{21}$$

The function ψ is Lipschitz (see [ADN], [BC]). Next we remark that the set D was chosen in such a way that $(\Omega\backslash E_1)\backslash \bar{D}_F \subset \Omega\backslash I[u_1^1]$. Therefore using again the basic properties of the solutions to the obstacle problems (see [KS]) we have that $Lu_1^1 = 0$ in $(\Omega\backslash E_1)\backslash \bar{D}_F$. Hence

$$L(u_1^1 - \psi) = 0 \quad \text{in } (\Omega\backslash \mathrm{E}_1)\backslash \bar{\mathrm{D}}_\mathrm{F}.$$

Moreover, we have

$$u_1^1 = \psi = \Psi_1 \quad \text{in } \partial\Omega,$$

which follows from the definition of extension of Ψ_1 and the constructions of $\tilde{\Psi}_1$ and ξ,

$$u_1^1 \geq \psi = \Psi_1 \quad \text{in } \partial E_1,$$

which follows from (16) and the constructions of $\tilde{\Psi}_1$ and ξ,

$$u_1^1 \geq \psi \quad \text{in } \partial D_F,$$

which follows from (20) and the construction of the set D_F. Then the maximum principle implies that

$$u_1^1 \geq \psi \quad \text{in } (\Omega\backslash \mathrm{E}_1)\backslash \bar{\mathrm{D}}_\mathrm{F}. \tag{22}$$

Finally we put

$$\Psi = \begin{cases} \Psi_1 & \text{in } E_1 \\ \delta & \text{in } \bar{D}_F \\ \psi & \text{in } (\Omega\backslash E_1)\backslash \bar{D}_F. \end{cases} \tag{23}$$

Clearly the function Ψ is Lipschitz continuous in Ω. Moreover,

$$u_1^1 \geq \Psi \quad \text{in } \Omega \tag{24}$$

since $u_1^1 \in K_1^1$, (22) holds and (20) is satisfied in $D \subset O_F \cap \overline{(\Omega\backslash E_1)}$.

Now we pass to the remaining part of the proof. We choose a Lipschitz function $\eta \in H_0^{1,\infty}(\Omega)$ such that $\eta > 0$ in $\Omega\backslash F_1$ and $\eta|_{\partial(\Omega\backslash F_1)} = 0$. Next we consider the Lipschitz function $\sigma \in H^{1,\infty}(\Omega)$ where we put $\sigma = \tilde{\Phi}_1 + \eta$. We know that $u_1^1 = \tilde{\Psi}_1 = \Psi_1$ in $I_E[u_1^1]$. It also satisfies the following:

$$u_1^1 < \sigma = \tilde{\Phi}_1 + \eta \quad \text{in } I_E[u_1^1] \tag{25}$$

as $\tilde{\Psi}_1 = \Psi < \tilde{\Phi}_1 + \eta$ in $I_E[u_1^1]$. From the continuity of u_1^1, η and $\tilde{\Phi}_1$ we state that there exists a neighbourhood O_E of $I_E[u_1^1]$ where the inequality (25) holds. Acting similarly as above we can choose a set D_E with smooth boundary such that:

$$I_E[u_1^1] \subset D_E \subset \bar{D}_E \subset O_E \cap (\Omega \backslash F_1).$$

Denoting by ϕ the solution of the problem

$$\begin{cases} L\phi = 0 & \text{in } (\Omega \backslash F_1) \backslash \bar{D}_E \\ \phi = \sigma & \text{in } \partial((\Omega \backslash F_1) \backslash \bar{D}_E), \end{cases} \tag{26}$$

we get that ϕ is Lipschitz provided σ is Lipschitz. Knowing that $(\Omega \backslash F_1) \backslash \bar{D}_E \subset \Omega \backslash I[u_1^1]$ and using the maximum principle we deduce that:

$$u_1^1 \leq \phi \quad \text{ in } (\Omega \backslash \mathrm{F}_1) \backslash \bar{\mathrm{D}}_\mathrm{E}. \tag{27}$$

Finally we put

$$\Phi = \begin{cases} \Phi_1 & \text{in } F_1 \\ \sigma & \text{in } \bar{D}_E \\ \phi & \text{in } (\Omega \backslash F_1) \backslash \bar{D}_E. \end{cases} \tag{28}$$

Clearly the function Φ is Lipschitz. Moreover,

$$u_1^1 \leq \Phi \quad \text{ in } \Omega \tag{29}$$

as $u_1^1 \in K_1^1$, (27) holds and (25) is satisfied in $D_E \subset O_E \cap \overline{\Omega \backslash F_1}$.

Conditions (24), (29) together with $u_1^1 \in H_0^1(\Omega)$ imply that $u_1^1 \in \tilde{K}_1^1$, where $\tilde{K}_1^1$ is defined in (12). Moreover, $\tilde{K}_1^1 \subset K_1^1$. Indeed, if $v \in \tilde{K}_1^1$ then $v \in H_0^1(\Omega)$, $v \geq \Psi = \Psi_1$ in E_1 and $v \leq \Phi = \Phi_1$ in F_1 which gives that $v \in K_1^1$.

Let us denote by $\tilde{u}_1^1$ the solution of the DGP with the impediments Ψ, Φ given by (23), (28), respectively and f being equal to zero. Using coercivity of the form $a(\cdot,\cdot)$ on $H_0^1(\Omega)$ we shall deduce that $u_1^1 = \tilde{u}_1^1$. This completes the proof (as $\tilde{u}_1^1$ is Lipschitz — see Lemma 3.1). □

The last theorem describes the Lipschitz continuity of the solutions in the most general case, i.e., $l, m - DIP$.

Theorem 3.5. *If the functions $\Psi_i \in H^{1,\infty}(E_i)$ (i=1,...,l), $\Phi_j \in H^{1,\infty}(F_j)$ (j=1,...,m) satisfy*

$$\Psi_i \leq \Phi_j \quad \text{ in } E_i \cap F_j, \tag{30}$$

then there exists a unique solution u_l^m to $l, m - DIP$ with the impediments Ψ_i, Φ_j. Moreover, if $f = 0$ this solution is Lipschitz continuous.

Proof. At the beginning similarly to what we did in the proof of Theorem 3.2 (see (14)) we construct the Lipschitz extensions of Ψ_i and Φ_j onto the whole domain Ω. Again in order to avoid too much complexity these extensions will be still called Ψ_i and Φ_j. Then we observe that $\max\{\Psi_1, \ldots, \Psi_l, 0\} + \min\{\Phi_1, \ldots, \Phi_m, 0\} \in K_l^m$ which gives us existence of the unique solution $u_l^m \in K_l^m$ to $l, m - DIP$.

It could be proved (see [ES]) that the following estimates for the solutions to the inner obstacle problems are satisfied

$$u_{1(i)}^m \le u_l^m \le u_l^{1(j)} \quad \text{in } \Omega, \tag{31}$$

where $u_{1(i)}^m$ denotes the solution of $1, m - DIP$ with an arbitrarily fixed impediment Ψ_i and m impediments Φ_j, $u_l^{1(j)}$ is the solution of $l, 1 - DIP$ with l impediments Ψ_i and an arbitrarily fixed Φ_j.

Firstly, we examine $1, m - DIP$ with the fixed impediment Ψ_i and m impediments Φ_j. Using the construction presented in the proof of Theorem 3.4 for each pair of the obstacles (Ψ_i, Φ_j) we construct m global obstacles $\tilde{\Phi}_j$ according to (28) and next we put $\Phi^i = \min_{1 \le j \le m} \tilde{\Phi}_j$. Then we consider $1, 1 - DIP$ with the admissible set

$$K_{1(i)}^1 = \{v \in H_0^1(\Omega) \ : \ v \ge \Psi_i \text{ on } E_i \wedge v \le \Phi^i \text{ on } \Omega\}. \tag{32}$$

Its solution exists and we denote it by $u_{1(i)}^1$.

It can be easily seen that $u_{1(i)}^m \in K_{1(i)}^1$ since (29) is satisfied for all $j = 1, \ldots, m$. On the other hand, $K_{1(i)}^1 \subset K_{1(i)}^m$. Indeed, if we take $v \in K_{1(i)}^1$ then $v \ge \Psi_i$ on E_i and $v \le \Phi^i = \min_{1 \le j \le m} \tilde{\Phi}_j$ and $\tilde{\Phi}_j = \Phi_j$ on F_j. Using the coerciveness of the form $a(\cdot, \cdot)$ on $H_0^1(\Omega)$ we get that $u_{1(i)}^m = u_{1(i)}^1$.

Having $1, 1 - DIP$ with the pair of obstacles (Ψ_i, Φ^i) we apply once again the construction described in the proof of the previous theorem and we create the function $\bar{\Psi}_i$ according to (23). Obviously

$$u_{1(i)}^m \ge \bar{\Psi}_i \quad \text{on } \Omega. \tag{33}$$

Then we consider $l, 1 - DIP$ with the fixed impediment Φ_j. For each pair of the obstacles (Ψ_i, Φ_j) we build l global obstacles $\tilde{\Psi}_i$ according to (23) and next we put $\Psi^j = \max_{1 \le i \le l} \tilde{\Psi}_i$. Then we consider

$1,1-DIP$ with the admissible set

$$K_1^{1(j)} = \{v \in H_0^1(\Omega) \ : \ v \le \Psi^j \text{ on } \Omega \wedge v \le \Phi_j \text{ on } F_j\}. \tag{34}$$

Its solution exists and we call it $u_1^{1(j)}$.

One can observe that $u_l^{1(j)} \in K_1^{1(j)}$ since (24) is satisfied for all $i = 1, \dots, l$. Moreover, $K_1^{1(j)} \subset K_l^{1(j)}$. Indeed, if we take $v \in K_1^{1(j)}$ then $v \le \Phi_j$ on F_j and $v \ge \Psi^j = \max_{1\le i\le l} \tilde{\Psi}_i$ and $\tilde{\Psi}_i = \Psi_i$ on E_i. Using the coerciveness of the form $a(\cdot,\cdot)$ on $H_0^1(\Omega)$ we get that $u_l^{1(j)} = u_1^{1(j)}$.

Concentrating once again on $1,1-DIP$ with the pair of obstacles (Ψ^j, Φ_j) we create using (28) the global impediment $\bar{\Phi}_j$ such that

$$u_l^{1(j)} \le \bar{\Phi}_j \quad \text{ on } \Omega. \tag{35}$$

At this moment we deal with two Lipschitz function $\bar{\Psi}_i$, $\bar{\Phi}_j$ satisfying (33), (35), respectively. Repeating for all $i = 1, \dots, l$ and for all $j = 1, \dots, m$ the constructions described above we can build l functions $\bar{\Psi}_i$ $(i = 1, \dots, l)$ and m functions $\bar{\Phi}_j$ $(j = 1, \dots, m)$. Then we define:

$$\Psi = \max_{i=1,\dots,l} \bar{\Psi}_i, \qquad \Phi = \min_{j=1,\dots,m} \bar{\Phi}_j.$$

From (31) we get the following estimates which hold for arbitrary i, j

$$\bar{\Psi}_i \le u_{1(i)}^m \le u_l^m \le u_l^{1(j)} \le \bar{\Phi}_j.$$

Thus

$$\Psi \le u_l^m \le \Phi. \tag{36}$$

This together with $u_l^m \in H_0^1(\Omega)$ implies that $u_l^m \in \tilde{K}_1^1$, where $\tilde{K}_1^1$ is defined in (13). Moreover, due to definition of Ψ and Φ, it is easy to see that $\tilde{K}_1^1 \subset K_l^m$.

Let us denote by $\tilde{u}_1^1$ the solution of DGP with the impediments Ψ and Φ. Using the coercivity of the form $a(\cdot,\cdot)$ in $H_0^1(\Omega)$ we deduce that u_l^m equals $\tilde{u}_1^1$ which is Lipschitz continuous (see Lemma 3.1). This completes the proof. □

Remark 3.6. It is well known that in case of solutions of the global problems one can expect their regularity up to $H^{2,p}$. For the inner problem the situation is much more complicated. Despite $H^{2,p}$ regularity of the obstacle the same class of the solution can not be obtained. However under certain assumptions it is possible to get $H^{2,p}$ regularity of the solutions (see [BS], [JOS]).

Remark 3.7. It is worth pointing out that it was possible to adopt the method of identification of the inner problem with the global one to obtain the results concerning continuous dependence on obstacle of solutions (see [OS], [JOS]).

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Cartan Subalgebras in C*-Algebras

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Abstract. According to J. Feldman and C. Moore's well-known theorem on Cartan subalgebras, a variant of the group measure space construction gives an equivalence of categories between twisted countable standard measured equivalence relations and Cartan pairs, i.e., a von Neumann algebra (on a separable Hilbert space) together with a Cartan subalgebra. A. Kumjian gave a C*-algebraic analogue of this theorem in the early eighties. After a short survey of maximal abelian self-adjoint subalgebras in operator algebras, I present a natural definition of a Cartan subalgebra in a C*-algebra and an extension of Kumjian's theorem which covers graph algebras and some foliation algebras.

1. Introduction

One of the most fundamental constructions in the theory of operator algebras, namely the crossed product construction, provides a subalgebra, i.e., a pair (B, A) consisting of an operator algebra A and a subalgebra $B \subset A$, where B is the original algebra. The inclusion $B \subset A$ encodes the symmetries of the original dynamical system. An obvious and naive question is to ask whether a given subalgebra arises from some crossed product construction. From the very construction of the crossed product, a necessary condition is that B is regular in A, which means that A is generated by the normalizer of B. In the case of a crossed product by a group, duality theory provides an answer (see Landstad [30]) which requires an external information, namely the dual action. Our question is more in line with subfactor theory, where one extracts an algebraic object (such as a paragroup or a quantum groupoid) solely from an inclusion

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of factors. Under the assumption that B is maximal abelian, the problem is somewhat more tractable. The most satisfactory result in this direction is the Feldman–Moore theorem [18, Theorem 1], which characterizes the subalgebras arising from the construction of the von Neumann algebra of a measured countable equivalence relation. These subalgebras are precisely the Cartan subalgebras, a nice kind of maximal abelian self-adjoint subalgebras (masas) introduced previously by Vershik in [47]: they are regular and there exists a faithful normal conditional expectation of A onto B. The Cartan subalgebra contains exactly the same information as the equivalence relation. This theorem leaves pending a number of interesting and difficult questions. For example, the existence or the uniqueness of Cartan subalgebras in a given von Neumann algebra. Another question is to determine if the equivalence relation arises from a free action of a countable group and if one can expect uniqueness of the group. There have been some recent breakthroughs on these questions: for example [36, 37, 38, 33]; in [33], Ozawa and Popa give the first examples of II_1 factors containing a unique Cartan subalgebra up to unitary conjugacy.

It was then natural to find a counterpart of the Feldman–Moore theorem for C^*-algebras. In [25], Kumjian introduced the notion of a C^*-diagonal as the C^*-algebraic counterpart of a Cartan subalgebra and showed that, via the groupoid algebra construction, they correspond exactly to twisted étale equivalence relations. A key ingredient of his theorem is his definition of the normalizer of a subalgebra (a definition in terms of unitaries or partial isometries would be too restrictive). His fundamental result, however, does not cover a number of important examples. For example, Cuntz algebras, and more generally graph algebras, have obvious regular masas which are not C^*-diagonals. The same is true for foliations algebras (or rather their reduction to a full transversal). The reason is that the groupoids from which they are constructed are topologically principal but not principal: they have some isotropy that cannot be eliminated. It seems that, in the topological context, topologically principal groupoids are more natural than principal groupoids (equivalence relations). They are exactly the groupoids of germs of pseudogroups. Groupoids of germs of pseudogroups present a technical difficulty: they may fail to be Hausdorff (they are Hausdorff if and only if the pseudogroup is quasi-analytical). For the sake of simplicity, our discussion will be limited to the Hausdorff case. We refer the interested reader to a forthcoming paper about the non-Hausdorff case. A natural definition of a Cartan subalgebra in the C*-algebraic context is that it is a masa which is regular and which admits a faithful conditional expectation. We show that in the reduced C*-algebra of a topologically principal Hausdorff étale groupoid (endowed with a twist), the subalgebra corresponding to the unit space is a Cartan subalgebra. Conversely, every Cartan subalgebra (if it exists!) arises in that fashion and completely determines the groupoid and the twist. Our proof closely follows Kumjian's. The comparison with Kumjian's theorem shows that a Cartan subalgebra has the unique extension property if and only if the corresponding groupoid is principal. As a corollary of the main result, we obtain that a Cartan subalgebra has a unique conditional expectation, which is clear when the subalgebra has the unique extension property but not so in the general case.

Here is a brief description of the content of this paper. In Section 2, I will review some basic facts about masas in von Neumann algebras, the Feldman–Moore theorem and some more recent results on Cartan subalgebras. In Section 3, I will review the characterization of topologically principal groupoids as groupoids of germs of pseudogroups of local homeomorphisms. In Section 4, I will review the construction of the reduced C*-algebra of a locally compact Hausdorff groupoid G with Haar system and endowed with a twist. I will show that, when G is étale, the subalgebra of the unit space is a masa if and only if G is topologically principal. In fact, this is what we call a Cartan subalgebra in the C*-algebraic context: it means a masa which is regular and which has a faithful conditional expectation. In Section 5, we show the converse: every Cartan subalgebra arises from an topologically principal étale groupoid endowed with a twist. This groupoid together with its twist is a complete isomorphism invariant of the Cartan subalgebra. We end with examples of Cartan subalgebras in C*-algebras.

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2. Cartan Subalgebras in von Neumann Algebras

The basic example of a masa in an operator algebra is the subalgebra D_n of diagonal matrices in the algebra M_n of complex-valued (n, n)-matrices. Every masa in M_n is conjugated to it by a unitary (this is essentially the well-known result that every normal complex matrix admits an orthonormal basis of eigenvectors). The problem at hand is to find suitable generalizations of this basic example.

The most immediate generalization is to replace $\mathbf{C}^n$ by an infinite dimensional separable Hilbert space H and M_n by the von Neumann algebra $\mathcal{B}(H)$ of all bounded linear operators on H. The spectral theorem tells us that, up to conjugation by a unitary, masas in $\mathcal{B}(H)$ are of the form $L^\infty(X)$, acting by multiplication on $H = L^2(X)$, where X is an infinite standard measure space. Usually, one distinguishes the case of $X = [0, 1]$ endowed with Lebesgue measure and the case of $X = \mathbf{N}$ endowed with counting measure. In the first case, the masa is called *diffuse* and in the second case, it is called *atomic*. Atomic masas A in $\mathcal{B}(H)$ can be characterized by the existence of a normal conditional expectation $P : \mathcal{B}(H) \to A$. Indeed, when $H = \ell^2(\mathbf{N})$, operators are given by matrices and P is the restriction to the diagonal. What we are looking for is precisely a generalization of these atomic masas.

There is no complete classification of masas in non-type I factors. In fact, the study of masas in non-type I factors looks like a rather formidable task. In 1954, J. Dixmier [13] discovered the existence of non-regular masas. A masa A in a von Neumann algebra M is called *regular* if its normalizer $N(A)$ (the group of unitaries u in M which normalize A, in the sense that $uAu^* = A$) generates M as a von Neumann algebra. On the other hand, it is called *singular* if $N(A)$ is contained in A. When $N(A)$ acts ergodically on A, the masa A is called *semi-regular*. Every masa in $\mathcal{B}(H)$ (or in a type I von Neumann algebra) is regular. Dixmier gave an example of a singular masa in the hyperfinite II_1 factor. Later, Popa has shown in [35] that singular masas do exist in every separable II_1 factor (as we shall see, this is in sharp contrast with regular masas). Moreover, every von Neumann subalgebra of a separable II_1 factor is the image of a normal conditional expectation. Thus, in order to generalize the atomic masas of $\mathcal{B}(H)$, it is natural to consider masas which are both regular and the image of a normal conditional expectation:

Definition 2.1. (Vershik [47], Feldman–Moore [18, Definition 3.1]) An abelian subalgebra A of a von Neumann algebra M is called a *Cartan subalgebra* if

(i) A is a masa;
(ii) A is regular;
(iii) there exists a faithful normal conditional expectation of M onto A.

Cartan subalgebras are intimately related to ergodic theory. Indeed, if M arises by the classical group measure construction from a free action of a discrete countable group Γ on a measure space (X, μ), then $L^\infty(X, \mu)$ is naturally imbedded in M as a Cartan subalgebra ([32]). Following generalizations by G. Zeller-Meier [50, Remarque 8.11], W. Krieger [21] and P. Hahn [20], J. Feldman and C. Moore give in [18] the most direct construction of Cartan subalgebras. It relies on the notion of a countable standard measured equivalence relation. Here is its definition: $(X, \mathcal{B}, \mu)$ is a standard measured space and R is an equivalence relation on X such that its classes are countable, its graph R is a Borel subset of $X \times X$ and the measure μ is quasi-invariant under R. The last condition means that the measures $r^*\mu$ and $s^*\mu$ on R are equivalent (where r, s denote respectively the first and the second projections of R onto X and $r^*\mu(f) = \int \sum_y f(x,y) d\mu(x)$ for a positive Borel function f on R). The orbit equivalence relation of an action of a discrete countable group Γ on a measure space (X, μ) preserving the measure class of μ is an example (in fact, according to [17, Theorem 1], it is the most general example) of a countable standard equivalence relation. The construction of the von Neumann algebra $M = W^*(R)$ mimicks the construction of the algebra of matrices M_n. Its elements are complex Borel functions on R, the product is matrix multiplication and involution is the usual matrix conjugation. Of course, in order to have an involutive algebra of bounded operators, some conditions are required on these functions: they act by left multiplication as operators on $L^2(R, s^*\mu)$ and we ask these operators to be bounded. The subalgebra A of diagonal matrices (functions supported on the diagonal of R), which is isomorphic to $L^\infty(X, \mu)$, is a Cartan subalgebra of M. When $X = \mathbf{N}$ and μ is the counting measure, one retrieves the atomic masa of $\mathcal{B}(\ell^2(\mathbf{N}))$. This construction can be twisted by a 2-cocycle $\sigma \in Z^2(R, \mathbf{T})$; explicitly, σ is a Borel function on $R^{(2)} = \{(x, y, z) \in X \times X \times X : (x, y), (y, z) \in R\}$

with values in the group of complex numbers of modulus 1 such that $\sigma(x,y,z)\sigma(x,z,t) = \sigma(x,y,t)\sigma(y,z,t)$. The only modification is to define as product the twisted matrix multiplication $f * g(x,z) = \sum f(x,y)g(y,z)\sigma(x,y,z)$. This yields the von Neumann algebra $M = W^*(R,\sigma)$ and its Cartan subalgebra $A = L^\infty(X,\mu)$ of diagonal matrices. The Feldman–Moore theorem gives the converse.

Theorem 2.2. [18, Theorem 1] *Let A be a Cartan subalgebra of a von Neumann algebra M on a separable Hilbert space. Then there exists a countable standard measured equivalence relation R on (X,μ), a $\sigma \in Z^2(R,\mathbf{T})$ and an isomorphism of M onto $W^*(R,\sigma)$ carrying A onto the diagonal subalgebra $L^\infty(X,\mu)$. The twisted relation (R,σ) is unique up to isomorphism.*

The main lines of the proof will be found in the C*-algebraic version of this result. This theorem completely elucidates the structure of Cartan subalgebras. It says nothing about the existence and the uniqueness of Cartan subalgebras in a given von Neumann algebra. We have seen that in $\mathcal{B}(H)$ itself, there exists a Cartan subalgebra, which is unique up to conjugacy. The same result holds in every injective von Neumann algebra. More precisely, two Cartan subalgebras of an injective von Neumann algebra are always conjugate by an automorphism (but not always inner conjugate, as observed in [18]). This important uniqueness result appears as [7, Corollary 11]. W. Krieger had previously shown in [22, Theorem 8.4] that two Cartan subalgebras of a von Neumann algebra M which produce hyperfinite ([17, Definition 4.1]) equivalence relations are conjugate (then, M is necessarily hyperfinite). On the other hand, it is not difficult to show that a Cartan subalgebra of an injective von Neumann algebra produces an amenable ([7, Definition 6]) equivalence relation. Since Connes–Feldman–Weiss's theorem states that an equivalence relation is amenable if and only if it is hyperfinite, Krieger's uniqueness theorem can be applied. The general situation is more complex. Here are some results related to Cartan subalgebras of type II_1 factors. In [8], A. Connes and V. Jones give an example of a II_1 factor with at least two non-conjugate Cartan subalgebras. Then S. Popa constructs in [36] a II_1 factor with uncountably many non-conjugate Cartan subalgebras. These examples use Kazhdan's property T. In [48], D. Voiculescu shows that for $n \geq 2$, the von Neumann algebra $L(\mathbf{F}_n)$ of the free group $\mathbf{F}_n$ on n generators has no Cartan subalgebra. Despite these rather negative results, it seems that the notion of Cartan subalgebra still has a rôle to play in the theory of II_1 factors. For example, S. Popa has recently (see [37, 38]) constructed and studied a large class of type II_1 factors (from Bernoulli actions of groups with property T) which have a distinguished Cartan subalgebra, unique up to inner conjugacy; moreover, these factors satisfy remarkable rigidity properties: isomorphisms of these factors essentially arise from conjugacy of the actions. Still more recently N. Ozawa and S. Popa give in [33] on one hand many examples of II_1 factors which do not have any Cartan subalgebra and on the other hand a new class of II_1 factors which have a unique Cartan subalgebra, in fact unique not only up to conjugacy but to inner conjugacy. This class consists of all the profinite ergodic probability preserving actions of free groups $\mathbf{F}_n$ with $n \geq 2$ and their products.

3. Topologically Principal Groupoids.

The purpose of this section is mainly notational. It recalls elementary facts about étale groupoids and pseudogroups of homeomorphisms. Concerning groupoids, we shall use the notation of [1]. Other relevant references are [40] and [34]. Given a groupoid G, $G^{(0)}$ will denote its unit space and $G^{(2)}$ the set of composable pairs. Usually, elements of G will be denoted by Greek letters as γ and elements of $G^{(0)}$ by Roman letters as x, y. The range and source maps from G to $G^{(0)}$ will be denoted respectively by r and s. The fibers of the range and source maps are denoted respectively $G^x = r^{-1}(x)$ and $G_y = s^{-1}(y)$. The inverse map $G \to G$ is written $\gamma \mapsto \gamma^{-1}$, the inclusion map $G^{(0)} \to G$ is written $x \mapsto x$ and the product map $G^{(2)} \to G$ is written $(\gamma, \gamma') \mapsto \gamma\gamma'$. The *isotropy bundle* is $G' = \{\gamma \in G : r(\gamma) = s(\gamma)\}$. It is the disjoint union of the isotropy subgroups $G(x) = G^x \cap G_x$ when x runs over $G^{(0)}$.

In the topological setting, we assume that the groupoid G is a topological space and that the structure maps are continuous, where $G^{(2)}$ has the topology induced by $G \times G$ and $G^{(0)}$ has the topology induced by G. We assume furthermore that the range and source maps are open. A topological groupoid G is called *étale* when its range and source maps are local homeomorphisms from G onto $G^{(0)}$.

We shall be exclusively concerned here with groupoids of germs. They are intimately connected with pseudogroups. Here are the definitions. Let X be a topological space. A homeomorphism $\varphi : U \to V$, where U, V are open subsets of X, is called a partial homeomorphism.

Under composition and inverse, the partial homeomorphisms of X form an inverse semigroup. A *pseudogroup* on X is a family $\mathcal{G}$ of partial homeomorphisms of X stable under composition and inverse. We say that the pseudogroup $\mathcal{G}$ is *ample* if every partial homeomorphism φ which locally belongs to $\mathcal{G}$ (i.e., every point in the domain of φ has an open neighborhood U such that $\varphi_{|U} = \beta_{|U}$ with $\beta \in \mathcal{G}$) does belong to $\mathcal{G}$. Given a pseudogroup $\mathcal{G}$, we denote by $[\mathcal{G}]$ the set of partial homeomorphisms which belong locally to $\mathcal{G}$; it is an ample pseudogroup called the ample pseudogroup of $\mathcal{G}$. Given a pseudogroup $\mathcal{G}$ on the topological space X, its *groupoid of germs* is

$$G = \{[x, \varphi, y], \quad \varphi \in \mathcal{G}, y \in dom(\varphi), x = \varphi(y)\}.$$

where $[x, \varphi, y] = [x, \psi, y]$ if and only if φ and ψ have the same germ at y, i.e., there exists a neighborhood V of y in X such that $\varphi_{|V} = \psi_{|V}$. Its groupoid structure is defined by the range and source maps $r[x, \varphi, y] = x, s[x, \varphi, y] = y$, the product $[x, \varphi, y][y, \psi, z] = [x, \varphi\psi, z]$ and the inverse $[x, \varphi, y]^{-1} = [y, \varphi^{-1}, x]$. Its topology is the topology of germs, defined by the basic open sets

$$\mathcal{U}(U, \varphi, V) = \{[x, \varphi, y] \in G : x \in U, y \in V\}$$

where U, V are open subsets of X and $\varphi \in \mathcal{G}$. Observe that the groupoid of germs G of the pseudogroup $\mathcal{G}$ on X depends on the ample pseudogroup $[\mathcal{G}]$ only.

Conversely, an étale groupoid G defines a pseudogroup $\mathcal{G}$ on $X = G^{(0)}$ as follows. Recall that a subset A of a groupoid G is called an *r-section* [resp. an *s-section*] if the restriction of r [resp. s] to A is injective. A *bisection* is a subset $S \subset G$ which is both an r-section and an s-section. If G is an étale topological groupoid, it has a cover of open bisections. The open bisections of an étale groupoid G form an *inverse semigroup* $\mathcal{S} = \mathcal{S}(G)$: the composition law is

$$ST = \{\gamma\gamma' : (\gamma, \gamma') \in (S \times T) \cap G^{(2)}\}$$

and the inverse of S is the image of S by the inverse map. The inverse semigroup relations, which are $(RS)T = R(ST)$, $(ST)^{-1} = T^{-1}S^{-1}$ and $SS^{-1}S = S$, are indeed satisfied. A bisection S defines a map $\alpha_S : s(S) \to r(S)$ such that $\alpha_S(x) = r(Sx)$ for $x \in s(S)$. If moreover G is étale and S is an open bisection, this map is a homeomorphism. The map $\alpha : S \mapsto \alpha_S$ is an inverse semigroup homomorphism of the inverse semigroup of open bisections $\mathcal{S}$ into the inverse semigroup of partial homeomorphisms of X. We call it the *canonical action* of $\mathcal{S}$ on X. The relevant pseudogroup is its range $\mathcal{G} = \alpha(\mathcal{S})$.

Proposition 3.1. *Let $\mathcal{G}$ be a pseudogroup on X, let G be its groupoid of germs and let $\mathcal{S}$ be the inverse semigroup of open bisections of G. Then*

(i) *The pseudogroup $\alpha(\mathcal{S})$ is the ample pseudogroup $[\mathcal{G}]$ of $\mathcal{G}$.*
(ii) *The canonical action α is an isomorphism from $\mathcal{S}$ onto $[\mathcal{G}]$.*

Proof. We have observed above that $\mathcal{G}$ and $[\mathcal{G}]$ define the same groupoid of germs G. Thus, every $\varphi \in [\mathcal{G}]$ defines the open bisection $S = S_\varphi = \mathcal{U}(X, \varphi, X)$. By construction, $\alpha_S = \varphi$. Conversely, let S be an open bisection of G. It can be written as a union $S = \cup_i \mathcal{U}(V_i, \varphi_i, U_i)$, where U_i, V_i are open subsets of X and $\varphi_i \in \mathcal{G}$. This shows that $\varphi = \alpha_S$ belongs to $[\mathcal{G}]$ and that $S_\varphi = S$. In other words, the maps $S \to \alpha_S$ and $\varphi \to S_\varphi$ are inverse of each other. □

Proposition 3.2. *Let G be an étale groupoid over X and let α be the canonical action of the inverse semigroup of its open bisections $\mathcal{S}$ on X. Let H be the groupoid of germs of the pseudogroup $\alpha(\mathcal{S})$. Then we have a short exact sequence of étale groupoids*

$$\mathrm{int}(G') \rightarrowtail G \twoheadrightarrow H$$

where $\mathrm{int}(G')$ *is the interior of the isotropy bundle.*

Proof. We define $\alpha_* : G \to H$ by sending $\gamma \in G$ into $[r(\gamma), \alpha_S, s(\gamma)]$, where S is an open bisection containing γ. This does not depend on the choice of S, because α_S, α_T and $\alpha_{S\cap T}$, where T is another open bisection containing γ, have the same germ at $s(\gamma)$. It is readily verified that α_* is a continuous and surjective homomorphism. Moreover, $\alpha_*(\gamma)$ is a unit in H if and only if the germ of S at $s(\gamma)$ is the identity. This happens if and only if γ belongs to the interior of G' because α_S is an identity map if and only if S is contained in G'. □

Corollary 3.3. *Let G be an étale groupoid over X and let $\mathcal{S}$ be the inverse semigroup of its open bisections. Let α be the canonical action of $\mathcal{S}$ on X. The following properties are equivalent:*

(i) *The map α is one-to-one.*
(ii) *The interior of G' is reduced to $G^{(0)}$.*

Proof. Assume that the map α is one-to-one. Then, the above map $\alpha_* : G \to H$ is one-to-one. Hence its kernel $\mathrm{int}(G')$ is reduced to $G^{(0)}$.

Conversely, if $\mathrm{int}(G') = G^{(0)}$, then G is isomorphic to H. Hence it is a groupoid of germs. Therefore, according to Proposition 3.1(ii), α is one-to-one. □

Definition 3.4. An étale groupoid which satisfies above equivalent conditions is called *effective*.

The reader will find a good discussion of this notion in the monograph [31, Section 5.5] by I. Moerdijk and J. Mrčun .

Definition 3.5. Let us a say that an étale groupoid G is

(i) *principal* if $G' = G^{(0)}$
(ii) *topologically principal* if the set of points of $G^{(0)}$ with trivial isotropy is dense.

The property (*ii*) appears under the name *essentially principal* in some previous articles (e.g., [3]). The present terminology agrees with the notion of a topologically free action introduced in [45, Definition 2.1].

The following proposition links effective groupoids and topologically principal groupoids.

Proposition 3.6. *Let G be an étale groupoid.*

(i) *If G is Hausdorff and topologically principal, then it is effective;*
(ii) *If G is a second countable effective groupoid and its unit space $G^{(0)}$ has the Baire property, then it is topologically principal.*

Proof. Let us introduce the set Y of units with trivial isotropy and its complement $Z = G^{(0)} \setminus Y$. Let us suppose that G is topologically principal. Then Z has an empty interior. Let U be an open subset of G contained in G'. Since G is Hausdorff, $G^{(0)}$ is closed in G and $U \setminus G^{(0)}$ is open. Therefore $r(U \setminus G^{(0)})$, which is open and contained in Z, is empty. This implies that $U \setminus G^{(0)}$ itself is empty and that U is contained in $G^{(0)}$.

Let us assume that G is second countable, that its unit space $G^{(0)}$ has the Baire property and that it is effective. We choose a countable family (S_n) of open bisections which covers G. We introduce the subsets $A_n = r(S_n \cap G')$ of $G^{(0)}$. By definition, for each n, $Y_n = int(A_n) \cup ext(A_n)$ is a dense open subset of $G^{(0)}$. By the Baire property, the intersection $\cap_n Y_n$ is dense in $G^{(0)}$. Let us show that

$\cap_n Y_n$ is contained in Y. Suppose that x belongs to $\cap_n Y_n$ and that γ belongs to $G(x)$. There exists n such that γ belongs to S_n. Then γ belongs to $S_n \cap G'$ and $x = r(\gamma)$ belongs to A_n. Since it also belongs to Y_n, it must belong to $int(A_n)$. Let V be an open set containing x and contained in A_n. Since r is a bijection from $S_n \cap G'$ onto A_n, the open set VS_n is contained in G'. According to condition (ii) of Corollary 3.3, it is contained in $G^{(0)}$ and $\gamma = xS_n$ belongs to $G^{(0)}$. Therefore x belongs to Y. □

There are easy examples of groupoids of germs which are not topologically principal. For example, the groupoid of germs of the pseudogroup of all partial homeomorphisms of **R**, which is transitive, is not topologically principal.

4. The Analysis of the Twisted Groupoid C*-Algebra.

Following [25], one defines a twisted groupoid as a central groupoid extension

$$\mathbf{T} \times G^{(0)} \rightarrowtail \Sigma \twoheadrightarrow G$$

where **T** is the circle group. Thus, Σ is a groupoid containing $\mathbf{T}\times G^{(0)}$ as a subgroupoid. One says that Σ is a twist over G. We assume that Σ and G are topological groupoids. In particular, Σ is a principal **T**-space and $\Sigma/\mathbf{T} = G$. We form the associated complex line bundle $L = (\mathbf{C}\times\Sigma)/\mathbf{T}$ over G, where **T** acts by the diagonal action $z(\lambda, \sigma) = (\lambda\overline{z}, z\sigma)$. The class of (λ, σ) is written $[\lambda, \sigma]$. We write $\dot{\sigma} \in G$ the image of $\sigma \in \Sigma$. The line bundle L is a Fell bundle over the groupoid G, as defined in [27] (see also [16]): it has the product $L_{\dot{\sigma}}\otimes L_{\dot{\tau}} \to L_{\dot{\sigma}\dot{\tau}}$, sending $([\lambda, \sigma], [\mu, \tau])$ into $[\lambda\mu, \sigma\tau]$ and the involution $L_{\dot{\sigma}} \to L_{\dot{\sigma}^{-1}}$ sending $[\lambda, \sigma]$ into $[\overline{\lambda}, \sigma^{-1}]$. An element u of a Fell bundle L is called unitary if u^*u and uu^* are unit elements. The set of unitary elements of L can be identified to Σ through the map $\sigma \in \Sigma \mapsto [1, \sigma] \in L$. In fact, this gives a one-to-one correspondence between twists over G and Fell line bundles over G (see [12]). It is convenient to view the sections of L as complex-valued functions $f : \Sigma \to \mathbf{C}$ satisfying $f(z\sigma) = f(\sigma)\overline{z}$ for all $z \in \mathbf{T}, \sigma \in \Sigma$ and we shall usually do so. When there is no risk of confusion, we shall use the same symbol for the function f and the section of L it defines.

In order to define the twisted convolution algebra, we assume from now on that G is locally compact, Hausdorff, second countable and that it possesses a Haar system λ. It is a family of measures $\{\lambda_x\}$ on G, indexed by $x \in G^{(0)}$, such that λ_x has exactly G^x as its support, which is continuous, in the sense that for every $f \in C_c(G)$, the function $\lambda(f) : x \mapsto \lambda_x(f)$ is continuous, and invariant, in the sense that for every $\gamma \in G$, $R(\gamma)\lambda_{r(\gamma)} = \lambda_{s(\gamma)}$, where $R(\gamma)\gamma' = \gamma'\gamma$. When G is an étale groupoid, it has a canonical Haar system, namely the counting measures on the fibers of s.

Let (G, λ) be a Hausdorff locally compact second countable groupoid with Haar system and let Σ be a twist over G. We denote by $C_c(G, \Sigma)$ the space of continuous sections with compact support of the line bundle associated with Σ. The following operations

$$f * g(\sigma) = \int f(\sigma\tau^{-1})g(\tau)d\lambda_{s(\sigma)}(\dot{\tau}) \qquad (1)$$

$$f^*(\sigma) = \overline{f(\sigma^{-1})} \qquad (2)$$

turn $C_c(G, \Sigma)$ into a $*$-algebra. Furthermore, we define for $x \in G^{(0)}$ the Hilbert space $H_x = L^2(G_x, L_x, \lambda_x)$ of square-integrable sections of the line bundle $L_x = L_{|G_x}$. Then, for $f \in C_c(G, \Sigma)$, the operator $\pi_x(f)$ on H_x defined by

$$\pi_x(f)\xi(\sigma) = \int f(\sigma\tau^{-1})\xi(\tau)d\lambda_x(\dot{\tau})$$

is bounded. This can be deduced from the useful estimate:

$$\|\pi_x(f)\| \le \|f\|_I = \max(\sup_y \int |f|d\lambda_y, \sup_y \int |f^*|d\lambda_y).$$

Moreover, the field $x \mapsto \pi_x(f)$ is continuous when the family of Hilbert spaces H_x is given the structure of a continuous field of Hilbert spaces by choosing $C_c(G, \Sigma)$ as a fundamental family of continuous sections. Equivalently, the space of sections $C_0(G^{(0)}, H)$ is a right C*-module over $C_0(G^{(0)})$ and π is a representation of $C_c(G, \Sigma)$ on this C*-module.

The reduced C*-algebra $C^*_{red}(G, \Sigma)$ is the completion of $C_c(G, \Sigma)$ with respect to the norm $\|f\| = \sup_x \|\pi_x(f)\|$.

Let us now study the properties of the pair $(A = C^*_{red}(G, \Sigma), B = C_0(G^{(0)}))$ that we have constructed from a twisted étale Hausdorff locally compact second countable groupoid (G, Σ).

The main technical tool is that the elements of the reduced C*-algebra $C^*_{red}(G, \Sigma)$ are still functions on Σ (or sections of the line bundle L).

Proposition 4.1. [40, II.4.1] *Let G be an étale Hausdorff locally compact second countable groupoid and let Σ be a twist over G. Then, for all $f \in C_c(G, \Sigma)$ we have:*

(i) $|f(\sigma)| \le \|f\|$ *for every* $\sigma \in \Sigma$ *and*
(ii) $\int |f|^2 d\lambda_x \le \|f\|^2$ *for every* $x \in G^{(0)}$.

Proof. This is easily deduced (see [40, II.4.1]) from the following equalities:

$$f(\sigma) = < \epsilon_\sigma, \pi_{s(\sigma)}(f)\epsilon_{s(\sigma)} >, \quad f_{|\Sigma_x} = \pi_x(f)\epsilon_x,$$

where $f \in C_c(G, \Sigma)$, $\sigma \in \Sigma$, $x \in G^{(0)}$ and $\epsilon_\sigma \in H_{s(\sigma)}$ is defined by $\epsilon_\sigma(\tau) = \overline{z}$ if $\tau = z\sigma$ and 0 otherwise. □

As a consequence ([40, II.4.2]) the elements of $C^*_{red}(G, \Sigma)$ can be viewed as continuous sections of the line bundle L. Moreover, the equations (1) and (2) defining $f * g$ and f^* are still valid for $f, g \in C^*_{red}(G, \Sigma)$ (the sum defining $f * g(\sigma)$ is convergent). It will be convenient to define the open support of a continuous section f of the line bundle L as

$$supp'(f) = \{\gamma \in G : f(\gamma) \neq 0\}.$$

Note that the unit space $G^{(0)}$ of G is an open (and closed) subset of G and that the restrictions of the twist Σ and of the line bundle L to $G^{(0)}$ are trivial. We have the following identification:

$$C_0(G^{(0)}) = \{f \in C^*_{red}(G, \Sigma) : supp'(f) \subset G^{(0)}\}$$

where $h \in C_0(G^{(0)})$ defines the section f defined by $f(\sigma) = h(x)\overline{z}$ if $\sigma = (x, z)$ belongs to $G^{(0)} \times \mathbf{T}$ and $f(\sigma) = 0$ otherwise. Then $B = C_0(G^{(0)})$ is an abelian sub-C*-algebra of $A = C^*_{red}(G, \Sigma)$ which contains an approximate unit of A.

Here is an important application of the fact that the elements of $C^*_{red}(G, \Sigma)$ can be viewed as continuous sections.

Theorem 4.2. [40, II.4.7] *Let (G, Σ) be a twisted étale Hausdorff locally compact second countable groupoid. Let $A = C^*_{red}(G, \Sigma)$ and $B = C_0(G^{(0)})$. Then*

(i) *an element $a \in A$ commutes with every element of B if and only if its open support $supp'(a)$ is contained in G';*
(ii) *B is a masa if and only if G is topologically principal.*

Proof. Since the elements of $C^*_{red}(G,\Sigma)$ are continuous sections of the associated line bundle L, it is straightforward to spell out the condition $ab = ba$ for all $b \in B$. It implies the given condition on the support of a. We refer to [40, II.4.7] for details. One deduces from (i) that B is a masa if and only if the interior of G' is $G^{(0)}$. According to Proposition 3.6, this is equivalent under our hypotheses to G being topologically principal. □

Another piece of structure of the pair $(A = C^*_{red}(G,\Sigma), B = C_0(G^{(0)}))$ is the restriction map $P : f \mapsto f_{|G^{(0)}}$ from A to B.

Proposition 4.3. [40, II.4.8] *Let (G,Σ) be a twisted étale Hausdorff locally compact second countable groupoid. Let $P : C^*_{red}(G,\Sigma) \to C_0(G^{(0)})$ be the restriction map. Then*

(i) *P is a conditional expectation onto $C_0(G^{(0)})$.*
(ii) *P is faithful.*
(iii) *If G is topologically principal, P is the unique conditional expectation onto $C_0(G^{(0)})$.*

Proof. This is proved in [40, II.4.8] in the principal case. The main point of (i) is that P is well defined, which is clear from the above. There is no difficulty checking that it has all the properties of an expectation map. Note that for $h \in C_0(G^{(0)})$ and $f \in C^*_{red}(G,\Sigma)$, we have $(hf)(\sigma) = h(r(\sigma))f(\sigma)$ and $(fh)(\sigma) = f(\sigma)h(s(\sigma))$. The assertion (ii) is also clear: for $f \in C^*_{red}(G,\Sigma)$ and $x \in G^{(0)}$, we have

$$P(f^* * f)(x) = \int |f(\tau)|^2 d\lambda_x(\dot\tau).$$

Hence, if $P(f^* * f) = 0$, $f(\tau) = 0$ for all $\tau \in \Sigma$. Let us prove (iii). Let $Q : C^*_{red}(G,\Sigma) \to C_0(G^{(0)})$ be a conditional expectation. We shall show that Q and P agree on $C_c(G,\Sigma)$, which suffices to prove the assertion. Let $f \in C_c(G,\Sigma)$ with compact support K in G. We first consider the case when K is contained in an open bisection S which does not meet $G^{(0)}$ and show that $Q(f) = 0$. If $x \in G^{(0)}$ does not belong to $s(K)$, then $Q(f)(x) = 0$. Indeed, we choose $h \in C_c(G^{(0)})$ such that $h(x) = 1$ and its support does not meet $s(K)$. Then $fh = 0$, therefore $Q(f)(x) = Q(f)(x)h(x) = (Q(f)h)(x) = Q(fh)(x) = 0$. Let $x_0 \in G^{(0)}$ be such that $Q(f)(x_0) \neq 0$. Then $Q(f)(x) \neq 0$ on an open neighborhood U of x_0. Necessarily, U contained in $s(S)$. Since G is topologically principal and S does not meet $G^{(0)}$, the induced homeomorphism $\alpha_S : s(S) \to r(S)$ is not the identity map on U.

Therefore, there exists $x_1 \in U$ such that $x_2 = \alpha_S(x_1) \neq x_1$. We choose $h \in C_c(G^{(0)})$ such that $h(x_1) = 1$ and $h(x_2) = 0$. We have $hf = f(h \circ \alpha_S)$. Therefore,

$$\begin{aligned}Q(f)(x_1) &= h(x_1)Q(f)(x_1) = Q(hf)(x_1)\\ &= Q(f(h \circ \alpha_S))(x_1) = Q(f)(x_1)h(x_2) = 0.\end{aligned}$$

This is a contradiction. Therefore $Q(f) = 0$. Next, let us consider an arbitrary $f \in C_c(G, \Sigma)$ with compact support K in G. We use the fact that $G^{(0)}$ is both open and closed in G. The compact set $K \setminus G^{(0)}$ can be covered by finitely many open bisections $S_1, \dots, S_n$ of G. Replacing if necessary S_i by $S_i \setminus G^{(0)}$, we may assume that $S_i \cap G^{(0)} = \emptyset$. We set $S_0 = G^{(0)}$. We introduce a partition of unity $(h_0, h_1, \dots, h_n)$ subordinate to the open cover $(S_0, S_1, \dots, S_n)$ of K: for all $i = 0, \dots, n$, $h_i : G \to [0, 1]$ is continuous, it has a compact support contained in S_i and $\sum_{i=0}^n h_i(\gamma) = 1$ for all $\gamma \in K$. We define $f_i \in C_c(G, \Sigma)$ by $f_i(\sigma) = h_i(\dot\sigma)f(\sigma)$. Then, we have $f = \sum_{i=0}^n f_i$, $f_0 = P(f)$ and f_i has its support contained in S_i for all i. Since $f_0 \in C_0(G^{(0)})$, $Q(f_0) = f_0$. On the other hand, according to the above, $Q(f_i) = 0$ for $i = 1, \dots, n$. Therefore, $Q(f) = f_0 = P(f)$. □

The C*-module $C_0(G^{(0)}, H)$ over $C_0(G^{(0)})$ introduced earlier to define the representation π and the reduced norm on $C_c(G, \Sigma)$ is the completion of A with respect to the B-valued inner product $P(a^*a')$; the representation π is left multiplication.

The conditional expectation P will be used to recover the elements of A as sections of the line bundle L:

Lemma 4.4. *Let (G, Σ) be a twisted étale Hausdorff locally compact second countable groupoid. Let $P : A = C^*_{red}(G, \Sigma) \to B = C_0(G^{(0)})$ be the restriction map. Then we have the following formula: for all $\sigma \in \Sigma$, for all $n \in A$ such that supp$'(n)$ is a bisection containing $\dot\sigma$ and all $a \in A$:*

$$P(n^*a)(s(\sigma)) = \overline{n(\sigma)}a(\sigma).$$

Proof. This results from the definitions. □

The last property of the subalgebra $B = C_0(G^{(0)}))$ of $(A = C^*_{red}(G, \Sigma)$ which interests us is that it is regular. This requires the notion of normalizer as introduced by A. Kumjian.

Definition 4.5. [25, 1.1] Let B be a sub C*-algebra of a C*-algebra A.

(i) Its *normalizer* is the set

$$N(B) = \{n \in A : nBn^* \subset B \quad \text{and} \quad n^*Bn \subset B\}.$$

(ii) One says that B is *regular* if its normalizer $N(B)$ generates A as a C*-algebra.

Before studying the normalizer of $C_0(G^{(0)}))$ in $C^*_{red}(G,\Sigma)$, let us give some consequences of this definition. We first observe that $B \subset N(B)$ and $N(B)$ is closed under multiplication and involution. It is also a closed subset of A. We shall always assume that B contains an approximate unit of A. This condition is automatically satisfied when B is maximal abelian and A has a unit but this is not so in general (see [49]). We then have the following obvious fact.

Lemma 4.6. *Assume that B be is a sub C*-algebra of a C*-algebra A containing an approximate unit of A. Let $n \in N(B)$. Then $nn^*, n^*n \in B$.*

Assume also that B is abelian. Let $X = \hat{B}$ so that $B = C_0(X)$. For $n \in N(B)$, define $dom(n) = \{x \in X : n^*n(x) > 0\}$ and $ran(n) = \{x \in X : nn^*(x) > 0\}$. These are open subsets of X.

Proposition 4.7. [25, 1.6] *Given $n \in N(B)$, there exists a unique homeomorphism $\alpha_n : dom(n) \to ran(n)$ such that, for all $b \in B$ and all $x \in dom(n)$,*

$$n^*bn(x) = b(\alpha_n(x))n^*n(x).$$

Proof. See [25]. The proof uses the polar decomposition $n = u|n|$ of n in the envelopping von Neumann algebra A^{**}. The partial isomorphism of B: $b \mapsto u^*bu$ implemented by the partial isometry u gives the desired homeomorphism α_n. □

Proposition 4.8. *Let (G,Σ) be a twisted étale Hausdorff locally compact second countable groupoid. Let $A = C^*_{red}(G,\Sigma)$ and $B = C_0(G^{(0)})$ be as above. Then*

(i) *If the open support $S = supp'(a)$ of $a \in A$ is a bisection of G, then a belongs to $N(B)$ and $\alpha_a = \alpha_S$;*

(ii) *If G is topologically principal, the converse is true. Namely the normalizer $N(B)$ consists exactly of the elements of A whose open support is a bisection.*

Proof. Suppose that $S = supp'(a)$ is a bisection. Then, for $b \in B$,

$$a^*ba(\sigma) = \int \overline{a(\tau\sigma^{-1})}b \circ r(\tau)a(\tau)d\lambda_{s(\sigma)}(\dot{\tau}).$$

The integrand is zero unless $\dot\tau \in S$ and $\dot{\overline{\tau\sigma^{-1}}} \in S$, which implies that $\dot\sigma$ is a unit. Therefore $supp'(a^*ba) \subset G^{(0)}$ and $a^*ba \in B$. Similarly, $aba^* \in B$. Moreover, if $\dot\sigma = x$ is a unit, we must have $\dot\tau = Sx$ and therefore

$$a^*ba(x) = a^*a(x)b \circ r(Sx) = a^*a(x)b \circ \alpha_S(x).$$

This shows that $\alpha_a = \alpha_S$.

Conversely, let us assume that a belongs to $N(B)$. Let $S = supp'(a)$. Let us fix $x \in dom(a)$. The equality

$$b(\alpha_a(x)) = \int \frac{|a(\tau)|^2}{a^*a(x)} b \circ r(\tau) d\lambda_x(\dot\tau)$$

holds for all $b \in B$. In other words, the pure state $\delta_{\alpha_a(x)}$ is expressed as a (possibly infinite) convex combination of pure states. This implies that $a(\tau) = 0$ if $r(\tau) \neq \alpha_a(x)$. Let

$$T = \{\gamma \in G : s(\gamma) \in dom(a), \text{and } r(\gamma) = \alpha_a \circ s(\gamma)\}.$$

We have established the containment $S \subset T$. This implies $SS^{-1} \subset TT^{-1} \subset G'$. If G is topologically principal, SS^{-1} which is open must be contained in $G^{(0)}$. Similarly, $S^{-1}S$ must be contained in $G^{(0)}$. This shows that S is a bisection. □

Corollary 4.9. *Let (G, Σ) be a twisted étale Hausdorff locally compact second countable groupoid. Let $A = C^*_{red}(G, \Sigma)$. Then $B = C_0(G^{(0)})$ is a regular sub-C*-algebra of A.*

Proof. Since G is étale, the open bisections of G form a basis of open sets for G. Every element $f \in C_c(G, \Sigma)$ can be written as a finite sum of sections supported by open bisections. Thus the linear span of $N(B)$ contains $C_c(G, \Sigma)$. Therefore, $N(B)$ generates A as a C*-algebra. □

We continue to investigate the properties of the normalizer $N(B)$.

Lemma 4.10. [25, 1.7] *Let B be a sub-C*-algebra of a C*-algebra A. Assume that B is abelian and contains an approximate unit of A. Then*

(i) *If $b \in B$, $\alpha_b = id_{dom(b)}$.*
(ii) *If $m, n \in N(B)$, $\alpha_{mn} = \alpha_m \circ \alpha_n$ and $\alpha_{n^*} = \alpha_n^{-1}$.*

This shows that $\mathcal{G}(B) = \{\alpha_a, a \in N(B)\}$ is a pseudogroup on X. By analogy with the canonical action of the inverse semigroup of open bisections of an étale groupoid, we shall call the map $\underline{\alpha}$:

$N(B) \to \mathcal{G}(B)$ such that $\underline{\alpha}(n) = \alpha_n$ the canonical action of the normalizer.

Definition 4.11. We shall say that $\mathcal{G}(B)$ is the *Weyl pseudogroup* of (A, B). We define the *Weyl groupoid* of (A, B) as the groupoid of germs of $\mathcal{G}(B)$.

Proposition 4.12. *Let B be a sub-C^*-algebra of a C^*-algebra A. Assume that B is abelian and contains an approximate unit of A. Then:*

(i) *The kernel of the canonical action $\underline{\alpha} : N(B) \to \mathcal{G}(B)$ is the commutant $N(B) \cap B'$ of B in $N(B)$.*
(ii) *If B is maximal abelian, then $ker\underline{\alpha} = B$.*

Proof. If $n \in N(B) \cap B'$, then for all $b \in B$, $n^*bn = bn^*n$. By comparing with the definition of α_n, we see that $\alpha_n(x) = x$ for all $x \in dom(n)$. Conversely, suppose that $n \in N(B)$ satisfies $n^*bn(x) = b(x)n^*n(x)$ for all $b \in B$ and all $x \in dom(n)$. We also have $n^*bn(x) = b(x)n^*n(x) = 0$ when $x \notin dom(n)$ because of the inequality $0 \leq n^*bn \leq \|b\|n^*n$ for $b \in B_+$. Therefore $n^*bn = bn^*n$ for all $b \in B$. As observed in [25, 1.9], this implies that $(nb - bn)^*(nb - bn) = 0$ for all $b \in B$ and $nb = bn$ for all $b \in B$. The assertion (ii) is an immediate consequence of (i). □

Let us study the normalizer $N(B)$ in our particular situation, where $A = C^*_{red}(G, \Sigma)$ and $B = C_0(G^{(0)})$.

Proposition 4.13. *Let (G, Σ) be a twisted étale Hausdorff locally compact second countable groupoid. Let $A = C^*_{red}(G, \Sigma)$ and $B = C_0(G^{(0)})$ be as above. Assume that G is topologically principal. Then,*

(i) *the Weyl pseudogroup $\mathcal{G}(B)$ of (A, B) consists of the partial homeomorphisms α_S where S is an open bisection of G such that the restriction of the associated line bundle L to S is trivializable;*
(ii) *the Weyl groupoid $G(B)$ of (A, B) is canonically isomorphic to G.*

Proof. Recall that $\mathcal{S}$ denotes the inverse semigroup of open bisections of G and $\mathcal{G}$ denotes the pseudogroup defined by $\mathcal{S}$. We have defined the canonical action $\alpha : \mathcal{S} \to \mathcal{G}$ and the canonical action $\underline{\alpha} : N(B) \to \mathcal{G}(B)$. We have seen that α and $\underline{\alpha}$ are related by $\underline{\alpha} = \alpha \circ supp'$, where $supp'(n)$ denotes the open support of $n \in N(B)$, Moreover, the restriction of the line bundle to $S = supp'(n)$ is trivializable, since it possesses a non-vanishing section. Conversely, let S be an open bisection such that the restriction $L_{|S}$ is trivializable. Let us choose a non-vanishing continuous section $u : S \to L$. Replacing $u(\gamma)$ by $u(\gamma)/\|u(\gamma)\|$, we may assume that $\|u(\gamma)\| = 1$ for all $\gamma \in S$. Then, we choose $h \in C_0(G^{(0)})$ such that $supp'(h) = s(S)$ and define the section $n : G \to L$ by $n(\gamma) = u(\gamma)h \circ s(\gamma)$ if $\gamma \in S$ and $n(\gamma) = 0$ otherwise. Let (h_i) be a sequence in $C_c(G^{(0)})$, with $supp(h_i) \subset s(S)$, converging uniformly to h. Then $uh_i \in C_c(G, \Sigma)$ and the sequence (uh_i) converges to n in the norm $\|.\|_I$ introduced earlier. This implies that n belongs to A. We have $S = supp'(n)$ as desired. This shows that $\mathcal{G}(B)$ is exactly the pseudogroup consisting of the partial homeomorphisms α_S such that S is an open bisection of G on which L is trivializable. According to a theorem of Douady and Soglio-Hérault (see Appendix of [16]), for all open bisection S and all $\gamma \in S$, there exists an open neighborhood T of γ contained in S on which L is trivializable. Therefore $\mathcal{G}(B)$ and the pseudogroup $\mathcal{G}$ defined by all open bisections have the same groupoid of germs, which is isomorphic to G by Corollary 3.6. □

Let us see next how the twist Σ over G can be recovered from the pair (A, B). This is done exactly as in Section 3 of [25]. Given an abstract pair (A, B), we set $X = \hat{B}$ and introduce

$$D = \{(x, n, y) \in X \times N(B) \times X : n^*n(y) > 0 \text{ and } x = \alpha_n(y)\}$$

and its quotient $\Sigma(B) = D/\sim$ by the equivalence relation: $(x, n, y) \sim (x', n', y')$ if and only if $y = y'$ and there exist $b, b' \in B$ with $b(y), b'(y) > 0$ such that $nb = n'b'$. The class of (x, n, y) is denoted by $[x, n, y]$. Now $\Sigma(B)$ has a natural structure of groupoid over X, defined exactly in the same fashion as a groupoid of germs: the range and source maps are defined by $r[x, n, y] = x, s[x, n, y] = y$, the product by $[x, n, y][y, n', z] = [x, nn', z]$ and the inverse by $[x, n, y]^{-1} = [y, n^*, x]$.

The map $(x, n, y) \to [x, \alpha_n, y]$ from D to $G(B)$ factors through the quotient and defines a groupoid homomorphism from $\Sigma(B)$ onto $G(B)$. Moreover the subset $\mathcal{B} = \{[x, b, x] : b \in B, b(x) \neq 0\} \subset \Sigma(B)$ can be identified with the trivial group bundle $\mathbf{T} \times X$ via the map $[x, b, x] \mapsto (b(x)/|b(x)|, x)$. In general, $\mathcal{B} \to \Sigma(B) \to G(B)$ is not an extension, but this is the case when B is maximal abelian.

Proposition 4.14. *Assume that B is a masa in A containing an approximate unit of A. Then*

$$\mathcal{B} \to \Sigma(B) \to G(B)$$

is (algebraically) an extension.

Proof. We have to check that an element $[x, n, y]$ of $\Sigma(B)$ which has a trivial image in $G(B)$ belongs to $\mathcal{B}$. If the germ of α_n at y is the identity, then $x = y$ and we have a neighborhood U of y contained in $dom(n)$ such that $\alpha_n(z) = \alpha_{n^*}(z) = z$ for all $z \in U$. We choose $b \in B$ with compact support contained in U and such that $b(x) > 0$ and we define $n' = nb$. Then $\alpha_{n'}$ is trivial. According to Proposition 4.12, n' belongs to B and $[x, n, x] = [x, n', x]$ belongs to $\mathcal{B}$. □

We shall refer to $\Sigma(B)$ as the *Weyl twist* of the pair (A, B).

Proposition 4.15. *Let (G, Σ) be a twisted étale Hausdorff locally compact second countable topologically principal groupoid. Let $A = C^*_{red}(G, \Sigma)$ and $B = C_0(G^{(0)})$ be as above. Then we have a canonical isomorphism of extensions:*

$$\begin{array}{ccccc}
\mathcal{B} & \longrightarrow & \Sigma(B) & \longrightarrow & G(B) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{T} \times G^{(0)} & \longrightarrow & \Sigma & \longrightarrow & G
\end{array}$$

Proof. The left and right vertical arrows have been already defined and shown to be isomorphisms. It suffices to define the middle vertical arrow and show that it is a groupoid homomorphism which makes the diagram commutative. Let $(x, n, y) \in D$. Since n belongs to $N(B)$ and G is topologically principal, $S = supp'(n)$ is an open bisection of G. The element $n(Sy)/\sqrt{n^*n(y)}$ is a unitary element of L because $n^*n(y) = \|n(Sy)\|^2$ and can therefore be viewed as an element of Σ. Let $(x, n', y) \sim (x, n, y)$. There exist $b, b' \in B$ with $b(y), b'(y) > 0$ such that $nb = n'b'$. This implies that the open supports $S = supp'(n)$ and $S = supp'(n)$ agree on some neighborhood of Sy. In particular, $Sy = S'y$. Moreover, the equality $n(Sy)b(y) = n'(Sy)b'(y)$ implies that $n(Sy)/\sqrt{n^*n(y)} = n'(Sy)/\sqrt{n'^*n'(y)}$. Thus we have a well-defined map $\Phi : [x, n, y] \mapsto (n(Sy)/\sqrt{n^*n(y)}, Sy)$ from $\Sigma(B)$ to Σ. Let us check that it is a groupoid homomorphism.

Suppose that we are given $(x,m,y),(y,n,z)\in D$. Let $S=supp'(m)$, $T=supp'(n)$. Then $supp'(mn)=ST$. We have to check the equality

$$\frac{mn(STz)}{\sqrt{(mn)^*mn(z)}}=\frac{m(Sy)}{\sqrt{m^*m(y)}}\frac{n(Tz)}{\sqrt{n^*n(z)}}.$$

It is satisfied because $mn(STz)=m(Sy)n(Tz)$ and

$$\begin{aligned}(mn)^*(mn)(z)&=(n^*(m^*m)n)(z)\\&=(m^*m)(\alpha_n(z))n^*n(z)=m^*m(y)n^*n(z).\end{aligned}$$

The image of $[x,n,y]^{-1}=[y,n^*,x]$ is

$$n^*(S^{-1}x)/\sqrt{nn^*(x)}=(n(xS))^*/\sqrt{nn^*(x)}.$$

It is the inverse of $n(Sy)/\sqrt{n^*n(y)}$ because $xS=Sy$ and $nn^*(x)=n^*n(y)$ and the involution agrees with the inverse on $\Sigma\subset L$. Let us check that we have a commutative diagram. The restriction of Φ to $\mathcal{B}$ sends $[x,b,x]$ to $(b(x)/|b(x)|,x)$. This is exactly the left vertical arrow. The image of $[x,n,y]$ in $G(B)$ is the germ $[x,\alpha_n,y]$. The image of $(n(Sy)/\sqrt{n^*n(y)},Sy)$ in G is Sy. The map $[x,\alpha_n,y]\mapsto Sy$ is indeed the canonical isomorphism from $G(B)$ onto G. □

In the previous proposition, we have viewed $\Sigma(B)$ as an algebraic extension. It is easy to recover the topology of $\Sigma(B)$. Indeed, as we have already seen, every $n\in N(B)$ defines a trivialization of the restriction of $\Sigma(B)$ to the open bisection $S=supp'(n)$. This holds in the abstract framework. Assume that B is a masa in A. Let $n\in N(B)$. Its open support is by definition the open bisection $S\subset G(B)$ which induces the same partial homeomorphism as n. We define the bijection

$$\varphi_n:\mathbf{T}\times dom(n)\to\Sigma(B)_{|S},$$

by $\varphi_n(t,x)=[\alpha_n(x),tn,x]$.

Lemma 4.16. (*cf.* [25, Section 3]) *Assume that B is a masa in A containing an approximate unit of A. With above notation,*

(i) *Two elements $n_1,n_2\in N(B)$ which have the same open support S define compatible trivializations of $\Sigma(B)_{|S}$.*
(ii) *$\Sigma(B)$ is a locally trivial topological twist over $G(B)$.*

Proof. For (i), assume that n_1 and n_2 have the same open support S. Then, according to Proposition 4.12, there exist $b_1,b_2\in B$, non vanishing on $s(S)$ and such that $n_1b_1=n_2b_2$. A simple computation from the relation $\varphi_{n_1}(t_1, x) = \varphi_{n_2}(t_2, x)$ and the fact that for $n \in N(B)$ and $b \in B$, the equality $nb = 0$ implies $b(x) = 0$ whenever $n^*n(x) > 0$ gives $t_2 = t_1 u(x)$ where $u(x) = \frac{b_2(x)|b_1(x)|}{|b_2(x)|b_1(x)}$. Therefore, the transition function is a homeomorphism. We deduce (ii). Indeed, we have given a topology to $\Sigma(B)_{|S}$ whenever S is a bisection arising from the Weyl pseudogroup $\mathcal{G}(B)$. This family, which is stable under finite intersection and which covers $\Sigma(B)$, is a base of open sets for the desired topology. □

5. Cartan Subalgebras in C*-Algebras

Motivated by the properties of the pair $\big(A = C^*_{red}(G, \Sigma),\ B = C_0(G^{(0)})\big)$ arising from a twisted étale locally compact second countable Hausdorff topologically principal groupoid, we make the following definition, analogous to [18, Definition 3.1] of a Cartan subalgebra in a von Neumann algebra. We shall always assume that the ambient C*-algebra A is separable.

Definition 5.1. We shall say that an abelian sub-C*-algebra B of a C*-algebra A is a *Cartan subalgebra* if

(i) B contains an approximate unit of A;
(ii) B is maximal abelian;
(iii) B is regular;
(iv) there exists a faithful conditional expectation P of A onto B.

We shall say that (A, B) is a *Cartan pair* when B is a Cartan subalgebra.

Let us give some comments about the definition. First, when A has a unit, a maximal abelian sub-C*-algebra necessarily contains the unit; however, as said earlier, there exist maximal abelian sub-C*-algebras which do not contain an approximate unit for the ambient C*-algebra. Since in our models, namely étale groupoid C*-algebras, the subalgebra corresponding to the unit space always contains an approximate unit of A, we have to make this assumption. Second this definition of a Cartan subalgebra should be compared to the Definition 1.3 of a C*-diagonal given by A. Kumjian in [25] (see also [41]): there it is assumed that B has the unique extension property, a property introduced by J. Anderson and studied by R. Archbold et al. If B has the unique extension property (and under the assumption that it contains an approximate unit of A), it is maximal abelian and there exists one and only one conditional expectation onto B. We shall say more about the unique extension property when we compare Theorem 5.9 and Kumjian's theorem. The analysis of the previous section can be summarized by the following result.

Theorem 5.2. *Let* (G, Σ) *be a twisted étale Hausdorff locally compact second countable topologically principal groupoid. Then* $C_0(G^{(0)})$ *is a Cartan subalgebra of* $C_r^*(G, \Sigma)$.

Given a Cartan pair (A, B), we construct the normalizer $N(B)$, the Weyl groupoid $G(B)$ on $X = \hat{B}$, the Weyl twist $\Sigma(B)$ and the associated line bundle $L(B)$. In fact, these constructions can be made under the sole assumption that B is a masa. Let us see how the elements of A define sections of the line bundle $L(B)$ or equivalently, functions $f : \Sigma \to \mathbf{C}$ satisfying $f(t\sigma) = \bar{t}f(\sigma)$ for all $t \in \mathbf{T}$ and $\sigma \in \Sigma(B)$. The answer is given by Lemma 4.4 (this formula also appears in [25]). Recall that $\Sigma(B)$ is defined as a quotient of

$$D = \{(x, n, y) \in X \times N(B) \times X : n^*n(y) > 0 \text{ and } x = \alpha_n(y)\}.$$

Lemma 5.3. *Given* $a \in A$ *and* $(x, n, y) \in D$*, we define*

$$\hat{a}(x, n, y) = \frac{P(n^*a)(y)}{\sqrt{n^*n(y)}}.$$

Then

(i) $\hat{a}(x, n, y)$ *depends only on its class in* $\Sigma(B)$*;*
(ii) $\hat{a}$ *defines a continuous section of the line bundle* $L(B)$*;*
(iii) *the map* $a \mapsto \hat{a}$ *is linear and injective.*

Proof. Assertion (i) is clear, since $\hat{a}(x, nb, y) = \hat{a}(x, n, y)$ for all $b \in B$ such that $b(y) > 0$. For (ii), the equality $\hat{a}(x, tn, y) = \bar{t}\hat{a}(x, n, y)$ for all $t \in \mathbf{T}$ shows that $\hat{a}$ defines a section of $L(B)$. To get the continuity, it suffices to check the continuity of $\hat{a}$ on the open subsets $\Sigma(B)_{|S}$, where S is the open support of $n \in N(B)$. But this is exactly the continuity of the function $y \mapsto P(n^*a)(y)/\sqrt{n^*n(y)}$ on $dom(n)$. The linearity in (iii) is clear. Let us assume that $\hat{a} = 0$. Let $n \in N(B)$. Then $P(n^*a)(y) = 0$ for all $y \in dom(n)$, hence also in its closure. If y does not belong to the closure of $dom(n)$, we can find $b \in B$ such that $b(y) = 1$ and $nb = 0$. Then $P(n^*a)(y) = P(b^*n^*a)(y) = 0$. Therefore $P(n^*a) = 0$ for all $n \in N(B)$. By regularity of B, this implies $P(a^*a) = 0$. By faithfulness of P, this implies that $a = 0$. □

Definition 5.4. The map $\Psi : a \mapsto \hat{a}$ from A to the space of continuous sections of $L(B)$ will be called the *evaluation map* of the Cartan pair (A, B).

Lemma 5.5. *Let (A, B) be a Cartan pair. For $n \in N(B)$ and $x \in dom(n)$ such that the germ of α_n at x is not trivial, we have $P(n)(x) = 0$.*

Proof. Since the germ of α_n at x is not trivial, there exists a sequence (x_i) in $dom(n)$ which converges to x and such that $\alpha_n(x_i) \neq x_i$. We fix i. There exist $b', b'' \in B$ such that $b'(x_i) = 1$, $b''(x_i) = 0$ and $b''n = nb'$. Indeed, there exists $b \in B$ with compact support contained in $ran(n)$ such that $b(\alpha_n(x_i))(n^*n)(x) = 1$ and $b(x_i) = 0$. Then $b' = (b \circ \alpha_n)(n^*n)$ and $b'' = (nn^*)b$ satisfy the conditions. We have

$$\begin{aligned} P(n)(x_i) &= P(n)(x_i)b'(x_i) = P(nb')(x_i) \\ &= P(b''n)(x_i) = b''(x_i)P(n)(x_i) = 0. \end{aligned}$$

By continuity of $P(n)$, $P(n)(x) = 0$. □

Corollary 5.6. *Let $a \mapsto \hat{a}$ be the evaluation map of the Cartan pair (A, B).*

(i) *Suppose that b belongs to B; then $\hat{b}$ vanishes off X and its restriction to X is its Gelfand transform.*
(ii) *Suppose that n belongs to $N(B)$; then the open support of $\hat{n}$ is the open bisection of $G(B)$ defined by the partial homeomorphism α_n.*

Proof. Let us show (i). If $\gamma = [\alpha_n(x), \alpha_n, x] \in G(B)$ is not a unit, the germ of α_n at x is not trivial. According to the lemma, for all $b \in B$, $P(n^*b)(x) = P(n)(x)b(x) = 0$. Therefore, $\hat{b}(\gamma) = 0$. On the other hand, if $\gamma = x$ is a unit, $\hat{b}(x) = P({b_1}^*b)(x) = {b_1}^*(x)b(x) = b(x)$ for $b_1 \in B$ such that $b_(x) = 1$. Let us show (ii). If $n \in N(B)$, the lemma shows that $\hat{n}[x, m, y] = 0$ unless $y \in dom(n)$ and α_m has the same germ as α_n at y. Then $[x, \alpha_m, y] = [x, \alpha_n, y]$ belongs to the open bisection S_n of $G(B)$ defined by the partial homeomorphism α_n. On the other hand, $\hat{n}(x, n, y) = n^*n(y)/\sqrt{n^*n(y)}$ is non zero for $y \in dom(n)$. □

Proposition 5.7. *The Weyl groupoid $G(B)$ of a Cartan pair (A, B) is a Hausdorff étale groupoid .*

Proof. By construction as a groupoid of germs, $G(B)$ is étale. Let us show that the continuous functions $\hat{a}$, where $a \in A$ separate the points of $G(B)$ in the sense that for all $\sigma, \sigma' \in \Sigma$ such that $\dot{\sigma} \neq \dot{\sigma}'$, there exists $a \in A$ such that $\hat{a}(\sigma) \neq 0$ and $\hat{a}(\sigma') = 0$. By construction, $\sigma = [x, n, y], \sigma' = [x', n', y']$ where $n, n' \in N(B)$ and $y \in dom(n), y' \in dom(n')$. If $y \neq y'$, we can take a of the form nb where $b(y) \neq 0$ and $b(y') = 0$. If $y = y'$, since α_n and $\alpha_{n'}$ do not have the same germ at y, we have by Lemma 5.5 that $P(n'^*n)(y) = 0$, which implies $\hat{n}(\sigma') = 0$. On the other hand, $\hat{n}(\sigma) = \sqrt{n^*n(y)}$ is non-zero. We can furthermore assume that $\hat{a}(\sigma) = 1$. Let $U = \{\tau : |\hat{a}(\tau) - 1| < 1/2\}$ and $V = \{\tau : |\hat{a}(\tau)| < 1/2\}$. Their images $\dot{U}, \dot{V}$ in $G(B)$ are open, disjoint and $\dot{\sigma} \in \dot{U}, \dot{\sigma}' \in \dot{V}$. □

Lemma 5.8. *Let (A, B) be a Cartan pair. Let $N_c(B)$ be the set of elements n in $N(B)$ such $\hat{n}$ has compact support and let A_c be its linear span. Then*

(i) *$N_c(B)$ is dense in $N(B)$ and A_c is dense in A;*
(ii) *the evaluation map $\Psi : a \mapsto \hat{a}$ defined above sends bijectively A_c onto $C_c(G(B), \Sigma(B))$ and $B_c = B \cap A_c$ onto $C_c(G^{(0)})$;*
(iii) *the evaluation map $\Psi : A_c \to C_c(G(B), \Sigma(B))$ is a $*$-algebra isomorphism.*

Proof. For (i), given $n \in N(B)$, there exists $b \in B$ such that $nb = n$. There exists a sequence (b_i) in B such that $\hat{b}_i \in C_c(G^{(0)})$ and (b_i) converges to b. Then nb_i belongs to $N_c(B)$ and the sequence (nb_i) converges to n. Note that $N_c(B)$ is closed under product and involution and that A_c is a dense sub-$*$-algebra of A. Let us prove (ii). By construction, $\Phi(A_c)$ is contained in $C_c(G, \Sigma)$. The injectivity of Φ has been established in Lemma 5.4. Let us show that $\Phi(A_c) = C_c(G, \Sigma)$. The family of open bisections $S_n = \{[\alpha_n(x), \alpha_n, x], x \in dom(n)\}$, where n runs over $N(B)$, forms an open cover of $G(B)$. If $f \in C_c(G, \Sigma)$ has its support contained in S_n, then $\hat{n}$ is a non-vanishing continuous section over S_n and there exists $h \in C_c(G^{(0)})$ such that $f = \hat{n}h$. Since $h = \hat{b}$ with $b \in B_c$, $f = \hat{a}$, where $a = nb$ belongs to $N_c(B)$. For a general $f \in C_c(G, \Sigma)$, we use a partition of unity subordinate to a finite open cover $S_{n_1}, \dots, S_{n_l}$ of the support of f. Let us prove (iii). By linearity of Ψ, it suffices to check the relations $\Psi(mn) = \Psi(m) * \Psi(n)$ and $\Psi(n^*) = \Psi(n)^*$ for $m, n \in N(B)$. According to Corollary 5.6, $\Psi(mn)(\sigma) = 0$ unless

$\sigma = t[x, mn, z]$ with $z \in dom(mn)$ and $t \in \mathbf{T}$; then we have

$$\Psi(mn)(t[x, mn, z]) = \bar{t}\sqrt{((mn)^*mn)(z)}.$$

On the other hand, $\Psi(m)\Psi(n)(\sigma) = 0$ unless σ is of the form

$$\sigma = t[x, m, y][y, n, z] = t[x, mn, z]$$

and then

$$\begin{aligned}\Psi(m)\Psi(n)(t[x, mn, z]) &= \Psi(m)(t[x, m, y])\Psi(n)([y, n, z] \\ &= \bar{t}\sqrt{(m^*m)(y)(n^*n)(z)}.\end{aligned}$$

The equality results from

$$\begin{aligned}(mn)^*(mn)(z) &= (n^*(m^*m)n)(z) \\ &= (m^*m)(\alpha_n(z))n^*n(z) = m^*m(y)n^*n(z).\end{aligned}$$

Similarly, $\Psi(n^*)(\sigma) = 0$ unless $\sigma = \bar{t}[y, n^*, x]$ with $x \in dom(n^*)$ and $t \in \mathbf{T}$ and then we have

$$\Psi(n^*)(\bar{t}[y, n^*, x]) = t\sqrt{(nn^*)(x)}.$$

On the other hand, $\Psi(n)^*(\sigma) = \overline{\Psi(n)(\sigma^{-1})} = 0$ unless $\sigma^{-1} = t[x, n, y]$ with $y \in dom(n)$ and $t \in \mathbf{T}$ and then we have

$$\Psi(n)^*(\bar{t}[y, n^*, x]) = t\sqrt{(n^*n)(y)}.$$

These numbers are equal because $nn^*(x) = n^*n(y)$. □

Theorem 5.9. *Let B be a Cartan sub-algebra of a separable C^*-algebra A. Then*

(i) *there exists a twist (G, Σ) where G is a second countable locally compact Hausdorff, topologically principal étale groupoid and an isomorphism of $C^*_r(G, \Sigma)$ onto A carrying $C_0(G^{(0)})$ onto B;*

(ii) *the above twist is unique up to isomorphism; it is isomorphic to the Weyl twist $(G(B), \Sigma(B))$.*

Proof. Let $(G, \Sigma) = (G(B), \Sigma(B))$. Let us show that the evaluation map $\Psi : A_c \to C_c(G, \Sigma)$ is an isometry with respect to the norms of A and $C^*_r(G, \Sigma)$. Since P is faithful, we have for any $a \in A$ the equality

$$\|a\| = \sup\{\|P(c^*a^*ac)\|^{1/2} : c \in A_c, P(c^*c) \leq 1\}.$$

If we assume that a belongs to A_c, then $\hat{a}$ belongs to $C^*_r(G,\Sigma)$ and satisfies a similar formula:

$$\begin{aligned}\|\hat{a}\| &= \sup\{\|\hat{P}(f^*(\hat{a})^*\hat{a}f)\|^{1/2} : f \in C_c(G,\Sigma), \hat{P}(f^*f) \leq 1\}\\ &= \sup\{\|\hat{P}((\hat{c})^*(\hat{a})^*\hat{a}\hat{c})\|^{1/2} : c \in A_c, \hat{P}((\hat{c})^*\hat{c}) \leq 1\}.\end{aligned}$$

Since $\Psi : A_c \to C_c(G,\Sigma)$ satisfies the relation $\hat{P} \circ \Psi = \Psi \circ P$, where $\hat{P}$ is the restriction map to $\hat{B}$, we have the equality of the norms: $\|\hat{c}\| = \|c\|$. Hence Ψ extends to a C*-algebra isomorphism $\tilde{\Psi} : A \to C^*_r(G,\Sigma)$. By continuity of point evaluation, $\tilde{\Psi}(a) = \Psi(a)$ as defined initially. Therefore, the evaluation map is a C*-algebra isomorphism of of $C^*_r(G,\Sigma)$ onto A carrying $C_0(G^{(0)})$ onto B. The separability of $C^*_r(G,\Sigma)$ implies that of $C_0(G,E)$. One deduces that G is second countable. Since $G = G(B)$ is a groupoid of germs, it results from Proposition 3.6 that G is topologically principal. The uniqueness of the twist up to isomorphism has been established in Proposition 4.15. □

We have mentioned earlier that the unique extension property of B implies the uniqueness of the conditional expectation onto B. The uniqueness still holds for Cartan subalgebras.

Corollary 5.10. *Let B be a Cartan subalgebra of a C*-algebra A. Then, there exists a unique expectation onto B.*

Proof. This results from the above theorem and Proposition 4.3. □

The following proposition is essentially a reformulation of Kumjian's theorem (see [25] and [41]). For the sake of completeness, we recall his proof. One says that the subalgebra B has the unique extension property if every pure state of B extends uniquely to a (pure) state of A. A C*-diagonal is a Cartan subalgebra which has the unique extension property.

Proposition 5.11. (*cf.* [25], [41]) *Let (A,B) be a Cartan pair. Then B has the unique extension property if and only if the Weyl groupoid $G(B)$ is principal.*

Proof. We may assume that $(A,B) = (C^*_r(G,\Sigma), C_0(G^{(0)}))$, where G is an étale topologically principal Hausdorff groupoid and Σ is a twist over G. Suppose that G is principal. A. Kumjian shows that this implies that the linear span of the set $N_f(B)$ of free normalizers is dense in the kernel of the conditional expectation P, where a normalizer $n \in N(B)$ is said to be free if $n^2 = 0$. Indeed, since an arbitrary element of the kernel can be approximated by elements in $C_c(G,\Sigma)\cap Ker(P)$, it suffices to consider a continuous section f with compact support which vanishes on $G^{(0)}$. Since the compact support of f does not meet the diagonal $G^{(0)}$, which is both open and closed, it admits a finite cover by open bisections U_i such that $r(U_i)\cap s(U_i)=\emptyset$. Let (h_i) be a partition of unity subordinate to the open cover (U_i). Then, $f=\sum g_i$, where $g_i(\sigma)=f(\sigma)h_i(\dot\sigma)$ is a free normalizer. Then, he observes that free normalizers are limits of commutators $ab-ba$, with $a\in A$ and $b\in B$. This show that $A=B+\overline{span}[A,B]$, which is one of the characterizations of the extension property given in Corollary 2.7 of [2]. We suppose now that B has the unique extension property and we show that the isotropy of G is reduced to $G^{(0)}$. It suffices to show that for $n\in N(B)$ and $x\in dom(n)$, the equality $\alpha_n(x)=x$ implies that the germ of α_n at x is trivial. According to Lemma 5.5, it suffices to show that $P(n)(x)\neq 0$. Given $n\in N(B)$ and $x\in dom(n)$, the states $x\circ P$ and $\alpha_n(x)\circ P$ are unitarily equivalent and their transition probability ([43]) is $\frac{|P(n)(x)|^2}{n^*n(x)}$. Indeed, let (H,ξ,π) be the GNS triple constructed from the state $x\circ P$. By construction, $x\circ P$ is the state defined by the representation π and the vector ξ. On the other hand, $\alpha_n(x)\circ P$ is the state of A defined by π and the vector $\eta=\pi(u)\xi$, where u is the partial isometry of the polar decomposition $n=u|n|$ of n in A^{**}. To show that, one checks the straightforward relation $b(\alpha_n(x))=(\eta,\pi(b)\eta)$ for $b\in B$ and one uses the unique extension property. The transition probability can be computed by the formula $|(\xi,\eta)|^2=\frac{|P(n)(x)|^2}{n^*n(x)}$. If $\alpha_n(x)=x$, the transition probability is 1. In particular, $P(n)(x)\neq 0$. □

6. Examples of Cartan Subalgebras in C*-Algebras

6.1. **Crossed products by discrete groups.** In his pioneering work [50] on crossed product C* and W*-algebras by discrete groups, G. Zeller-Meier gives the following necessary and sufficient condition (Proposition 4.14) for B to be maximal abelian in the reduced crossed product $C^*_r(\Gamma;B;\sigma)$, where Γ is a discrete group acting by automorphisms on a commutative C*-algebra B and σ is a 2-cocycle: the action of Γ on $X=\hat B$ must be topologically free, meaning that for all $s\in\Gamma\setminus\{e\}$, the set $X_s=\{x\in X: sx=x\}$ must have an empty interior in X. This amounts to the groupoid $G=\Gamma\ltimes X$ of the action being topologically principal. Proposition 2.4.7 of [40] extends this result. Note that G is principal if and only if the action is free, in the sense that for all $s \in \Gamma \setminus \{e\}$, the set $X_s = \{x \in X : sx = x\}$ is empty. The particular case of the group $\Gamma = \mathbf{Z}$ is well studied (see for example [45]) and we consider only this case below.

Irrational rotations and minimal homeomorphisms of the Cantor space are examples of free actions. The C*-algebras of these dynamical systems are well understood and completely classified. I owe to I. Putnam the remark that the C*-algebra of a Cantor minimal system may contain uncountably many non-conjugate Cartan subalgebras (which are in fact diagonals in the sense of Kumjian). Indeed, according to [19], such a C*-algebra depends only, up to isomorphism, on the strong orbit equivalence class of the dynamical system; however, two minimal Cantor systems which are strongly orbit equivalent need not be flip conjugate (flip conjugacy amounts to groupoid isomorphism). More precisely, Boyle and Handelman show in [5] that the strong orbit equivalence class of the dyadic adding machine contains homeomorphisms of arbitrary entropy. These will give the same C*-algebra but the corresponding Cartan subalgebras will not be conjugate.

On the other hand, two-sided Bernoulli shifts are examples of topologically free actions which are not free. They provide examples of Cartan subalgebras which do not have the extension property. In [45], J. Tomiyama advocates the view that in relation with operator algebras, the notion of topologically free action, rather than that of free action, is the counterpart for topological dynamical systems of the notion of free action for measurable dynamical systems. The comparison of Theorem 5.9 and of the Feldman–Moore theorem completely supports this view.

6.2. **AF Cartan subalgebras in AF C*-algebras.** Approximately finite dimensional (AF) C*-algebras have privileged Cartan subalgebras. These are the maximal abelian subalgebras obtained by the diagonalization method of Strătilă and Voiculescu ([44]). In that case, the twist is trivial and the whole information is contained in the Weyl groupoid. The groupoids which occur in that fashion are the AF equivalence relations. These are the equivalence relations R on a totally disconnected locally compact Hausdorff space X which are the union of an increasing sequence of proper equivalence relations (R_n). The proper relations R_n are endowed with the topology of $X \times X$ and R is endowed with the inductive limit topology. As shown by Krieger in [23], AF C*-algebras and AF equivalence relations share the same complete invariant, namely the dimension group. One deduces that these privileged Cartan subalgebras, also called AF Cartan subalgebras, are conjugate by an automorphism of the ambient AF algebra. However, AF C*-algebras may contain other Cartan subalgebras. An example of a Cartan subalgebra in an AF C*-algebra without the unique extension property is given in [40, III.1.17]. A more striking example is given by B. Blackadar in [4]. He constructs a diagonal in the CAR algebra whose spectrum is not totally disconnected. More precisely, he realizes the CAR algebra as the crossed product $C(X)\rtimes\Gamma$ where $X = \mathbf{S}^1 \times$ Cantor space and Γ is a locally finite group acting freely on X. Note that the groupoid $X\rtimes\Gamma$ is also an AP equivalence relation, in the sense that it is the union of an increasing sequence of proper equivalence relations (R_n).

6.3. **Cuntz-Krieger algebras and graph algebras.** The Cuntz algebra $\mathcal{O}_d$ is the prototype of a C*-algebra which has a natural Cartan subalgebra without the unique extension property. By definition, $\mathcal{O}_d$ is the C*-algebra generated by d isometries $S_1, \dots, S_d$ such that $\sum_{i=1}^d S_i S_i^* = 1$. The Cartan subalgebra in question is the sub C*-algebra $\mathcal{D}$ generated by the range projections of the isometries $S_{i_1} \dots S_{i_n}$. It can be checked directly that $\mathcal{D}$ is a Cartan subalgebra of $\mathcal{O}_d$; however, it is easier to show first that $(\mathcal{O}_d, \mathcal{D})$ is isomorphic to $(C^*(G), C(X))$, where $X = \{1, \dots, d\}^{\mathbf{N}}$ and $G = G(X,T)$ is the groupoid associated to the one-sided shift $T : X \to X$ (see [40, 11, 42]):

$$G = \{(x, m-n, y) : x, y \in X, m, n \in \mathbf{N}, T^m x = T^n y\}.$$

This groupoid is not principal but it is topologically principal. In fact, the groupoid $G(X,T)$ associated to the local homeomorphism $T : X \to X$ is topologically principal if and only if T is topologically free, meaning that for all pairs of distinct integers (m,n), the set $X_{m,n} = \{x \in X : T^m x = T^n x\}$ must have an empty interior in X.

Condition (I) introduced by Cuntz and Krieger in their fundamental work [10] ensures that the subalgebra $\mathcal{D}_A$ is a Cartan subalgebra of $\mathcal{O}_A$. Here, A is a $d \times d$ matrix with entries in $\{0,1\}$ and non-zero rows and columns. The associated dynamical system is the one-sided subshift of finite type (X_A, T_A); condition (I) guarantees that this system is topologically free. In subsequent generalizations, in terms of infinite matrices in [14] and in terms of graphs in [29], exit condition (L) replaces condition (I). On the topological dynamics side, it is a necessary and sufficient condition for the relevant groupoid to be topologically principal. On the C*-algebraic side, it is the condition which ensures that the natural diagonal subalgebra $\mathcal{D}$ is maximal abelian, hence a Cartan subalgebra. Moreover, it results from [29] that this subalgebra has the extension property if and only if the graph contains no loops. Condition (II) of [9] or its generalization (K) in [28] implies that each reduction of the groupoid to an invariant closed subset is topologically principal and therefore that the image of $\mathcal{D}$ in the corresponding quotient is still maximal abelian.

6.4. **Cartan subalgebras in continuous-trace C*-algebras.** Let us first observe that a Cartan subalgebra of a continuous-trace C*-algebra necessarily has the unique extension property. The proof given in [15, Théorème 3.2] for foliation C*-algebras is easily adapted.

Proposition 6.1. *Let B be a Cartan subalgebra of a continuous-trace C*-algebra A. Then B has the unique extension property.*

Proof. From the main theorem, we can assume that $(A, B) = (C_r^*(G, \Sigma), C_0(G^{(0)}))$, where G is an étale topologically principal Hausdorff groupoid and Σ is a twist over G. Since A is nuclear, we infer from [1, 6.2.14, 3.3.7] that G is topologically amenable and from [1, 5.1.1] that all its isotropy subgroups are amenable. Since A is CCR, we infer from [6, Section 5,] that $G^{(0)}/G$ injects continuously in $\hat{A}$ and that all the orbits of G are closed (the presence of a twist does not affect this result nor its proof). Since G is étale, these closed orbits are discrete. Now, each $h \in C_c(G^{(0)})$ belongs to the Pedersen ideal $K(A)$. Therefore, it defines a continuous function on $\hat{A}$ whose value at $[x] \in G^{(0)}/G$ is

$$\overline{h}[x] = \sum_{y \in [x]} h(y).$$

Suppose that $G(x)$ is not reduced to $\{x\}$. Then there exists an open neighborhood V of x such that $[x] \cap V = \{x\}$ and $[y] \cap V$ contains at least two elements for $y \neq x$. For $h \in C_c(G^{(0)})$ supported in V and equal to 1 on a neighborhood of x, we would obtain $\overline{h}[x] = 1$ and $\overline{h}[y] \geq 2$ for y close to x, which contradicts the continuity of $\overline{h}$. Hence G is principal and B has the unique extension property. □

When a Cartan subalgebra B of a continuous-trace C*-algebra A exists, the cohomology class $[\Sigma(B)]$ of its twist is essentially the Dixmier–Douady invariant of A. Indeed, just as in the group case, the groupoid extension $\Sigma(B)$ defines an element of the cohomology group $H^2(G(B), \mathbf{T})$ (see [46] for a complete account of groupoid cohomology). Since $G(B)$ is equivalent to $\hat{B}/G(B) = \hat{A}$, this can be viewed as an element of $H^2(\hat{A}, \mathcal{T})$, where $\mathcal{T}$ is the sheaf of germs of $\mathbf{T}$-valued continuous functions. Its identification with the Dixmier–Douady invariant is done in in [25, 41, 39]. Moreover, a simple construction shows that every Čech cohomology class in $H^3(T, \mathbf{Z})$, where T is a locally compact Hausdorff space, can be realized as the Dixmier–Douady invariant of a continuous-trace C*-algebra of the above form $C^*(G, \Sigma)$.

However, Cartan subalgebras B of a continuous-trace C*-algebra A do not always exist. It has been observed (see [2, Remark 3.5.(iii)]) that there exist non-trivial n-homogeneous C*-algebras which do not have a masa with the unique extension property. Therefore, these C*-algebras do not have Cartan subalgebras. In [24, Appendix], T. Natsume gives an explicit example. Given a Hilbert bundle H over a compact space T, let us denote by A_H the continuous-trace C*-algebra defined by H. Let B be a Cartan subalgebra of A_H. The inclusion map gives a map $\hat{B} \to T$ which is a local homeomorphism and a surjection. If T is connected and simply connected, this is a trivial covering map and B decomposes as a direct sum of summands isomorphic to $C(T)$. Therefore H decomposes as a direct sum of line bundles. This is not always possible. For example there exists a vector bundle of rank 2 on the sphere $\mathbf{S}^4$ which cannot be decomposed into a direct sum of line bundles.

6.5. **Concluding remarks.** Just as in the von Neumann setting, the notion of Cartan subalgebra in C*-algebras provides a bridge between the theory of dynamical systems and the theory of operator algebras. Examples show the power of this notion, in particular to understand the structure of some C*-algebras, but also its limits. This notion has to be modified if one wants to include the class of the C*-algebras of non-Hausdorff topologically principal étale groupoids. In the case of continuous-trace C*-algebras, we have seen that the twist attached to a Cartan subalgebra is connected with the Dixmier–Douady invariant. It would be interesting to investigate its C*-algebraic significance in other situations.

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The Undergraduate Ambassadors Scheme in Ireland

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Abstract. In this article I will briefly describe the Undergraduate Ambassadors Scheme which has been running in the UK since 2002. This scheme provides university departments with a framework for running a module that awards academic credit to undergraduates for developing transferable skills, while working with teachers in local schools. I will explain why I chose to coordinate an Undergraduate Ambassadors Scheme module in mathematics in University College Dublin and discuss what is involved in setting-up, running, and assessing such a module.

1. Introduction

The Undergraduate Ambassadors Scheme (UAS) first started in the United Kingdom in 2002. The scheme was founded by Simon Singh and Hugh Mason out of a concern over teacher shortages in mathematics, science and technology, and the declining number of university applicants to these subject areas. The basic idea behind the UAS is that undergraduate students gain academic credit for developing their transferable skills, while working with a teacher in a local school. The scheme has grown rapidly since 2002 when 28 students from four university departments took part. In the academic year 2007–2008, 107 departments from 41 universities across the UK and Ireland participated in the scheme—with the School of Mathematical Sciences, University College Dublin (UCD) offering a UAS Module to its undergraduates this year for the first time.

In this article, I will explain why I chose to coordinate a UAS module in mathematics in UCD and discuss what is involved in setting-up, running, and assessing a module of this kind. It is my hope that should a colleague want to offer a similar module to his or her undergraduate students, this article will act as a check-list of some of the main issues to be considered.

2. Why Offer a UAS Module?

In 2004, on reading an article by Ray d'Inverno and Paul Cooper in *Educational Studies in Mathematics* [2], I decided that a UAS module was something I'd like to introduce in UCD. In this article the authors describe setting-up and running one of the first UAS modules in 2002 at the University of Southampton. (For the interested reader, there are some additional articles written on the scheme [1, 3, 4].)

My main motivation for offering a UAS module in mathematics was because of the benefits it offers the final-year undergraduate student. Firstly, it gives the undergraduate a chance to explore a career in teaching by giving him or her the opportunity of gaining some experience in the classroom before graduation. Each year, a number of final year students on the BA in Mathematical Studies Degree in UCD apply for a place on a Postgraduate Diploma in Education (PGDE). The UAS module can give them the chance to decide if this is right graduate programme for them. The module might also encourage undergraduates, who may not already have done so, to seriously consider a career in teaching. We need enthusiastic, interested, well-qualified mathematics teachers in our second-level schools and this module has the potential to help address this issue, if only in a small way.

Secondly, the student gains academic credit for developing his or her key transferable skills. These can include communication and presentation skills; the ability to work in a team; to manage time and prioritise; to improvise and take initiative when appropriate; to give and receive feedback; and to critically reflect on one's strengths and weaknesses. A skill I place particular importance on in the UCD module, is the ability to communicate mathematics effectively to others. Every mathematics department wants its graduates, not just to have good mathematical knowledge, but to present well in interviews and to act professionally in the workplace. For this reason I feel, as mathematics lecturers, we cannot ignore the issue of how we might help our undergraduates develop these key skills.

Of course the undergraduate student is not the only stakeholder in this module—the second-level mathematics teachers and students, along with the mathematics department and the university, should also benefit from taking part. Benefits to the second-level teacher might include teaching assistance in a large class; support in providing additional assistance to students at either end of the ability spectrum; support in providing after-school activities such as mathematics study groups; lasting classroom resources as a result of the undergraduate's *Special Project*; and links with mathematicians at a local university. For the second-level students, benefits from taking part in the module might include the opportunity to gain individual or small-group mathematics support; to avail of after-school mathematics support; and to learn about mathematics and university life from an enthusiastic undergraduate. University departments have the potential to benefit from having more confident, employable, professional graduates; from creating and maintaining links with local second-level schools; and from an improvement in the number of students entering undergraduate degrees in mathematics (perhaps!).

3. Setting-up a UAS Module

I took advantage of the introduction of modularisation to UCD in order to offer MST30060—*Undergraduate Ambassadors Scheme in Mathematics*—as an *optional* or *elective* module to final year BA in Mathematical Studies students in the second semester of the academic year 2007-2008. I would not advocate making a module of this nature compulsory or *core* for all final year mathematics undergraduates—placing a student with a bad attitude or poor people skills in a local school, could do immense harm to the scheme and reflect very badly on the mathematics department.

And so it was, that in September 2007, I found myself drawing up a list of things I needed to do before students embarked on the module in January 2008. I had three immediate concerns: firstly, I needed to find schools and teachers willing to take part; secondly, I needed to find *suitable* undergraduates who wanted to enrol on the module; and thirdly, I had to find out if my students needed Garda Clearance in order to work with children under 18. I will now address each of these in turn.

In order to find teachers and schools willing to take part, I contacted the UCD New ERA Office who run the widening participation programme in UCD. By planning interventions at the primary and second level in local government designated-disadvantaged schools, the New ERA Office aims to encourage more students from socially-disadvantaged backgrounds to consider obtaining a third-level education. Fiona Sweeney and Michelle King from this office were very keen to have some of these schools participate in the UAS module. Of the 18 schools contacted, 14 wanted to participate and each wrote a short paragraph explaining why. In most cases, the schools were keen to host an undergraduate in order to provide additional mathematics support for their Junior and Leaving Certificate students at either end of the ability spectrum.

With 14 schools willing to participate, I then needed to find *suitable* undergraduates who wanted to enrol in the module. Any final-year Mathematical Studies student interested in taking part had to complete and submit an application form by 5 October. On the application form, students had to explain why they wanted to take part in the module, and to describe any relevant experience they had that might benefit them on the module. I received nine applications—just under half the class.

All nine students were invited to interviews which were held in mid-October. Each interview lasted approximately 15 minutes, and was jointly conducted by the module tutor, Cathy Paolucci, and me in as formal an atmosphere as possible in order to make the experience more realistic for the student. Based on their application forms, interviews, and mathematics grades, six students were offered places on the module. All accepted and were officially enrolled.

One criticism at this point may be that it was not worth offering this module, if only six students were taking it. However, given that I was running it for the first time, I wanted to treat the module more as a pilot project. In the coming year, I would hope to double the number of students on the module.

Now that I had six undergraduates, I needed to chose six schools from the fourteen that wanted to participate. Primarily I made the decision based on the proximity of the school from UCD, and access to the school from the university on public transport. However, one student did agree to go to a school in Killinarden near Tallaght, another went to Ballyfermot, and yet another to Bray. It is important to consider whether the undergraduate has a car, when assigning him or her to a particular school.

At the start of November, I visited each school and spoke to the host mathematics teacher. While this was time-consuming, I wanted to meet each teacher face-to-face in order to establish a working relationship with him or her. Ray d'Inverno, the Director of the UAS in the UK, has suggested that the teachers could all be invited into the university for a presentation and lunch, and this might be more practical in the future should the number of schools participating increase. I asked each teacher what advice I should give the undergraduates before entering the school, and I received some great suggestions which were then incorporated into the UAS Handbook and training day materials.

My third concern at the start of the semester had been in relation to Garda Clearance. Since 2006, Garda vetting has become a condition of employment for new teachers and appointees who have unsupervised access to children and vulnerable adults (see the website at `www.teachingcouncil.ie`). One cannot apply directly to the Garda Central Vetting Unit—there is usually a contact person in each university through which all applications must be submitted. The UCD contact supplied me with the application forms and advised me that they can take up to three months to be processed. In our case, it took four.

By mid-November, my three concerns had been addressed—each undergraduate was assigned to a school and had submitted a Garda vetting application form. The UAS in the UK advise that the undergraduates attend a training day before they enter the schools. Consequently, two training sessions for the undergraduates were arranged from 5–8pm on the first two days of the second semester. All was now in order until after Christmas!

4. UAS Training Sessions and Handbook

At the start of January I had to think about what should be included in the training sessions, and about developing an accompanying handbook, which would act both as a training and resource manual. Here the UAS website (`www.uas.ac.uk`) proved to be an excellent starting point as one can access and edit a number of handbooks and materials already developed by others. With the UAS handbook as a template, Cathy and I developed a handbook more suitable for the Irish context. It contains five chapters. After a brief introduction to the UAS in Chapter 1, the second chapter entitled *What's Involved in the UAS Module?* describes the learning outcomes of the module, and elaborates on how the module will be assessed.

A key point about the assessment of this module, is that students simply don't get an A+ because they have spent 20–30 hours working in a local school. They are assessed on whether or not they have achieved the learning objectives, which in the case of the UCD module were as follows:

> On completion of this module the student should be able to provide evidence of how he or she has:
> - Communicated mathematics effectively to others;
> - Developed key transferable skills.

Evidence is provided by the undergraduate in a *reflective journal* that is kept throughout the semester (worth 30%); a *final report* of 2,000 words to be submitted at the end of the semester (worth 30%); and a 15-*minute presentation* held during the last week of the semester (worth 30%). The final 10% is awarded by the host teacher who writes a few lines of feedback on each student and gives him or her a mark out of 10.

Given that the typical mathematics undergraduate may not be used to these forms of assessment, an hour and a half of the first training session was spent outlining the learning objectives of the module and what was expected from each assessment. An additional part of the assessment required each undergraduate to design a Special Project for use in the school, and to describe this in the final report. Some time in the training session was spent discussing what might be involved in a Special Project.

An hour and a half of the second training session was spent covering the material in Chapters 3 and 4 of the handbook. Chapter 3—*Second-level Mathematics Education in Ireland*—describes the Irish second-level education system, and outlines current issues in second-level mathematics education in this country. Chapter 4—*The Classroom Situation*—outlines the advice given to me by the participating teachers; advises the undergraduates on how to work with the host teacher; and encourages the student to think about possible situations that may arise when working with second-level students.

Chapter 5—*Teaching Mathematics*—discusses points to consider when planning a mathematics lesson. During the first training session, Cathy spent an hour and a half on this topic, and during the second training session the undergraduates were asked to put the theory into practice. In pairs, they had to consider how they would plan a lesson on graphing a line, and then present the lesson to the rest of the class. This proved to be a great success, and a tremendous learning experience for the students.

With the training sessions completed, the module had officially begun.

5. During a UAS Module

As the module coordinator, the demands on my time for the duration of the twelve weeks of the semester were actually quite low. Cathy and I met twice with the students to get some feedback on what type of experiences they were having in the schools, and to discuss any issues or problems which might have arisen. Mid-way through the semester, I also rang each teacher to get some initial feedback on how each student was doing. Two weeks before the semester ended, I wrote to the teachers requesting formal feedback on the undergraduates, and I also invited them to write a testimonial on their UAS experiences. The reflective journals and final reports had to be submitted by the Monday of the last week of the semester, and the presentations were arranged from 5–7pm on the Wednesday of the same week.

If the demands on my time during the second semester were quite low, the same could not be said of the undergraduates! They were requested to spend a minimum of 20 hours in their schools, but almost all spent more than this. All were given the opportunity to observe and assist teachers in a variety of mathematics classes, from first year through to sixth year. Most got the opportunity to teach a whole class or a subset of a class. Two of the undergraduates set-up and ran after-school study groups for Higher Level Junior and/or Leaving Certificate students. One undergraduate also taught (one-on-one) a fifth year student taking Higher Level Leaving Certificate Mathematics a complete topic from the higher level curriculum.

With regard to Special Projects, one of the undergraduates designed a mathematics quiz for first year students and ran it on an interactive whiteboard that the school had just acquired; another designed and ran a mathematics competition with a first year class; and yet another held a sudoku competition with a class. The other three undergraduates designed or developed revision packs or sessions for Junior and Leaving Certificate students.

6. Assessing a UAS Module

In grading the assessments, the main thing I was looking for was that the students had given serious reflection to how they had communicated mathematics to others and developed their transferable skills. Cathy also graded the three assessment components independently, and then we agreed on a mark. For anyone coordinating a UAS module, I feel it is essential that there is a second grader for each assessment—at least until the coordinator gains a few years of experience at grading these types of assessments.

For me the highlight of the module was the students' presentations at the end of the semester. Most of the undergraduates had little experience at making a presentation, and very few had used powerpoint before. At the start of the semester I felt that they were very nervous about this assessment component, and for that reason, I only invited the Head of the UCD School of Mathematical Sciences and the host teachers along (one teacher came). However, despite initial nervousness, the presentations were of an extremely high standard. All had mastered powerpoint, all kept within the allocated 15-minute time-slot, and the confidence, passion and enthusiasm with which they all spoke was overwhelming.

Next year, I would not invite along the host teachers as I would be afraid that an undergraduate, in all sincerity, might express a view about teaching mathematics or school discipline, which a teacher may take umbrage with. However I would invite along any undergraduate interested in taking part in the UAS the following year—the presentations would give them an idea of what to expect and, if this year's standard was anything to go by, would set the bar very high for future students.

In relation to the assessment component graded by the teacher, there was only one instance where, given the teacher's written feedback and comparing it with the written feedback from the other teachers, we felt that the undergraduate deserved a slightly higher mark out of ten and we duly altered it.

Overall, given the rigorous selection process and the small number of students chosen to take part in the module, it was not surprising that in the end, no student received an overall grade lower than a B+.

7. Conclusions

From the point of view of a module coordinator, setting-up and running a UAS module for the first time requires a substantial investment of time and effort, at least in the initial year. However, the UAS website and the UAS manager in the UK, Brian Lockwood, proved invaluable to me in terms of providing materials and support respectively. If any colleague would like to set-up such a module, I am more than happy to send all the materials developed to anyone to use or edit as he or she wishes.

Throughout the article, I have mentioned a few things I would do differently next year, but on the whole I was quite happy with the framework that I employed. Again, I feel this is due to the shared nature of advice and materials on the UAS website—others before me had ironed out many of the major issues and difficulties with running a module of this nature.

From the perspective of the six undergraduates who took part, in an end-of-semester questionnaire all felt that participating in the module had improved their overall key/transferable skills. They were given a list of ten transferable skills, and asked to select the skills which they felt had improved as a result of the module. The results are given in Table 1 below.

Transferable skill	**No. of students**
Communication skills on a one-to-one basis	6
Communication skills with a group	4
Presentation skills	5
Planning skills	4
Team working	2
Time management	5
Ability to prioritise tasks	5
Ability to negotiate	1
Confidence	5
Essay/report writing	4

Table 1

Five of the six undergraduates said they were interested in a career in teaching, with two already accepted onto PGDE programmes. All six said they would recommend the module to others. In an informal discussion with the six undergraduates, they told me that the module may have involved more work than some of their other mathematics modules. However because the work was so different in nature, they really didn't mind. They also liked the fact that there was no final examination involved.

I will conclude this article with some testimonials from the undergraduates and teachers who took part:

"I am in the final year of a Mathematical Studies degree in UCD and have always wanted to be a teacher. This module has given me the opportunity to experience first hand, what teaching mathematics at a secondary level entails and has given me an accurate idea of how much work would be involved in this career choice. I am now more certain than ever that I want to pursue a career in teaching mathematics as I have had a chance to witness, first hand, the lack of teachers in this area and the lack of understanding of maths within the classroom."

Jennifer Keeler, Undergraduate Ambassador 2008

"As a final year student who was thinking about teaching as a career, this module has been invaluable to me. The amount of knowledge I have gained could never be equalled in a lecture situation. The hands-on aspect was so beneficial. The school and I both benefited immensely."

Annette Larkin, Undergraduate Ambassador 2008

"This module gave me a great opportunity to start making the transition into the professional working world. I would strongly recommend this module to anyone interested in teaching, as it helped me develop key skills needed in the classroom."

Rebekah Holmes, Undergraduate Ambassador 2008

"We, at St. Kilian's C.S., have had a very positive first experience of the UAS. Our assigned student developed a Maths Study Club, which our students greatly enjoyed, and from which they benefited greatly. He developed revision material which the students will use while preparing for their J. Cert. examinations. There was excellent communication between the student, the facilitator, and our school which ensured that participation in the scheme was hassle-free."

Mr John Murphy, St. Kilian's CS, Bray

"St. Tiernan's greatly appreciated the interest and support given to our maths students by a young 3rd level student. Motivating our students to have the confidence to pursue maths and science at honours level is in line with our learning objectives in terms of ongoing school development planning."

Mr Declan Hughes, St. Tiernan's, Dundrum

"[Our undergraduate] provided practical assistance by offering after-school classes to a group of third year students sitting honours level in their Junior Cert. She also took a 5th year honours student and covered a complete topic with him over the weeks she was in the school. [She] also provided general classroom assistance which was of great benefit to the students, teachers and school."

Ms Anne Brogan, Killinarden CS, Killinarden

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