Abstracts of PhD Theses at Irish Universities 2007

On the Asymptotic Behaviour of Deterministic and Stochastic Volterra Integro–Differential Equations

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This is an abstract of the PhD thesis *On the asymptotic behaviour of deterministic and stochastic Volterra integro–differential equations* written by Siobhán Devin under the supervision of Dr John Appleby and Dr David Reynolds at the School of Mathematical Sciences, Dublin City University and submitted in March 2007.

This thesis examines a question of stability in stochastic and deterministic systems with memory, and involves studying the asymptotic properties of Volterra integro–differential equations. The type of stability that has been established for this class of equations is important in a variety of real–world problems which involve feedback from the past, and are subject to external random forces. These include modelling endemic diseases, and more particularly the modelling of inefficient financial markets.

The theme of the thesis is to subject a dynamical system with memory to increasingly strong and unpredictable external noise. Firstly, a fundamental deterministic Volterra equation is considered:

$$R'(t) = AR(t) + \int_0^t K(t-s)R(s)\,ds, \quad t > 0; \quad R(0) = I,$$

where the solution R is known as the resolvent, A is a constant $n \times n$-dimensional matrix, K is $n \times n$-dimensional function-valued matrix and I is the $n \times n$-dimensional identity matrix. Necessary and sufficient conditions for the solution to approach a nontrivial limit are known. A strengthened version of these conditions is shown to be necessary and sufficient for exponential convergence to a nontrivial limit.

Next, a Volterra equation with a fading stochastic perturbation is studied:

$$dX(t) = \left[AX(t) + \int_0^t K(t-s)X(s)\,ds\right] dt + \Sigma(t)\,dB(t), \quad t > 0;$$
$$X(0) = X_0.$$

Here Σ is a $n \times d$-dimensional function-valued matrix known as the noise term and the random behaviour is introduced using a d-dimensional Brownian motion B. The influence of the noise fades by allowing $\Sigma(t) \to 0$ as $t \to \infty$.

Two types of stochastic convergence are considered: mean square and almost sure convergence. Conditions are found which ensure that the solution converges to a non-equilibrium random limit. Moreover, the rate at which this limit is approached is established. In the mean square case, necessary and sufficient conditions on the resolvent, kernel and noise are determined to ensure this rate of convergence. In the almost sure case, the same conditions are found to be sufficient; furthermore, it is shown that the conditions on the resolvent and the kernel are necessary. A corresponding result was also found to hold for a more general class of weakly singular kernels. As in the deterministic case, necessary and sufficient conditions for the solution to converge exponentially fast to its limit are found.

Finally, a stochastic Volterra equation with constant noise intensity is considered. This gives rise to the process analogous to Brownian motion, which has applications to mathematical finance. It can be shown that the distribution of the increments of the process converge to a stationary statistical distribution. The conditions under which such convergence can take place are completely characterised. In fact, a solution of a corresponding Volterra equation with infinite memory is shown to have exactly stationary increments which match the limiting distribution of the increments of the general solution.

Convergence Properties of Bimodules over Maximal Abelian Self-adjoint Algebras

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This is an abstract of the PhD thesis *Convergence properties of bimodules over maximal abelian self-adjoint algebras* written by Joseph

Habgood under the supervision of Ivan Todorov at Queen's University Belfast and submitted in September 2007.

Let H_1 and H_2 be separable Hilbert spaces. Given maximal abelian self-adjoint algebras (*masas*) $\mathcal{D}_1 \subseteq B(H_1)$ and $\mathcal{D}_2 \subseteq B(H_2)$ and a $\mathcal{D}_2, \mathcal{D}_1$-bimodule (*masa-bimodule*) $\mathcal{U}$ the *projection bilattice* of $\mathcal{U}$ is the set

$$\mathrm{bil}\mathcal{U} = \{(P_1, P_2) \in \mathrm{Proj}\mathcal{D}_1 \times \mathrm{Proj}\mathcal{D}_2 \,:\, P_2\mathcal{U}P_1 = 0\}.$$

If the masas are given coordinate representations $\mathcal{D}_1 = L^\infty(X, m)$ (resp. $\mathcal{D}_2 = L^\infty(Y, n)$) for some standard Borel spaces (X, m) and (Y, n) then the *support* of $\mathcal{U}$ is defined, up to marginal equivalence, by taking a countable strongly dense subset $\mathcal{S} \subseteq \mathrm{bil}\mathcal{U}$ and putting

$$\mathrm{supp}\mathcal{U} = \Big(\bigcup_{(M_{\chi_\alpha}, M_{\chi_\beta}) \in \mathcal{S}} \alpha \times \beta \Big)^c.$$

Indeed, any subset of $X \times Y$ whose complement is marginally equivalent to a countable union of Borel rectangles is the support of some masa-bimodule; such sets are said to be *ω-closed*.

If $\kappa \subseteq X \times Y$ is an ω-closed set then there is a largest weak* closed masa-bimodule whose support is κ, denoted by $M_{max}(\kappa)$; and, less obviously, a smallest weak* closed masa-bimodule whose support is κ, which is denoted by $M_{min}(\kappa)$.

The main results of this thesis are (semi-)continuity theorems for the map sending a masa-bimodule to its support and the maps which take an ω-closed set $\kappa \subseteq X \times Y$ to $M_{max}(\kappa)$ and $M_{min}(\kappa)$. For a precise meaning of the word continuity the reader is referred to [2]. It suffices here to note that continuity theorems of this kind have been obtained by several authors in the past, notably: Haagerup and Winslow's [1] result on the continuity of the map sending a von Neumann algebra to its commutant; and, in the setting of non-self-adjoint operator algebras, the various results of Shulman and Todorov [3] on the continuity of the map sending a weak* closed unital operator algebra to its invariant subspace lattice. Indeed, it is easy to extend a continuity theorem of [3], for operator algebras that contain a masa, to show that the map sending a weak* closed masa-bimodule to its projection bilattice is continuous. Difficulties seem to arise however in translating this into a useful continuity result for the support.

This problem is addressed by introducing a natural topology on the collection of ω-closed sets in terms of a family of set-functions on

$X \times Y$. The collection of ω-closed sets, endowed with this topology, is homeomorphic to the collection of weakly closed convex hulls of projection bilattices endowed with a convergence derived from the weak operator topology. This fact is used to show that, with this topology on the codomain and a convergence on the domain derived from the weak* and strong* topologies on $B(H_1, H_2)$, the map sending a bimodule to its support is continuous.

On the other hand, the maps M_{max} and M_{min} are both shown to be discontinuous, with discontinuities occurring at the 'non-synthetic' ω-closed sets (these are the ω-closed sets κ at which $M_{max}(\kappa)$ and $M_{min}(\kappa)$ differ). Semi-continuity results are obtained for these maps: M_{max} is upper-semi-continuous and, in a weaker sense, M_{min} is lower-semi-continuous. In the special case when the bimodules are ranges of weak* continuous masa-bimodule projections all relevant ω-closed sets are synthetic, hence the discontinuities mentioned above do not occur. Using different techniques, specific to this situation, a related continuity theorem is obtained.

References

[1] U. Haagerup and C. Winslow, *The Effros–Maréchal topology in the space of von Neumann algebras,* Amer. J. Math. **120** (1998), 156–617.

[2] J. L. Habgood and I. G. Todorov, *On convergence of bimodules over maximal abelian self-adjoint algebras,* to appear in J. Operator Theory.

[3] V. S. Shulman and I. G. Todorov, *On subspace lattices II. Continuity of lat,* J. Operator Theory **52** (2004), 371–384.

Degree Patterns of Projective Representations of Finite Groups

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This is an abstract of the PhD thesis *Degree patterns of projective representations of finite groups* written by Donal Healy under the supervision of Dr Russell Higgs at the School of Mathematical Sciences, University College Dublin and submitted in November 2006.

The study of projective representations was initiated by Schur [6] in 1904, and the subject received renewed impetus in 1937, when Clifford [1] showed that projective representations and projective characters arise naturally in the study of ordinary representations and characters of a finite group G, and in particular in the restriction of such characters to normal subgroups of G.

It is very well known that if all the ordinary characters of a group G have equal degree, then G is abelian. We consider the projective case, that is, if a group G has a fixed 2-cocycle α such that the set of irreducible α-projective characters of G all have the same degree, then is G solvable? This has been an open conjecture for at least the last twenty years. The main result of this thesis verifies this conjecture in one special case.

In 1964 Iwahori and Matsumoto [4] conjectured that a group G with a fixed 2-cocycle α such that G has exactly one irreducible α-projective character is solvable. This was verified by Howlett and Isaacs [3] in 1982 after much work was done by Liebler and Yellen [5] in a paper published in 1979.

In [2], Higgs proved that a group G with a fixed 2-cocycle α such that G has exactly two irreducible α-projective characters is solvable. The case where G has exactly three irreducible α-projective characters is an open conjecture. This thesis considers the more restricted case that G has exactly three irreducible α-projective characters all of the same degree and we prove that G is solvable under this hypothesis.

In the thesis we provide the structure of the Schur multiplier for each group of order less than 128. We also give one covering group for most groups of order less than 128 and list all the different covering groups for nearly all groups of order less than 45. We also present a Magma program for computing projective character tables of groups of order less than 32.

Next we work with projective character degree patterns of non-abelian groups of order p^n for p an odd prime. We deal initially with the case $n = 4$, and establish the structure of the Schur multiplier of the four groups of order p^4 not available in the existing literature. We also find all the groups of order p^4 which realise all their possible projective character degree patterns for non-trivial cohomology classes. We then prove that no group of order p^n for $n > 4$ realises all its projective character degree patterns for non-trivial cohomology classes.

Finally we introduce a concept that will play a pivotal role in the rest of the thesis, that is, a group G with a fixed 2-cocycle α such that G has exactly three irreducible α-projective characters of equal degree. We say that such a group is of 3α-central type. We prove that the property that a group G is of 3α-central type is 'inherited' by the Sylow 3-subgroup of G as well as certain factor groups of G under certain conditions. Finally we prove using the Classification of the Finite Simple Groups that a group G of 3α-central type is solvable.

References

[1] A. H. Clifford. Representations induced in an invariant subgroup. *Ann. of Math. (2)*, 38:533–550, 1937.
[2] R. J. Higgs. Groups with two projective characters. *Math. Proc. Cambridge Philos. Soc.*, 103:5–14, 1988.
[3] R.B. Howlett and I.M. Isaacs. On groups of central type. *Math. Z.*, 179:555–569, 1982.
[4] N. Iwahori and H. Matsumoto. Several remarks on projective representations of finite groups. *J. Fac. Sci. Univ. Tokyo Sect. I*, 10:129–146, 1964.
[5] R.A. Liebler and J.E. Yellen. In search of nonsolvable groups of central type. *Pacific J. Math.*, 82:485–492, 1979.
[6] I. Schur. Über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen. *J. Reine Angew. Math.*, 127:20–50, 1904.

Difference Sets with Classical Parameters in Abelian Groups

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This is an abstract of the PhD thesis *Difference Sets with Classical Parameters in Abelian Groups* written by Kevin Jennings under the supervision of Rod Gow at the UCD School of Mathematical Sciences, University College Dublin and submitted in April 2007.

Let G be a group of order v. Let $D = \{d_1, d_2, \ldots, d_k\}$ be a k-subset of G. Then D is a (v, k, λ)-difference set for G if each non-identity $g \in G$ can be expressed in exactly λ ways as a product $g = d_i d_j^{-1}$ where d_i and d_j are elements of D.

If G is abelian and additive notation is used, the condition can be read that each non-zero $g \in G$ has exactly λ expressions as a *difference* $g = a - b$ where $a, b \in D$. Hence the origin of the term 'difference set'. For example, the set $\{1, 2, 4, 10\}$ is a difference set in $\mathbb{Z}_{13}$ as can be easily verified by computing the differences modulo 13.

Difference sets were introduced by James Singer in 1938. His family of difference sets have parameters

$$(v, k, \lambda) = \left(\frac{q^d - 1}{q - 1}, \frac{q^{d-1} - 1}{q - 1}, \frac{q^{d-2} - 1}{q - 1}\right)$$

where $d \geq 2$ and q is a prime power. Any difference sets with these parameters are said to have classical parameters and there are several known families. These difference sets correspond to sequences with ideal autocorrelation properties and are currently (2007) of interest to applied mathematicians. There are several enticing but extremely difficult conjectures on these difference sets. Firstly, it is conjectured that q must be a prime power. Secondly, if such a difference set is in an abelian group, it is conjectured that the group must be cyclic. Such questions are not addressed in this thesis.

In this thesis we investigate difference sets with classical parameters in abelian groups. In particular, we study how a difference set in a group interacts with subgroups and consequently how the parameters of a difference set can restrict the structure of the underlying group. The thesis should be accessible to any mathematics graduate as the arguments used are relatively elementary. The group theoretic arguments are delicate but since we are working in abelian groups, the problems are more combinatoric by nature. There is a geometric interpretation underlying the work but we have not drawn insight from this and it is rarely called upon here, although our results are sometimes stated with the geometry in mind.

Our first two chapters introduce the topic and describe the classical constructions of difference sets, with the necessary techniques from field theory and linear algebra.

Chapter 3 involves Hall's multipliers and we reach here our first result, proving the existence of a subgroup inside a certain difference set. In Chapter 4 we generalise this observation, employing the more technical tools of the Mann Test. We also prove that the Sylow 2-subgroup often must be cyclic for an abelian group to support a classical difference set.

In Chapter 5 we focus on the family of difference sets with parameters of the 3-dimensional finite projective geometries. These are one dimension up from projective planes and we can discern precise details of their structure from the mere parameters of these objects.

In Chapter 6 we present our main result that a planar difference set is embedded in a natural but not obvious way inside a difference set with the parameters of a 5-dimensional projective geometry.

Models Of Rimming Flows

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This is an abstract of the PhD thesis *Models Of Rimming Flows* written by Seán M. Lacey under the supervision of Prof. Eugene Benilov and Prof. Stephen O'Brien at the Department of Mathematics and Statistics, University of Limerick and submitted in October 2007.

The dynamics of a thin film of viscous fluid on the inside of a cylinder with horizontal axis, rotating about this axis, are examined in this thesis. The stability of this film has been previously explored using the leading order lubrication approximation, under which it was found to be neutrally stable.

In this thesis, the effect of higher-order corrections (such as inertia, described by the material derivatives in the Navier–Stokes equations, surface tension, and the hydrostatic pressure gradient) on the stability of the film is examined. Assuming that these correction terms are weak, an asymptotic equation is derived which takes into account these effects as perturbations. This equation is used to examine the stability of the steady-state distribution of the film around the cylinder (rimming flow) with respect to linear disturbances with harmonic dependence on the axial variable and on time (normal modes).

It has been shown by [3], in two-dimensional motion, that the hydrostatic pressure gradient does not affect the stability of normal modes at all, and the effect of surface tension is weak — whereas inertia always causes instability. This thesis will investigate the effect of these three higher-order corrections terms in three-dimensional motion.

The leading order three-dimensional case has been investigated by [1], where it was shown that there are infinitely many normal modes, which are neutrally stable and where the eigenfunctions form a complete set. [1] also showed that the film is nonetheless unstable with respect to non-harmonic disturbances, which develop singularities in a finite time. [2] coined the phrase 'explosive' instability to describe these singularities.

In [2], the explosion occurred in the azimuthal direction, while in [1], it occurred in the axial direction. In this thesis, both the azimuthal and axial components of the hydrostatic pressure gradient are taken into account. One aim of the thesis is to determine if the film thickness of the liquid can explode in both directions, and if so, can the characteristics of this type of explosion be deduced. It will also be shown that two types of explosions can occur — i.e., explosions with zero net mass and explosions with non-zero net mass.

References

[1] Benilov E.S. 2004 *Explosive instability in a linear system with neutrally stable eigenmodes. Part 2: Multi-dimensional disturbances*, J. Fluid Mech. **501**, 105–124.

[2] Benilov E.S., O'Brien S.B.G., Sazonov I.A. 2003 *A new type of instability: explosive disturbances in a liquid film inside a rotating horizontal cylinder*, J. Fluid Mech. **497**, 201–224.

[3] Benilov E.S., O'Brien S.B.G. 2004 *Inertial instability of a liquid film inside a rotating horizontal cylinder*, Phys. Fluids **17**, 052106.

Gauss' Method for the Determination of Cyclotomic Numbers

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This is an abstract of the PhD thesis *Gauss' Method for the Determination of Cyclotomic Numbers* written by Ciarán Mac an Bhaird under the supervision of Dr. Pat McCarthy at the Department of Mathematics, NUI Maynooth and submitted in October 2007.

In this work we have shown that Gauss' method for the determination of Cyclotomic numbers leads to a series of functional equations. We then obtain necessary and sufficient conditions for these functional equations to have integer solutions. This leads to a finite Diophantine system — the number of equations is independent of the prime p. We have shown that this purely Diophantine system has precisely $\phi(l)$ solutions which correspond to the Cyclotomic numbers of order l, where $p \equiv 1 \bmod l$.

This is the first purely Diophantine characterisation of the cyclotomic numbers and the coefficients of the minimal polynomial of the

Gaussian periods, and in fact, the problem is solved for all orders. A purely Diophantine description has the advantage that if one obtains alternative formulae for the Cyclotomic numbers, the result can be proved by simply verifying that these formulae satisfy the system of equations. There are other descriptions of the general Cyclotomic numbers which involve Diophantine systems, but these descriptions all employ a rejection criterion and so cannot be considered to be purely Diophantine.

Most of the work involves proving that the Galois group of a related polynomial acts cyclically on its roots and therefore the polynomial is irreducible. It is then not too difficult to show that the polynomial is in fact the minimal polynomial of the Gaussian Periods.

It was commonly believed that Gauss' method for the determination of Cyclotomic Numbers, and thus the determination of the minimal polynomial of Gaussian Periods, was unwieldy for the general case. See for example, the remarks of André Weil [1]. The prevailing wisdom seemed to be that the determination of the minimal polynomial of the Gaussian Periods, using Jacobi Sums etc., should be done first and then the Cyclotomic numbers be determined as a consequence. In fact, this was suggested by Weil [1]. It appeared that Gauss' original method was abandoned.

We have shown that Gauss' method works. In view of Weil's comments, it is interesting that Jacobi sums appear nowhere in our argument and furthermore, that the determination of the Cyclotomic numbers can be taken as a starting point for the determination of the Gaussian periods. This of course was Gauss' motivation for his work on Cyclotomic numbers of orders 3 and 4.

We demonstrate the beauty of this method by determining the Cyclotomic numbers

$$\frac{c(\mathbf{e}_0 + \mathbf{e}_i + \mathbf{e}_j)}{\mu}$$

of order 5. We have also characterised the algebraic variety for all values, there are, in fact $\phi(l)$ integer points on a torus of dim $\frac{l-1}{2}$.

REFERENCES

[1] A. Weil, *Numbers of solutions of equations in finite fields.* Bull. A.M.S., v. 55 (1949), 497–508.

The Paradigm Shift from Euclid to a Composite System of Geometry in Intermediate Certificate Mathematics in Ireland, 1966–1973

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This is an abstract of the PhD thesis *The Paradigm Shift from Euclid to a Composite System of Geometry in Intermediate Certificate Mathematics in Ireland, 1966–1973*, written by Susan M. C. Mac Donald under the supervision of Dr Richard O. Watson at the Department of Mathematics, National University of Ireland, Maynooth and Prof. Emeritus John Coolahan at the Department of Education, National University of Ireland, Maynooth, and submitted in February 2007.

For over a twenty-one year period, from the mid 1960s, logically flawed geometry was prescribed to over one million second-level students in Ireland.

A change took place in second-level geometry in Ireland in the mid twentieth century that was influenced by international and national factors including the reaction to an OECD-promulgated declaration that 'Euclid must go!' [1] and the Department of Education's distinctive approach to geometry development. It resulted in the replacement of a syllabus based on the 2300-year-old synthetic geometry of Euclid by one based on a foundationally imperfect, composite system of geometry consisting partly of the traditional geometry of Euclid and partly of the transformation geometry of the Belgian mathematician Georges Papy [2]–[4].

This change was a paradigm shift that came about in two main stages with the introduction of a new Intermediate Certificate Mathematics syllabus in 1966 and again in 1973. In each stage, parts of the geometry of Papy were combined with some parts of traditional synthetic geometry; the difficulty was that the parts did not cohere, and the result, in particular in the 1973 syllabus, was an approach to geometry without a solid, logical foundation, having flawed definitions, axioms and proofs.

The methods employed that brought about this paradigm shift were determined by a system of syllabus development that was unsuited to the task of preparing second-level mathematics syllabuses.

In particular, the lack of formal university representation on Intermediate Certificate syllabus committees together with the control of syllabus committees exercised by the Irish Government Department of Education resulted in the adoption of the defective geometry syllabuses.

Ramifications of these syllabuses remain evident in second-level geometry in Ireland: successive syllabuses have been influenced and expertise in traditional deductive geometry will soon disappear.

REFERENCES

[1] OEEC, New Thinking in School Mathematics, Paris: OEEC, 1961.
[2] Papy, G., Methods and Techniques of Explaining New Mathematical Concepts in the Lower Forms of Secondary Schools, in: Mathematics To-day: A Guide for Teachers, OECD, 99–147, Paris: OECD, 1964.
[3] Papy, G., Modern Mathematics: Volume 1, London: Collier-Macmillan, 1968.
[4] Papy, G., Modern Mathematics: Volume 2, Real Numbers and the Vector Plane, London: Collier-Macmillan, 1968.

A Priori Bounds on Derivatives of Solutions to Singularly Perturbed Convection-Diffusion Problems

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This is an abstract of the PhD thesis *A priori bounds on derivatives of solutions to singularly perturbed convection-diffusion problems* written by Aidan Naughton under the supervision of Prof. Martin Stynes at the School of Mathematical Sciences, University College Cork and submitted in September 2006.

Mathematical models involving convection and diffusion occur frequently throughout science, engineering and economics. Examples range from fluid flows to oil extraction and the Black–Scholes financial model (see [3, Chapter 1] for an extensive list of examples). Often the diffusion term is quite weak relative to the convection term. Consider the example of pollutant flowing into a river given in [4, Introduction]. The random motion of the molecules of water and pollutant will cause the pollutant to diffuse through the river. Obviously the effects of this process will be dwarfed by the convective current in a fast flowing river. To reflect its relative weakness a small parameter ε multiplies the diffusion term. This is a singularly perturbed convection-diffusion problem. (A historical survey of singularly perturbed problems is given in [5]).

This thesis is concerned with finding sharp a priori bounds on derivatives of solutions to singularly perturbed convection-diffusion problems. Such bounds are of great importance to numerical analysts for the construction of numerical methods and for error analysis.

The thesis commences with the analysis of some one-dimensional problems. Both convection-diffusion and reaction-diffusion problems are dealt with. The methods used are short, relatively simple and result in sharp bounds.

The focus then moves to convection-diffusion problems posed in two-dimensional domains. A two-dimensional domain is more realistic in a physical sense and therefore of greater interest, but it does introduce several complications not present in one-dimensional problems. One such issue in rectangular domains is the effect of compatibility of the data of the problem at the corners of the domain. A convection-diffusion problem posed on the unit square, with Dirichlet boundary conditions, is considered under the assumption of compatibility at the corners. Then two further problems are analysed (one with Neumann outflow boundary data, one with Neumann characteristic boundary data) without any assumption of data compatibility at the corners. Thus corner singularities are possible and the interaction of these singularities with the singularly perturbed nature of the differential operator is a challenge to analyse. The techniques used here are a continuation of those used in [1] and [2]. Sharp pointwise bounds are derived in all cases.

References

[1] A. Brandt. Interior estimates for second-order elliptic differential (or finite-difference) equations via the maximum principle. *Israel J. Math.*, 7:95–121, 1969.

[2] R. Bruce Kellogg and Martin Stynes. Corner singularities and boundary layers in a simple convection-diffusion problem. *J. Differential Equations*, 213(1):81–120, 2005.

[3] K. W. Morton. *Numerical solution of convection-diffusion problems*, volume 12 of *Applied Mathematics and Mathematical Computation*. Chapman & Hall, London, 1996.

[4] H.-G. Roos, M. Stynes, and L. Tobiska. *Numerical methods for singularly perturbed differential equations*, volume 24 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1996.

[5] Shagi-Di Shih and R. Bruce Kellogg. Asymptotic analysis of a singular perturbation problem. *SIAM J. Math. Anal.*, 18(5):1467–1511, 1987.

Cylindrically Symmetric Models of Gravitational Collapse

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This is an abstract of the PhD thesis *Cylindrically Symmetric Models of Gravitational Collapse* written by Louise V. Nolan under the supervision of Dr. Brien Nolan at the School of Mathematical Sciences, Dublin City University and submitted in September 2007.

In this thesis we examine two main problems. Firstly, we attempt to match the most general cylindrically symmetric vacuum spacetime with a Robertson–Walker interior. The matching conditions show that the interior must be dust filled, the boundary must be comoving and the vacuum region must be polarized. We use a result of Thorne's to simplify the line element. We can then prove that the matching is impossible. This demonstrates the impossibility of generalising the Oppenheimer–Snyder model of gravitational collapse to the cylindrically symmetric case. The second problem is an analysis of cylindrically symmetric spacetimes with self-similarity modelling gravitational collapse. The field equations and regularity conditions are examined firstly for a vacuum spacetime and then for a dust filled spacetime. The vacuum case leads to an explicit solution but no solutions that are of relevance to gravitational collapse. In the dust case, the solution of the field equations reduces to the solution of a non-linear third-order ordinary differential equation. A dynamical systems approach is then adopted, and an autonomous three-dimensional system is obtained. A unique solution is found to emanate from the regular axis $\{r = 0, t < 0\}$, where t and r are time and radial coordinates which emerge naturally from the analysis. This solution persists up to $\{t = 0, r > 0\}$, which we define as Σ_0. The solution coming from Σ_0 has one parameter (a bifurcation has occurred) and propagates up to the future null cone, $\mathcal{F}$, through the scaling origin p_o, where $p_o = \{(r, t) = (0, 0)\}$. We describe the physical invariants of the system and discuss the nature of such a spacetime in terms of its global structure.

Levels and Sublevels of Composition Algebras

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This is an abstract of the PhD thesis *Levels and Sublevels of Composition Algebras* written by James O'Shea under the supervision of Dr. Thomas Unger at the School of Mathematical Sciences, University College Dublin and submitted in June 2007.

Although quadratic forms can be traced back to the era of the Babylonians, the development of a theory regarding their algebraic properties has been a relatively recent phenomenon. Ernst Witt laid the foundations for this theory in a seminal work dated 1937. That it subsequently flourished however, is primarily due to Albrecht Pfister's elegant classification of the level of fields, some three decades later.

The level is an important invariant in the theory, measuring the least number of squares in a given algebraic structure which sum to -1. Interest first arose in this concept in 1927, with its appearance in the Artin–Schreier Theorem, a key stepping-stone result to Artin's solution of Hilbert's Seventeenth Problem that same year.

Quaternion algebras represent the simplest algebraic structure for which a complete classification of the level remains outstanding. It was David Lewis who first considered this topic, in the mid-eighties, showing the existence of quaternion algebras of level 2^k and $2^k + 1$ for all integers $k \geq 0$.

This thesis is similarly concerned with addressing this problem. In addition, since the level has inequivalent generalisations in a non-commutative setting, we additionally regard the classification of the sublevel. Furthermore, given the striking similarities between sums of squares in quaternion and octonion algebras, it makes sense to gather these structures together under the umbrella of composition algebras, and treat the classification of their levels and sublevels in tandem.

We tackle these problems in a variety of manners, from the exploration of the relationship between bounds on the level and sublevel of composition algebras and the behaviour of associated quadratic forms, through to the explicit construction of algebras of prescribed, and hitherto unknown, level and sublevel values, via function fields of quadratic forms. Moreover, we additionally devote attention to the classes of composition algebras which have transcendental generators, as well as to those defined over certain fields.

REFERENCES

[1] D. W. Lewis, Levels of quaternion algebras, *Rocky Mountain J. Math.* **19**, 787–792 (1989).

[2] J. O'Shea, Levels and sublevels of composition algebras, *Indagationes Mathematicae* **18**, no. 1, 147–159 (2007).

[3] J. O'Shea, New values for the level and sublevel of composition algebras, preprint (see http://www.math.uni-bielefeld.de/LAG/man/240.pdf) (2007).

[4] S. Pumplün, Sums of squares in octonion algebras, *Proc. Amer. Math. Soc.* **133**, 3143–3152 (2005).

A Model for Wave Formation in Bubbly Flows

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This is an abstract of the PhD thesis *A Model for Wave Formation in Bubbly Flows* written by Marguerite Robinson under the supervision of Professor Stephen O'Brien at the Department of Mathematics and Statistics, University of Limerick, and submitted in September 2007.

When a gas-liquid mixture flows through a vertical pipe the two phases can arrange themselves into different patterns known as flow regimes. At low gas flow rates small gas bubbles are dispersed in the liquid (bubbly flow) and at higher flow rates large slugs of gas separate liquid regions (slug flow). Modelling bubbly flows is complicated by the presence of complex characteristics and the resultant ill-posedness of the two-fluid equation system, which is thought to herald the bubble-to-slug regime transition (Pauchon and Banerjee, 1986). Laboratory experiments of gas-liquid flows have identified a relationship between this transition and instability of the uniform bubbly flow, leading to wave formation (Kytomaa and Brennen, 1991). However, it is not clear from experiments if the instability leads to wave formation or a regime transition.

In this thesis we examined the onset and propagation of waves in a bubbly flow, in the context of a properly constituted hyperbolic two-fluid model. We focused specifically on the downward travelling waves seen in a glass of Guinness shortly after pouring and derived a condition for a well-posed system. We showed that our simplified model is, in a mathematical sense, a generalization of the St. Venant equations used in hydraulic engineering to model downstream periodic travelling waves (known as roll-waves) in open channels (Dressler, 1949). We showed that waves in bubbly flows are multi-phase analogues of the single-phase roll-wave phenomenon and result from the manifestation of a similar instability of the uniform flow.

Our model predicts that the uniform bubbly flow will always become unstable before the onset of complex characteristics, and wave generation in the bubbly flow can be seen as a precursor to a transition to the slug flow regime. Observations of high speed digital video clips of the Guinness settling process indicate waves of bubble-free liquid, which appear as a series of dark lines travelling down the inside of the glass. These pure liquid regions are a manifestation of a regime transition and support our finding that wave formation, resulting from the instability, precedes the transition.

References

[1] R. F. Dressler (1949) Mathematical solution of the problem of roll-waves in inclined open channels. Comm. Pure Appl. Maths. **2**, 149–194.

[2] C. Pauchon, S. Banerjee (1986) Interphase momentum interaction effects in the averaged multifield model. Part I: Void propagation in bubbly flows. Int. J. Multiphase Flow **12**, 559–573.

[3] H.K. Kytomaa, C.E. Brennen (1991) Small amplitude kinematic wave propagation in two-component media. Int. J. Multiphase Flow **17**, 13–26.

[4] M. Robinson, A.C. Fowler, A.J. Alexander, S.B.G. O'Brien (2007) Waves in Guinness. Physics of Fluids, submitted.

Ordinary Differential Equation Models of Opiate-Use: The Treatment-Relapse Cycle and HIV Infection

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This is an abstract of the PhD thesis *Ordinary Differential Equation (ODE) Models of Opiate-Use: The Treatment-Relapse Cycle and HIV Infection* written by Emma White under the supervision of Dr. Catherine Comiskey at the Department of Mathematics, National University of Ireland, Maynooth and submitted in October 2007.

Problem opiate-use is globally acknowledged to be a large-scale problem with serious negative effects for individuals and society [1],

[2], [3], [4]. Latest figures estimate the problem opiate-using population in Europe at 1.7 million people [1] and in Ireland at 14,000 people [5], [6]. This research presents a first-order non-linear ODE system that models problem opiate-use. It is based on the drug-using career concept, which includes a treatment-relapse cycle and removal from the opiate-using population. An important assumption is that contact with problem opiate-users not in treatment results in relapse to opiate-use. The basic reproduction ratio, R_0, a measure of the number of secondary cases arising from a single infection introduced into a susceptible population, is derived. Sensitivity analysis is then used to identify the most important parameter on which it depends. Values of R_0 less than, equal to and greater than its threshold value of one are used to perform detailed local stability analysis for the system. Parameters for the Irish population are then estimated. A numerical simulation is carried out to verify the validity of the model by establishing its fit with available epidemiological data. A key result derived from the model is that prevention of opiate-use is more effective in reducing prevalence than treatment. Since problem opiate-users are known to be at high risk of HIV infection [1], the model is extended to include HIV positive opiate-users and the risk that HIV negative opiate-users may acquire HIV. R_0 is calculated and the extended model is applied, using European data, to two populations of opiate-injecting drug-users. The impact of opiate-use treatment effectiveness rates on HIV incidence is examined, as is the impact of increased HIV testing independent of opiate treatment services. It is concluded that high relapse rates from treatment for opiate-use do not increase HIV incidence (and thus prevalence) and that increasing HIV testing rates for opiate-users outside problem opiate-use treatment reduces HIV incidence, irrespective of initial HIV prevalence.

References

[1] European Monitoring Committee for Drugs and Drug Addiction (EMCDDA): Annual Report, 2006: the state of the drugs problem in Europe. *EMCDDA, 2006*, `http://ar2006.emcdda.europa.eu/download/ar2006-en.pdf`

[2] Goldstein, A.: Addiction, From Biology to Drug Policy. *Oxford University Press, UK, 2001.*

[3] United Nations Office on Drugs and Crime (UNODC): World Drug Report, 2007. *UNODC, 2007.*

[4] World Health Organisation (WHO): Bi-regional Strategy for Harm Reduction, 2005-09, HIV& Injecting Drug Use. *WHO, 2005.*

[5] Comiskey, C. M.: Methods for estimating prevalence of opiate use as an aid to policy and planning. *Substance Use and Misuse: Emerging Issues for the 21st Century, Vol. 36, No. 1 & 2, 131-151. May 2001.*

[6] Kelly, A., Carvalho, M. and Teljeur, C.: Prevalence of Opiate Use in Ireland 2000–2001. A 3-source capture recapture study. *Report to the NACD. Small Area Health Research Unit, Dept. of Public Health & Primary Care, Trinity College Dublin, 2003.*