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EDITORIAL

It is very positive to see that the number of mathematical conferences organised in Ireland continues to grow steadily and the variety of topics expands as well. The column announcing such conferences in the Bulletin *is no true reflection of these activities, since, alas, few organisers care to inform the editor of their plans in time for inclusion. In fact, it may be time to close this section altogether and leave the announcements to much faster media like the web; after all the Society has a dedicated webpage at*

`http://www.maths.tcd.ie/pub/ims/Calendar-ie/`

for this purpose.

The contributions to these meetings are, however, a wonderful source of survey papers which can give the non-specialist insights into novel developments in all areas of mathematics, pure and applied. Once again I would like to ask the organisers of conferences to bring this possibility to the attention of their speakers: the Bulletin *is always looking for good survey articles. These need not to be long and treat a subject comprehensively. Often a shorter paper focussing on some recent exciting new directions can catch the eye, and can be digested, more easily. Fortunately, this issue contains an example of a well-written and informative survey paper.*

The new section on abstracts of PhD theses has started well in the previous issue, and I hope that supervisors will continue to encourage their students to submit an abstract to the Bulletin *following the instructions which, together with a template, are on the IMS website. Please do adhere to the page limit.*

—MM

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1. The Irish Mathematical Society has reciprocity agreements with the American Mathematical Society, the Irish Mathematics Teachers Association, and the Real Sociedad Matemática Española.

2. The current subscription fees (as from 1 January 2002) are given below:

Institutional member	130 euro
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5. Any ordinary member who has reached the age of 65 years and has been a fully paid up member for the previous five years may pay at the student membership rate of subscription.

6. Subscriptions normally fall due on 1 February each year.

7. Cheques should be made payable to the Irish Mathematical Society. If a Eurocheque is used then the card number should be written on the back of the cheque.

8. Any application for membership must be presented to the Committee of the I.M.S. before it can be accepted. This Committee meets twice each year.

9. Please send the completed application form with one year's subscription to:

   The Treasurer, I.M.S.
   Department of Mathematics
   National University of Ireland
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ANNOUNCEMENTS OF MEETINGS AND CONFERENCES

This section contains announcements of meetings and conferences as supplied by organisers. The Editor does not take any responsibility for the accuracy of the information provided.

LMS Workshop on Motives, Quadratic Forms and Algebraic Groups

Queen's University Belfast

August 27–31, 2007

The workshop is planned to bring together experts, young and old, on various aspects of research in Chow motives, quadratic forms and algebraic groups, as well as graduate students, postdocs and others who wish to learn about the subject areas. It is organised by Roozbeh Hazrat and supported by the London Mathematical Society.

Further information will be posted on the conference website
`http://queensworkshop.googlepages.com/`

Elliptic Curve Cryptography Workshop 2007

University College Dublin

September 5–7, 2007

The workshop is hosted by the Claude Shannon Institute for Discrete Mathematics, Coding and Cryptography at University College Dublin. For registration (free for students and postdocs) see the conference webpage. On the evening of Tuesday 4 September at 7.30, Joseph H. Silverman will deliver a special public lecture entitled "The Ubiquity of Elliptic Curves".

ECC 2007 is the eleventh in a series of annual workshops dedicated to the study of elliptic curve cryptography and related areas. Over the past years the ECC conference series has broadened its scope beyond elliptic curve cryptography and now covers a wide range of areas within modern cryptography. As with past ECC conferences, there will be about 15 invited lectures (and no contributed talks) delivered by internationally leading experts. There will be both state-of-the-art survey lectures as well as lectures on latest research developments.

The workshop is sponsored by Intel, Certicom and the Claude Shannon Institute. The local organiser is Gary McGuire; for the full list of scientific organisers and all other conference details see the website at

http://www.shannoninstitute.ie/conferences.htm

Operator Theory and Operator Algebras in Cork.
In Memory of Gerard J. Murphy.
University College Cork
May 7–9, 2008

A three-day conference to commemorate the life and mathematical achievements of Gerard J. Murphy who was a lecturer and professor of Mathematics at University College Cork from 1984 until his untimely passing in October 2006 will be held in the National University of Ireland, Cork focusing on operator theory and operator algebras, the two areas in which Gerard made major contributions. There will be plenary talks by two principal speakers and a number of invited talks by other participants, the emphasis being on modern developments in these fields.

The principal speakers are

Laurent Marcoux, University of Waterloo, Canada

Ryszard Nest, University of Copenhagen, Denmark

The Scientific Committee consists of M. Mathieu (QUB), R. M. Timoney (Trinity College Dublin) and S. Wills (UCC), and the Local Organising Committee is formed by Donal Hurley and Stephen Wills.

For further conference details please see
http://euclid.ucc.ie/pages/staff/wills/GJMconf/home.html

Fifth European Congress of Mathematics
Amsterdam
July 14–18, 2008

The Fifth European Congress of Mathematics (5ECM) will be organized in Amsterdam, from 14 to 18 July 2008, under the auspices of the European Mathematical Society. This congress is the fifth in a series of successful four-yearly European congresses that cover the whole range of the mathematical sciences, from pure to applied. The series started in Budapest, in 1992, followed by meetings in Paris (1996), Barcelona (2000), and Stockholm (2004). The ECM congresses alternate with the IMU world congresses, organized every (2 mod 4) year.

Next year's ECM congress will be organized under the special patronage of the Koninklijk Wiskundig Genootschap (Royal Dutch Mathematical Society, KWG), and will include the yearly meeting of the members of KWG. The 5ECM Local Organizing Committee consists of André Ran (Free University Amsterdam, chairman), Herman te Riele (CWI Amsterdam, secretary), and Jan Wiegerinck (University of Amsterdam, treasurer).

An outstanding Scientific Committee with representatives from all over Europe, chaired by Lex Schrijver (CWI and University of Amsterdam), has composed an interesting scientific program consisting of ten plenary lectures, three (also plenary) science lectures, about thirty (parallel) invited lectures, and twenty-one (parallel) Minisymposia. In addition, ten prize lectures will be presented by outstanding young European mathematicians, selected by a Prize Committee chaired by Rob Tijdeman (Leiden University).

For more information on the conference, such as grants, up-to-date information on the program, and for registration, please visit our website at

www.5ecm.nl

Gerard J. Murphy (1948–2006)

FINBARR HOLLAND

1. Introduction

Following an illness that lasted for about one year, Gerard Murphy, MRIA, Associate Professor of Mathematics at University College, Cork, died on October 12, 2006, of colonic and liver cancer. What follows is an account of his life and scholarly work, that is based on information given to me by his wife, Mary, his sister, Carol, Des MacHale, David Simms, Roger Smyth, Richard Timoney, Trevor West and Stephen Wills, to whom I express thanks.

2. The Early Years

Gerard John Murphy was the first-born of Mary and Laurence Murphy, a window-cleaning contractor. He was born on November 12, 1948, and had two brothers and five sisters. The family resided in Drimnagh, Dublin 12, and Gerard and his siblings attended their local school—Our Lady of Good Counsel, Mourne Road.

Along with many other boys of his generation and social background, Gerard left school at the age of fourteen to supplement the family income, and took up his first job with the Post Office, working as a telegram-boy out of the GPO, O'Connell St. But he soon tired of this, and went to work for his father instead. But this too failed to satisfy him, and, to ease the daily drudgery, he began reading whatever books he could lay his hands on, and became a voracious reader. As luck would have it, the Simms family, who had been his father's customers since 1949, had built up a good relationship with the Murphys, and they very willingly lent Gerard books, including, in particular, a set of encyclopedias, which he absorbed. Interestingly, his brothers, who were close to him in age, also became avid readers, and later on, one studied music, and the other, art. But his sisters also were influenced by his success and love of learning, and just recently his sister Carol took her PhD in Psychology.

As time went by, Gerard became more and more disillusioned with manual labour, and decided to better himself by furthering his education. After working as a window cleaner for about five years, he decided to quit his job, brushed aside all opposition to this course of action, and proceeded to educate himself at home, with a single-minded approach that was one of his distinguishing traits. To help him achieve his goal, which was to do Engineering at TCD, he signed on with the International Correspondence Schools to prepare for A-level courses in Mathematics and Computer Science. According to Carol, he was in a welter of excitement when his first batch of study material arrived, and couldn't wait to get started! It is perhaps noteworthy, too, that, from time to time, when he was studying for A-level Mathematics, to qualify for entry to TCD, he received help from David Simms, whenever he needed it.

When he was ready to sit his A-level examinations he had to take himself to London to do them, a daunting enough task for someone who had never been outside Ireland before. He completed these successfully, and subsequently satisfied the Matriculation requirements for TCD, and applied to do Engineering there. But this was before the points system came into operation, and places in the Engineering School at TCD in those days were allocated on the basis of examination results and headmasters' reports, and, because he hadn't been to a secondary school, Gerard failed to satisfy the admission criteria. So, he couldn't do Engineering. But, by this time, he had developed a taste for Mathematics, and applied to do a degree in Honours Mathematics instead. But here, too, in not having English or another language as a matriculation subject, he fell short of the entry requirements for this programme as well, which the then Senior Lecturer wouldn't waive, despite David's protests. So, he was initially forced to register for a General Studies degree, which involved doing Mathematics and Applied Mathematics at a level well below his capabilities.

Thus, after overcoming these various hurdles, rather unusually for a future mathematical scholar of distinction, Gerard came late to the "groves of academe", and entered the portals of TCD in October 1970, a little short of his 22nd birthday, to do a BA (General) degree. But, once inside, he appears to have been allowed by the Professor of Pure Mathematics, Brian Murdoch, to attend the Special Honor Mathematics course, and take an examination in it at the end of the first term, at which he excelled, so much so that the Senior

Lecturer was persuaded to transfer him to Honours Mathematics in January, 1971. Thereafter, it was plain sailing for him. He joined the Special Honor class, which included, among others, Paul Barry (WIT), Colm Ó Dúnlaing (TCD) and Ray Ryan (NUIG), made rapid progress, and was subsequently awarded a Foundation Scholarship, which took care of his College fees, and board and lodging.

After a brilliant undergraduate career, he graduated from TCD in 1974, with a First Class Honours degree, earning a Gold Medal for the quality of his answering. Once the results of the final examination were known, he received a memorandum from Brian Murdoch, who congratulated him on his "superb performance", and noted "that it was probably the best year we have ever had in Mathematics". According to David Simms, Gerard showed an inclination for Pure Mathematics, when, while studying Synge and Griffith, he became puzzled by the way a mathematical concept was introduced!

3. Cambridge Days

Following his success at TCD, which singled him out as a special mathematical talent, Gerard was awarded a Gulbenkian Research Studentship by Churchill College, Cambridge, which he held for the next three years. This covered all his University and College fees, and, in addition, provided a maintenance allowance of 715 pounds *per annum*. Thus, in the Autumn of 1974, he was able to enroll at Churchill College, Cambridge, and study there for the degree of Doctor of Philosophy, unencumbered by financial considerations.

Non-Archimedean Banach Algebras is the title of Gerard's doctoral thesis [7], which he wrote under the guidance of Dr. G. A. Reid, and submitted in the month of April, 1977, after just two and a half years of study.

The theory of non-Archimedean Functional Analysis was begun in the 1940s, and, in the succeeding decades, efforts were made to extend the standard theorems of classical Functional Analysis by replacing the underlying field of real or complex numbers with a non-Archimedean field, namely, a field F that is equipped with a non-trivial *valuation*, i.e., a mapping $|\cdot| : F \to [0,\infty)$, that assumes at least three different values, that is multiplicative, and induces a complete ultrametric on F, so that

$$|x-y| \le \max\{|x-z|, |z-y|\} \qquad (x,y,z \in F).$$

(In what follows immediately, F will denote such a field.) The standard example of such a field is provided by the p-adic numbers[1], and, no doubt, this served to motivate the study of other algebraic structures over a non-Archimedean field. (It seems to me, though, that people who investigated such concepts were, perhaps unwittingly, merely following Darwin's dictum that one should carry out a damn-fool experiment every so often, a suggestion with which J. E. Littlewood seemingly concurred [5]!)

By the early 1970s the theory of non-Archimedean Analysis had been extended to such areas as Banach Spaces, Harmonic Analysis and Complex Analysis, and two books had appeared, one in 1970 by A. F. Monna [6], and another in 1971 by L. Narici, E. Beckenstein and G. Bachmann [20], where the basic theory of Banach algebras over a field F is worked out, and the differences between this and Gelfand's theory are highlighted. (A Banach algebra over F is an associative algebra that is endowed with a sub-multiplicative, ultrametric, complete norm.) To get an overview of the subject of non-Archimedean spaces, and the impact it has made, see also [19].

While efforts had also been made to extend the theory of C^*-algebras to a non-Archimedean setting, these were not terribly successful, apparently; and the area was ripe for further development when Gerard was admitted to Churchill College in 1974. His supervisor, Dr. Reid, set Gerard the task of developing a more satisfactory theory of these structures, a project he successfully completed, unaided, winning the Knight Research Prize in his second year of study on foot of an essay he wrote at the time.

Confining himself almost entirely to commutative C^*-algebras with a unit, Gerard obtained an appropriate analogue of the Stone–Weierstrass theorem—thereby extending Kaplansky's version of it [4]—which was an important first step, and introduced the concepts of bundles, L-algebras, Boolean spaces and idempotents into the subject. For example, he used the concept of idempotent to overcome a marked deficiency that a field F possesses, namely, it lacks the notion of a non-trivial 'conjugation-like' self-map. As a result, it wasn't clear what the 'correct' definition of a C^*-algebra should be in this new framework.

[1] I first learnt about such things from P. B. Kennedy as a fresher in UCC, but he never gave the context, and I didn't fully understand such matters until much later. However, he invariably set a question about p-adic valuations on the examination paper.

A *bundle* is simply a family of Banach algebras over F indexed by a topological space. If the latter is compact, Hausdorff and totally disconnected, we get a Boolean bundle. The notion of a bundle gives rise to that of an L-algebra on the bundle. Gerard's version of the Stone–Weierstrass theorem in a non-Archimedean setting reads: If A is a separating Banach algebra on a Boolean bundle, then it is an L-algebra on the bundle. Perhaps somewhat surprisingly, this does include the classical theorem!

As defined by Gerard in his thesis, a C^*-algebra is a Banach algebra A over a field F with the properties that all non-zero idempotents have norm 1, and each maximal ideal of A is generated by its idempotents. Thus, F itself is a C^*-algebra, as is the algebra $C(K, F)$ of continuous functions on a compact set K that take their values in F, with the supremum norm, the idempotents being the characteristic functions of the clopen subsets of K. Gerard developed a satisfactory theory of such C^*-algebras that runs parallel with the classical Gelfand theory, and, as well, discusses fully a list of some interesting examples. All of this, and more, is contained in his Ph. D. dissertation.

His first research paper [8], which appeared in 1978, contains a very readable account of the main ideas touched on above. Indeed, as far as I'm aware, this was the only paper he ever published on the topic, even though he was occupied with the theory of C^*-algebras for the rest of his life. Remarkably, too, he never mentions the subject of non-Archimedean algebras in his book [11], not even amongst the exercises. But already in this paper one can discern early signs of his ability to present difficult ideas in a clear and cogent manner, a skill which was another of his hall-marks. Aside from this, moreover, one learns from his thesis his penchant for algebraic methods and axiomatics, his sense of mathematical aesthetics, his ability to deal with abstract concepts, and his knowledge and understanding of several different areas of Algebra, Topology and Functional Analysis, skills which he displayed in abundance later in the seventy or so research papers he subsequently wrote.

4. Back in Trinity College, Dublin

Following his stint at Cambridge, Gerard returned in the Autumn of 1977 to Trinity College, Dublin, where he held a Government Postdoctoral Research Fellowship for the next three years; and also did some teaching there. There, too, he commenced an active and fruitful, but discontinuous, collaboration with Trevor West, with whom he subsequently wrote six research articles, only five of which appear to have been reviewed. (Trevor presented their joint paper "Removing the Interior of the Spectrum—Silov's Example" at the Second International Symposium in West Africa on Functional Analysis and its Applications, in Kumasi, 1979, but it was not reviewed.) Their joint paper [18] contains, *inter alia*, a formula for the spectral radius, $\rho(a)$, of an element a in a C^*-algebra[2] A, viz.,

$$\rho(a) = \inf\{||e^h a e^{-h}|| : h \in A, h = h^*\},$$

a very beautiful result for which they will both be remembered, although Trevor attributes it wholly to Gerard.[3]

The "Little Red Book", to which Gerard often referred, had its origins in discussions Roger Smyth and Trevor had in the mid 1970s, prior to Gerard's second coming to TCD, about the possibility of extending the notion of finite rank operators to Banach Algebras. Roger told me that they were inspired to select bi-ideals of algebraic elements as suitable candidates by reading [2], and were further motivated by Rien Kaashoek, whom they met in Amsterdam in 1975 on their way to Oberwolfach, who told Roger that the concepts of finite rank and Riesz elements would command far wider interest if they could be extended to take in Fredholm elements as well. Following up this suggestion, Trevor arranged for Bruce Barnes from Oregon—who had written on such matters—to come on sabbatical to TCD, and collaborate with them to develop their ideas; and, later on, the three of them were joined by Gerard. Their joint venture led to the publication in 1982 of the research monograph [1], the aim of which the authors state was "to highlight the interplay between algebras and spectral theory which emerges in any penetrating analysis of compact, Riesz and Fredholm operators on Banach spaces". According to both Trevor and Roger, Gerard's main contribution to

[2]From now on, by a C^*-algebra is meant a Banach algebra with a *-operation satisfying the same algebraic properties as the adjoint map for Hilbert space operators plus the key property $\|a^*a\| = \|a\|^2$. They are the norm closed self-adjoint algebras of operators on a Hilbert space, considered up to *-isomorphism.

[3]*Editor's Note*: This paper has very recently been made good use of once again in [M. Mathieu, A. R. Sourour, Hereditary properties of spectral isometries, Arch. Math. (Basel) **82** (2004), 222–229].

the production of this book was his first-class grasp of Spectral Theory, his knowledge of, and expertise in, C^*-algebras, and his ability to solve whatever problems arose in his area of speciality during the course of their joint investigation.

5. Abroad in North America

To gain further professional experience, Gerard took up a post of Research Associate in Dalhousie University, Halifax, Canada, which was funded by a two-year fellowship from the Canadian Government. There he also lectured to undergraduates on a part-time basis. While there he came into contact with Heydar Radjavi and Peter Fillmore—who was primarily responsible for arranging his fellowship—two world-renowned mathematicians who have made significant contributions to the field of C^*-algebras.

He spent two rewarding years in Halifax, and, during the second year, married Mary O'Hanlon, formerly a nurse and midwife who worked in her professional capacity in Cork, the Channel Islands, the United States of America and Dublin, where they met for the first time. Theirs was to be a fruitful and very happy union, which was blessed with four lovely children, Alison, Adele, Neil and Elaine.

He then moved to the United States of America, and held two one-year appointments at Associate Professor level, first at New Hampshire, and then at Oregon State University, where he again linked up with Bruce Barnes. At both these places, Gerard lectured full-time to undergraduate students, and continued his research activities. During their stay in New Hampshire, Gerard and Mary renewed their marriage vows in Church, an event that was celebrated by their families.

His period in North America was also a very productive time for Gerard, and he completed at least seven papers, one of them with Radjavi, and two with C. K. Fong, who held successive positions at the Universities of Toronto and Ottawa. He was also invited to lecture on his research at other universities, such as Toronto, Illinois, Indiana and Vancouver. In this way, he spent four years in North America, gaining invaluable teaching and research experience and making important research contacts, as well as acquiring an understanding of different systems of university education, lessons which were to stand him in good stead when he returned home in 1984. He was especially impressed by the quality of the teaching he experienced there, and this motivated him to strive for excellence in his own teaching.

6. Cork, 1984–1990

Gerard returned to Ireland in 1984 to take up a permanent appointment as a Lecturer in the Department of Mathematics in University College, Cork, where he worked for the rest of his career. He taught and examined a wide range of courses, delivering every type of course at every level, from first year calculus courses to very large classes, through to advanced postgraduate courses to small groups of students. He was a versatile teacher, and as well as giving general level courses to Arts, Commerce and Science students, he taught Control Theory to Fourth Year Electrical Engineers, gave undergraduate courses on Topology, Functional Analysis and Measure Theory to Honours students, and delivered courses on Banach Algebras and Operator Theory to postgraduate students. His principal objective in teaching was to further the students' understanding and appreciation of the intellectual beauty and depth of Mathematics and its power as a tool for understanding other disciplines. He put a lot of thought and preparation into his courses, which were designed to reflect his own approach and ideas in terms of selection of material, examples, homework and student motivation. Never content to use a colleague's lecture notes, he always designed his own. These were models of clarity and precision, and are as fresh and novel today as they were when he delivered them.

From early on he organized weekly research seminars for his postgraduate students, postdoctoral assistants and interested staff, and exposed his own and other researcher's thoughts on contemporary results in his field of interest. While these seminars were largely for the benefit of his four PhD students[4] and other postgraduates whose theses he supervised for their Master's degree, they were very informative, and kept the rest of us abreast of current developments in his speciality. Indeed, it was at these that I, and, I'm sure, many of his former students became acquainted with such topics as noncommutative geometry and quantum groups, which occupied him for

[4]Mícheál Ó Searcóid, (Fredholm theory in rings, 1987), Tadhg Creedon (Derivations that map into the radical, 1995), Kamaledin Abodayeh (Compact topological semigroups, 1998) and Adel Bashir Badi (Index theory for generalized Toeplitz operators, 2005)

the last decade of his life. He took delight in the success of all his students, and took great interest in their subsequent careers. He was especially proud that one of his MA students, Thomas Cooney, won the prestigious NUI Traveling Studentship in Mathematics in 2002 for his thesis “Amenability and coamenability of quantum groups”.

From the moment he set foot on the UCC campus, he was eager to host regular International Mathematical Conferences here, and the first one, entitled “Aspects of Analysis”, co-organised by Gerard, Brian Twomey and myself, was held in mid-May, 1986. This was well-attended, and attracted experts in the fields of Operator Theory and Function Theory. Buoyed by its success, Gerard was encouraged to organise an international conference on his own on “Operator Theory and Operator Algebras” in each of the following three years. These were very well organised, immensely successful, and very popular with the participants, numbering between 30 and 40, who came from all over the world. Indeed, some people were disappointed when none was held in 1990! But, by then, he had decided to ease the burden on himself, and hold them less frequently. And, for that reason, the next one was held in 1991, to be followed by others at two-yearly intervals until they lapsed again for a period after 1995. They weren’t held again until 2003.

As well as carrying out his normal every-day duties during his first seven years at UCC, and busying himself organising conferences, he found time to continue to produce a steady stream of high quality research articles on Toeplitz operators and C^*-algebras, and prepare a textbook about the latter subject for postgraduate students. So, it’s only right that, at this point, I should interrupt this narrative, and try to describe his contributions in these areas.

6.1. **Toeplitz operators and Toeplitz algebras.** Beginning in 1987 Gerard wrote about 20 research papers on these topics. Indeed, his last published paper dealt with them, as does another long paper [17] which has yet to appear. It seems fitting therefore to describe briefly the salient points of a subject that occupied his attention for two-thirds of his active research life, and to mention some of his contributions to this important area.

The study of Toeplitz operators begins with Toeplitz’s discovery in 1911 that if $\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$ are the Fourier coefficients of a bounded function, then the bilinear form $\sum_{i,j=1}^{\infty} a_{i-j} x_i y_j$ is bounded on the sequence space ℓ_2. Much later, it was realised that the converse statement is true, and that the corresponding operator could best be investigated by treating it as an operator on the classical Hardy space H^2 consisting of square integrable functions on the unit circle whose Fourier coefficients vanish on the negative integers. While much of the early work focused on the analytical properties of such operators, Brown and Halmos [3] gave the subject a new impetus in 1963 when they showed, *inter alia*, that the class of Toeplitz operators on H^2 coincides with the commutant of the unilateral shift, itself a Toeplitz operator, and emphasised the algebraic approach, a point of view that Gerard adopted in his own investigations.

He divided his attention between two of several possible generalizations of the classical theory of Toeplitz operators. On the one hand, he considered them as compressions of bounded multiplication operators acting on $L^2(\hat{G}, m)$ to an abstract Hardy space $H^2(\hat{G})$—the subspace of functions in $L^2(\hat{G}, m)$ whose Fourier transforms vanish on $\{x \in G : x < 0\}$—where $\hat{G}$ is the Pontryagin dual of a fixed ordered abelian group G, and m is normalised Haar measure on $\hat{G}$. In other words, the objects of interest for him, here, were those operators T defined on $H^2(\hat{G})$ by $Tf = P(\phi f)$, where ϕ is bounded on $\hat{G}$, and P is the orthogonal projection of $L^2(\hat{G}, m)$ onto $H^2(\hat{G})$. In this context, much of the theory of classical Hardy spaces carries over, and, thence, that of the corresponding Toeplitz operators, but, as Gerard explains in his survey article [10], many new insights emerge about C^*-algebras in general as a result of examining the particular C^*-algebra generated by Toeplitz operators. This point of view dominated his thinking throughout his career, and was one of the reasons that motivated him in his pursuit of an abstract theory—the belief that an abundance of concrete examples would act as guiding principles in the formation of an abstract theory, which, in turn, would lead to a better understanding of the classical theory.

He formulated a different class of examples of Toeplitz operators as follows. Let A be a function algebra on a compact Hausdorff space K. Suppose m is a probability measure on K that determines a continuous multiplicative linear functional τ on A such that $\tau(f) = \int_K f\,dm,\ f \in A$. Define the Hardy space H^2 to be the norm closure of A in $L^2(K, m)$. If $\phi \in L^\infty(K)$, the Toeplitz operator with symbol ϕ is then defined by $T_\phi f = P(\phi f)$, where P is the orthogonal projection of $L^2(K, m)$ onto $H^2(K)$. (We recover the classical theory, when K is the unit circle, A is the algebra of polynomials, $\tau(f) = f(0)$, $f \in A$, and m is normalised Lebesgue measure on K.) Rather remarkably, as Gerard has ably demonstrated in a series of papers that he produced about such operators in the last twenty or so years, many of the classical results about Toeplitz operators extend to this more abstract setting. Indeed, in his review of one of Gerard's papers [12], Sheldon Axler writes: "Most of the classical results hold, although often new proofs are needed in this context. The author has come up with proofs that are clean and sometimes add new insight to the classical case. Even when the classical results fail to generalize, the author has usually found an interesting substitute. For example, in the classical case, if ϕ is real-valued and $\phi \neq 0$, then T_ϕ has no eigenvalues. This fails in the more general context, but the author shows that any eigenspace of T_ϕ must be infinite-dimensional."

In his last published paper [16], that appeared in 2006, Gerard was inspired by Connes' quantization of classical mathematics to develop the essential properties of a still more abstract concept of a Toeplitz operator, for which he constructs a far-reaching index theorem that includes several classical index theorems that pertain to, for example, the Wiener–Hopf integral operator, and almost periodic functions. But he didn't think this was the end of the story, and suspected that one could prove an index theorem in the general setting of what he calls a unimodular algebra on a compact Hausdorff space K, i.e., a function algebra A on K such that every function in $C(K)$ can be approximated by elements of the form $f\theta$, where $f, \theta \in A$ and $|\theta| = 1$.

He also had it in his head in 2005 to draw together his results about Toeplitz operators in a book sometime in the future, but, alas, fate intervened.

6.2. ***C*-algebras.*** If for nothing else, Gerard will surely be remembered for his postgraduate-level textbook entitled "C*-Algebras and Operator Theory", which appeared in 1990. As he writes in the Preface "This book is aimed at the beginning graduate student and the specialist in another area who wishes to know the basics of this subject. The reader is assumed to have a good background in real and complex analysis, point set topology, measure theory, and elementary functional analysis." The book was very well received by the mathematical community worldwide and warmly reviewed, and, according to him, became a standard textbook in many countries. A Russian translation of it appeared in 1997. It's still on sale, and to-date, 1,892 copies of it have been sold.

Not only did its appearance mark Gerard's "arrival" on the international stage, it acted as a springboard for his subsequent professional career, and as a stepping stone to further advancement within and without UCC.

The book deals with the general theory of C^*-algebras, the unifying theme that courses through his work, and was one of his main areas of specialisation. While he studied such structures for their own sake, he was well aware of their origins and importance within Mathematics, their wide range of applications, and the reasons for considering them. Indeed, about a third of his published papers have "C^*-algebras" in their title, and, significantly, about two-thirds of these were published after the appearance of his book. Because these papers are readily identified, a reader wishing to know his major achievements in this area, is recommended to read the book to learn the foundations, and then use MathSciNet to locate his papers, and learn about the recent development of the subject and the directions it has taken as a result of his pioneering investigations.

But to give a flavour of his work which had a major impact in the theory of general C^*-algebras, it seems only right that I should comment on one topic which he studied, and which grew out of his investigations of non-classical Toeplitz operators. In his first paper about these objects [9], his most important idea was the identification of a certain C^*-algebra, $C^*(\Gamma^+)$, where Γ^+ is the positive cone of a discrete ordered abelian group Γ, with a corner of a crossed product of a commutative C^*-algebra by Γ. This led him to generalise the notion of a crossed product of a C^*-algebra A by an abelian group G, which he proceeded to develop in a series if papers, focusing on a theory of semigroup crossed products. His 1996 paper [13] is possibly the most influential of these. In the classical theory and for the simplest situations, one has a given group homomorphism α from G to the group $\mathrm{Aut}(A)$ of *-automorphisms of A. The crossed product should be a C^*-algebra B that contains A and admits a representation τ of G into the unitary elements of B so that $\alpha_g(a) = \tau(g)a\tau(g)^*$. The algebra B should be generated by A and the image $\tau(G)$. Gerard was motivated to his generalisation by more or less contemporary work of P. J. Stacey and of Iain Raeburn with various coauthors. They were in turn partly motivated by a desire to find an underlying theory for the 1977 construction by Cuntz of new simple C^*-algebras, and another motivation for Gerard was in connections with Toeplitz algebras. In [13] he found an appropriate generalisation to the case when the group G was replaced by a cancellative abelian semigroup M with a zero element and the map α is replaced by an action $x \mapsto \alpha_x$ of M as injective *-endomorphisms of the C^*-algebra A. A key step is to find an appropriate larger C^*-algebra where the more classical case of automorphisms reappears and the group G is the Grothendieck group for the semigroup M. This is achieved by an inductive limit construction. And furthermore the crossed product can be twisted by a multiplier of M.

7. Advancement in Cork, 1990–2000

Within a short time after the appearance of his book, Gerard was promoted to Statutory Lecturer in Mathematics at UCC, and shortly after that, in recognition of his scholarly standing, he was honoured by members of the Royal Irish Academy who elected him to membership of this venerable body. He was immensely proud of his membership of the Academy, and, later on, he became joint Editor-in-Chief of its Mathematical Proceedings, helping to change its format and production, which gave it greater visibility and raised its profile as an international journal.

In 1991, he re-commenced the organisation of two-yearly international conferences in UCC on "Operator Theory and Operator Algebras" for which he received funding from a variety of different sources. In the early years he obtained small amounts of money from the Royal Irish Academy and the Irish Mathematical Society, but his principal source of funding in those days was, somewhat surprisingly, the US Air Force. In later years, he received financial support from EOLAS, FORBAIRT and the EU, which over time became the major sponsor.

The conference he organised in 1995 was one of the events held to mark the 150th anniversary of the founding of University College, Cork, and attracted upwards of 100 participants. Coincidentally, in the same year he was promoted to the rank of Associate Professor in Cork in recognition of the quality and quantity of his research output, the calibre of his teaching, and the overall contribution he made to the running of the Department of Mathematics and the well-being of the College.

Around about the same time, he was also invited to join the EU Operator Algebras Network, and, over two four-year periods, succeeded in attracting substantial funding from the EU which provided conference support, and enabled him to offer worthwhile Scholarships to his postgraduate students, and invite several postdoctoral research assistants to come to UCC and work with him. As a result of his establishing a node of this network here, Cork became an internationally recognised centre of excellence, with Gerard as its leading investigator, not only for the promotion of Operator Algebras, but also for the development of Noncommutative Geometry and Quantum Groups, new subjects of great intrinsic importance, both for Mathematics and Physics.

7.1. **Noncommutative Geometry and Quantum Groups.** Together with a succession of postdoctoral research assistants, Tom Hadfield, Johan Kustermans, Deepak Parashar and Lars Tuset, from the late 1990s onwards, Gerard gave seminars in UCC about these fields, and made important contributions to them, but, before attempting to describe these, I invite you to read Gerard's own descriptions of these subjects:

Regarding noncommutative geometry, he says: "Noncommutative geometry was invented by Alain Connes in the 1980s to provide a quantized calculus that extends the usual de Rham calculus and to provide a geometric tool to deal with the so-called singular spaces that arise so frequently in advanced mathematics and quantum physics. A singular space is a space that is poorly behaved from the point of view of classical mathematics in that the usual tools—measure theory, topology, differential geometry, group theory—do not apply. Examples of singular spaces are the spaces of irreducible representations of discrete groups, spaces of orbits of group actions, spaces of leaves of foliations of smooth manifolds, and the phase space of quantum physics. The solution offered by noncommutative geometry is to replace these spaces by associated noncommutative algebras that encode in a better way the problem one wishes to study. For example, instead of studying a space of group orbits of an action of a group G on a compact space X in the pathological case that the natural quotient topology on the space of orbits is the coarse topology (this frequently happens), one studies a corresponding C^*-algebra $C(X) \rtimes G$, the crossed product by G of the algebra $C(X)$ of continuous functions on X. One can then, for example, study the algebraic topology of the space of orbits by studying the K-theory of $C(X) \rtimes G$.

An important feature of Connes' theory is his quantization of the calculus. Given a function f (or more generally, an element in a suitable noncommutative algebra), the differential of f is defined by $df = [F, f] = Ff - fF$, where F is a suitable operator on a Hilbert space. Note that this is well defined even for functions that are not differentiable in the classical sense. The calculi obtained from this construction are associated to objects called Fredholm modules and these in turn, by means of a Chern character construction, give rise to cyclic cocycles (however, not all cyclic cocycles arise in this fashion). The cyclic cocycles in turn form the cycles for an important new cohomology theory, cyclic cohomology, that generalizes in a profound way classical de Rham homology. This theory has already had deep and important applications to classical mathematics and physics."

He said this about quantum groups: "The theory of quantum groups had its origins in attempts to extend Pontryagin duality theory from the context of abelian locally compact groups to non-abelian ones—it turns out that the dual of a non-abelian group is not itself a group; rather, it is a new kind of object called a quantum group. However the class of quantum groups is much more extensive than merely the class of group duals—quantum groups arise in many other ways, for example, by quantizing classical groups to obtain deformations, something that is very important in applications to physics. Quantum groups also arise as symmetry objects for quantum spaces. To explain this latter idea, note that in the framework of noncommutative geometry spaces are replaced by noncommutative algebras that are viewed as quantum spaces. The symmetries of a classical space are analyzed in terms of groups but the symmetries of a quantum space require a quantum group formulation. In the 1980s revolutionary work by S. L. Woronowicz and the Fields Medalist V. G. Drinfeld, arising from considerations in theoretical physics, led to major advances in the theory of quantum groups and to its being regarded as one of the most important subjects in contemporary mathematics. It is envisaged by some physicists that quantum groups will provide the mathematical framework for the solution of the outstanding difficulty of modern physics: the problem of unifying the presently inconsistent theories of general relativity and quantum physics.

In Woronowicz's approach a quantum group is a C^*-algebra with additional structure, such as a comultiplication. Corresponding to a locally compact group the C^*-algebra encodes the topological or geometric aspect and the comultiplication corresponds to the group operation. This theory has been most successfully worked out in the case of compact quantum groups, where Woronowicz has shown the existence of a Haar integral and developed the corepresentation theory. He has also considered the problem of endowing these quantum groups with suitable differential structures. This is an aspect of the theory that is still very mysterious and it is one of the aspects of quantum group theory in which my research is based."

Working jointly with J. Kustermans and L. Tuset, both of whom spent time at UCC, Gerard introduced certain linear functionals called *twisted graded traces*, and developed an extensive theory for them and their integrals. For example, they showed that if one associated a multilinear function φ to a triple $(\Omega, d, \int)$, where (Ω, d) is an N-dimensional differential calculus and $\int$ a suitable twisted graded trace, by setting

$$\varphi(a_0, \ldots, a_N) = \int a_0 da_1 \cdots da_N,$$

then φ is a cycle for a new cohomology theory. They also showed that the new theory had all the basic features of Connes' cyclic cohomology theory and contained it as a special case. Thus, it was shown that a significant portion of Connes' noncommutative geometry can be extended to the case of the differential calculi that arise in the quantum group setting. It should be mentioned, too, that Gerard attached great significance to this new "twisted cyclic cohomology" that they had introduced, and had every intention of following it up before his untimely illness. He seemed to think that the most important thing to be done to further develop the theory was to construct a Chern character.

In a different direction, together with E. Bédos and L. Tuset, he wrote three papers about the concepts of amenability and co-amenability in compact quantum groups and algebraic quantum groups, framing the definition of amenability in a way that is analogous to the classical one—a locally compact group G is *amenable* if there is a positive linear functional $m : L^\infty(G) \to \mathbb{C}$ of norm one such that

$$m(\lambda_x f) = m(f) \qquad (f \in L^\infty(G)),$$

where $\lambda_x f(g) = f(x^{-1}g)$, $g \in G$. (Compact groups and abelian groups are amenable.) Their notion of co-amenability stems from a property possessed by a certain C^*-algebra associated with a discrete group. They gave several equivalent formulations of this concept, involving a C^*-algebra. *Inter alia* it turns out that co-amenability of an algebraic quantum group G implies amenability of its dual $\hat{G}$. It's not known if the converse holds.

8. Miscellaneous Research Items

8.1. **Occasional papers.** About 24 of Gerard's published papers have nothing much in common with either Toeplitz operators or C^*-algebras, and about a third of these were joint efforts with others, such as T. T. West (5), M. Mathieu (1), C. K. Fong (2) and K. Abodayeh (1). While it's hard to classify them, they fall into the general area of Spectral Theory of operators on Banach spaces, and show his versatility to work with others on diverse problems. He had the capacity to spot connections between different areas, and oftentimes produce easier proofs of known results. As an illustration, let me single out [14]. According to its reviewer, Qing Lin, "In this elegant short paper, Murphy provides an elementary and much simpler proof" of a theorem of P. Y. Wu to the effect that, in an infinite-dimensional Hilbert space, any unitary operator is a product of 16 positive operators. This is a by-product of the spectral theorem, and a result that, if u is a unitary element in a C^*-algebra, then the matrix $\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$ is a product of eight positive ones.

8.2. **Survey Articles.** Gerard clearly enjoyed writing expository papers, and he published at least eight of these, most of them in the Bulletin of the Irish Mathematical Society. These were mainly concerned with surveys of areas that he was currently working in or about to explore, but were aimed at non-specialists. For instance, in issues of the Bulletin he wrote about "Extensions and K-theory of C^*-algebras" (1987), "Toeplitz operators" (1989), "Dimension theory and stable rank" (1990), "C^*-dynamical systems and invariance algebras" (1991), "Partially ordered groups" (1992), and "Function algebras" (1993). An exposition of his work on Quantum Groups is given in [15]. While a common thread runs through these papers, as with most of his work, namely, the abstract theory of C^*-algebras, nevertheless, these are distinctly different, and contain not only historical summaries of separate aspects of the origins and early development of this subject, but serve as useful signposts of further developments. As well, they illustrate his breadth of knowledge and understanding of many areas of mathematics and physics, and provide a valuable insight to his thinking and mode of work.

9. Final Years, 2000–2006

Gerard did his fair share of departmental and College administration during his time in UCC. He served on the College's Promotions Board for many years, and, following the retirement of Paddy Barry from the Chair of Mathematics in 1999, he became Head of Department, a role he filled with quiet efficiency for the next five years, during which time he oversaw the development of new management structures and the delivery of new mathematical degree programmes which attracted bright students from the start, and have grown in popularity, becoming the flagship degree programmes of the School of Mathematical Sciences at UCC. All the while, too, while performing his administrative duties, he continued to teach his courses, produce a steady stream of research papers, maintain the link with the EU Operator Algebras Network, do his editorial work for the Proceedings of the Royal Irish Academy, supervise research students and, in 2003, revive the Cork international conferences on C^*-algebras. Gerard organised his last international conference[5] in June, 2005. A few months later he was diagnosed with cancer.

Gerard was very widely read, and delved deeply into History and Economics, especially. Indeed, he had every intention, apparently, of writing an Economic History of Ireland, and had written copious notes in print form—which was his style of writing—which he hoped to pull together in book form at some stage. Another plan of his was to write children's stories, many of which he composed for his own children, of whom he was exceedingly proud.

[5]Stephen Wills, who co-organised the 2003 and 2005 conferences with Gerard, will run the next one in 2008 with the support of Martin Mathieu and Richard Timoney.

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Factoring Generalized Repunits

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Abstract. Twenty-five years ago, W. M. Snyder extended the notion of a repunit R_n to one in which for some positive integer b, $R_n(b)$ has a b-adic expansion consisting of only ones. He then applied algebraic number theory in order to determine the pairs of integers under which $R_n(b)$ has a prime divisor congruent to 1 modulo n. In this paper, we show how Snyder's theorem follows from existing theory pertaining to the Lucas sequences.

1. Introduction

A *repunit* R_n is any integer written in decimal form as a string of 1's. The numbers 1, 11, 111, 1111, 11111, etc., are examples of repunits. In [7], S. Yates alludes to a letter dated June, 1970 that he received from A. H. Beiler in which Beiler claims to have invented the term years earlier. An interesting characteristic regarding repunits is the apparent scarcity of primes among them. Letting R_n denote the *nth* repunit, only R_2, R_{19}, R_{23}, R_{317}, and R_{1031} have thus far been identified as prime. In fact, they are the only repunit primes for $n \leq 16000$. Although it is necessary for n to be prime in order for R_n to be prime, this is not a sufficient condition as $R_5 = 11111 = 41 \cdot 271$ is composite.

In [6], W. M. Snyder extended the notion of a repunit to one in which for some integer $b > 1$, $R_n(b)$ has a b-adic expansion consisting of only ones. In other words, $R_n(b) = \Sigma_{i=0}^{n-1} b^i = (b^n - 1)/(b - 1)$, where $n > 0$. Examples of these "generalized repunits" include the Mersenne numbers, $M_n = 2^n - 1 = 1 + 2^1 + 2^2 + \ldots + 2^{n-1}$, for $n \geq 2$. Snyder's admitted objective was to apply algebraic number theory in cyclotomic fields in order to determine the pairs of integers n and b under which $R_n(b)$ has a prime divisor congruent to 1 modulo n. To this purpose, Snyder demonstrated the following proposition.

Theorem 1 (Snyder)**.** *$R_n(b)$ has a prime divisor congruent to* 1 (mod n) *if and only if $n \neq 2$, or $n = 2$ and $b \neq 2^e - 1$, for all integers e greater than* 1*.*

In this paper, we illustrate how Theorem 1 may be derived from the existing theory of the Lucas sequences, upon which, we then introduce a primality test for base-10 repunits.

2. The Lucas Sequences

Let P and Q be any pair of relatively prime integers. We define the *Lucas* and *companion Lucas sequences*, respectively, as

$$U_{n+2}(P,Q) = PU_{n+1} - QU_n,\ U_0 = 0,\ U_1 = 1,\ n \in \{0,1,\ldots\} \quad (1)$$

$$V_{n+2}(P,Q) = PV_{n+1} - QV_n,\ V_0 = 2,\ U_1 = P,\ n \in \{0,1,\ldots\}. \quad (2)$$

Now, (1) and (2) are linear, and hence, solvable. Letting $D = P^2 - 4Q$ be the discriminant of $X^2 - PX + Q = 0$, the roots of the said characteristic equation are $\theta = (P + \sqrt{D})/2$ and $\phi = (P - \sqrt{D})/2$. Thus, the Lucas and companion Lucas sequences are given explicitly by

$$\begin{aligned} U_n(P,Q) &= \frac{\theta^n - \phi^n}{\theta - \phi}, \qquad n \in \{0,1,\ldots\} \\ V_n(P,Q) &= \theta^n + \phi^n, \qquad n \in \{0,1,\ldots\}. \end{aligned} \quad (3)$$

The *rank of apparition* of a prime is the index of the first term in the sequence with nonnegative index in which N occurs as a divisor. We let $\omega(p)$ denote the rank of apparition of p in $\{U_n\}$ and $\lambda(p)$ the corresponding rank of apparition of p in $\{V_n\}$. Also, we say that p is a *primitive* prime factor of the term in which it has rank of apparition. The next lemma contains results that are found in [5].

Lemma 1. *Let p be an odd prime.*

(1) *If $p \nmid P$, $p \nmid Q$, and $p \mid D$, then $p \mid U_k$ exactly when $p \mid k$.*
(2) *If $p \nmid PQD$, then $p \mid U_{p-(D/p)}$, where (D/p) denotes the Legendre symbol.*
(3) *$p \mid U_n$ if and only if $n = k\omega$, for some positive integer k.*

3. Generalized Repunits by the Lucas Sequences

Now, we show that for any base $b > 1$, $\{R_n(b)\}$ is a Lucas sequence.

Theorem 2. *Let b be any integer > 1. Then,*

$$U_n(b+1, b) = (b^n - 1)/(b - 1).$$

Proof. Let $P = b + 1$ and $Q = b$. Since b and $b + 1$ are relatively prime, then by (3),

$$U_n = \frac{\left(\frac{P+\sqrt{D}}{2}\right)^n - \left(\frac{P-\sqrt{D}}{2}\right)^n}{\left(\frac{P+\sqrt{D}}{2}\right) - \left(\frac{P-\sqrt{D}}{2}\right)} = \frac{\left(\frac{b+1+(b-1)}{2}\right)^n - \left(\frac{b+1-(b-1)}{2}\right)^n}{b-1} = \frac{b^n - 1}{b - 1}.$$

□

4. An Elementary Proof of Theorem 1

In this section, we shall demonstrate Snyder's Theorem 1 by first establishing that every term of the Lucas sequence $\{R_n(b)\} = \{U_n(b+1, b)\}$ has a primitive prime factor. The latter result rests upon Carmichael's generalization of K. Zsigmondy's theorem for numbers of the form $a^n \pm b^n$ to the family of Lucas sequences [3]. We also point out that Zsigmondy's result given in [8] is an extension of a theorem of A. S. Bang, who in 1886, proved the special case $b = 1$ [1]. A further discussion of these results is found in [5]. The following is Carmichael's result, which will lead us to Lemma 3.

Lemma 2 (Carmichael)**.** *Let $\{U_n(P, Q)\}$ be a Lucas sequence and $D = P^2 - 4Q$.*

(1) *Let $D > 0$. Then, for all $n \neq 1, 2, 6$, U_n has a primitive prime factor, unless $n = 12$, $P = \pm 1$, and $Q = -1$.*
(2) *Let D be a square. Then, for all n, U_n has a primitive prime factor unless $n = 6$, $P = \pm 3$, and $Q = 2$.*

Lemma 3. *Every term of $\{R_n(b)\}$ =$\{U_n(b + 1, b)\}$, with the exception of $b = 2$ and $n = 6$ has a primitive prime factor.*

Proof. Since $P = b + 1$, $Q = b$, and $D = P^2 - 4Q = (b + 1)^2 - 4b = (b - 1)^2$, it follows by Lemma 2 that $R_n(b)$ has a primitive prime factor unless $b = 2$ and $n = 6$. □

Remark. If n is odd, say $2k + 1$ ($k \geq 1$), then

$$R_n = b^{2k} + b^{2k-1} + \ldots + b + 1$$

is odd. On the other hand, if n is even, say $2k$ $(k \geq 2)$, then

$$R_n = (b^{2k} - 1)/(b - 1) = [(b^k + 1)(b^k - 1)]/(b - 1)$$
$$= (b^k + 1)(b^{k-1} + b^{k-2} + \ldots + 1).$$

Hence, for $k \geq 2$, we have $b^k + 1 \neq 2^\alpha$ for all integral values of α. Thus, if $n \geq 3$ then there exists at least one odd prime factor of R_n regardless of the parity of n. It is also without loss of generality we assume that $b = 2$ and $n = 6$ do not simultaneously hold for otherwise, $R_6(2) = 63$ has the prime factor 7 congruent to $1 \pmod 6$. Under this stipulation, it then follows that every term of $\{R_n\}$ has a primitive prime factor. Moreover, for generalized repunits, we further extend the reach of Lemma 3 to include the existence of an odd primitive prime factor. This is so, because if $k = 2$ then $2 \mid R_4$. Hence, if b is odd then $R_2 = b+1$ and $\omega(2) = 2$ and if b is even then the odd factor $b^2 + 1$ necessarily contains an odd prime factor that divides neither $R_2 = b + 1$ nor $R_3 = (b + 1)^2 - b$.

We now give our alternative demonstration of Theorem 1.

Proof of Theorem 1. Let's assume that either $n \neq 2$, or $n = 2$ and $b \neq 2^e - 1$ is not true for all integers $e > 1$. Then, $n = 2$ and $b = 2^e - 1$ for some $e > 1$. Therefore, $R_2(b) = R_2(2^e - 1) = 2^e$, which has no prime divisors congruent to $1 \pmod 2$. To prove necessity, we may assume that $n > 1$. Otherwise, every prime is trivially congruent to $1 \pmod 1$.

Case 1: Let $n = 2$. Then, $R_n = R_2 = b+1$. Hence, if $b = 2^e - 1$ then $R_n = 2^e$, which does not have a prime factor congruent to $1 \pmod n$.

Now, assume that $n \geq 3$. By the previous remark, we let p be an odd primitive prime factor of R_n. In turn, this implies that $\omega(p) = n$.

Case 2: Let $p \nmid PQD$. Since $D = P^2 - 4Q = (b - 1)^2$, it follows that $(D/p) = 1$. So, by the second statement of Lemma 1, $p \mid U_{p-1}$, from which it follows from the third conclusion of the same lemma that $\omega(p) \mid p - 1$. Therefore, $\omega(p)k = p - 1$, for some integer k. In other words, $p = \omega(p)k + 1$.

Case 3: Let $p \mid P = b+1$. Then, $U_2 = b+1$ and $\omega(p) = 2$, which is impossible, as p is a primitive prime factor of R_n $(n \geq 3)$.

Case 4: Let $p \mid Q = b$. But this implies that $p \mid R_n = b^{n-1} + b^{n-2} + \ldots + b + 1$, which is also impossible, as $p \neq 1$.

Case 5: Let $p \nmid PQ$ and $p \mid D = P^2 - 4Q$. By (1) of Lemma 1, $p \mid R_n$ exactly when $p \mid n$. As $R_{n+1} > R_n$ and $R_3 = P^2 - Q$, it then follows that $\omega(p) < 3$, which under our assumptions cannot happen. □

5. Testing the Primality of Base-10 Repunits

A corollary to Fermat's Little Theorem tells us that for any integer a, $a^n \equiv a \pmod{n}$ if n is a prime. Thus, if we can identify an integer a for which this congruence does not hold, then we may conclude that n is composite. For example, by taking $a = 3$ and $n = 8$, we see that $3^8 \equiv 1 \pmod{8}$. This proves that 8 is composite. So, Fermat's Theorem is a way to determine if a number n is composite without having to first extract a factor. Nonetheless, for large values of n the number of computations involved is prohibitively large.

A method that is sometimes used for making an educated guess as to the prime or composite character of an integer n is the *Miller–Rabin test*. The idea of this algorithm is to write $n = 2^h m + 1$, where m is odd. Then, for a particular base $a : 1 < a < n - 1$, we consider the sequence of terms $a^m, a^{2m}, a^{4m}, \ldots, a^{2^h m} = a^{n-1}$ modulo n. The number n is said to "pass the test" if the first occurrence of 1 is either the first term or -1 precedes it. An odd prime will pass this test for all bases. To help us decide if n is prime or composite, we may randomly select k integers, say, a_i, $1 \le i \le k$. If n fails the Miller–Rabin test for any one of these bases, we immediately conclude that n is composite. On the other hand, if n passes the test for all a_i, then n is dubbed a *probable prime*. Although we can never be certain that n is prime after conducting the Miller–Rabin test, the probability of a composite number surviving k applications of the algorithm is at most $(1/4)^k$ [2]. In 1999, Harvey Dubner announced that R_{49081} is a probable repunit prime [4] and in 2000, Lew Baxter added R_{86453} to the short list.

We now arrive at our final objective—to construct a definitive Lucas-type test for deciding the primality of any base-10 repunit. To this purpose, we introduce the Legendre symbols $\sigma = (P^2/p)$, $\epsilon = (D/p)$, and $\tau = (Q/p)$. The following lemmas will be alluded to and may be found in [5].

Lemma 4. *The gcd(U_n, V_n) is* 1 *or* 2.

Lemma 5. *Suppose that ω is odd. Then, $V_n(\sqrt{R}, Q)$ is not divisible by p for any value of n. If n is even, say $2k$, then $V_{(2n+1)k}(\sqrt{R}, Q)$ is divisible by p for every n but no other term of the sequence contains p as a factor.*

Lemma 6. $U_{(p-\sigma\epsilon)/2}(P, Q) \equiv 0 \pmod{p}$ *if and only if $\sigma = \tau$.*

Lemma 7. *If $N \pm 1$ is the rank of apparition of N then N is prime.*

Base-10 repunits are generated by the Lucas sequence $\{U_n(11, 10)\}$. Thus, celebrated properties of the Lucas sequences such as the necessity of the index being prime in order for U_n to be prime, $U_m \mid U_n$ if $m \mid n$, and if $d \mid U_m$ and $d \mid U_n$ then $d \mid U_{m+n}$ are also attributable to base-10 repunits. Furthermore, we point out that $10 \mid R_n - 1$. This enables us to state and prove the following necessary and sufficient condition for the primality of an arbitrary R_n, bearing in mind that R_n is prime only if n is prime. We remark that although there are infinitely many Lucas sequences that can be used for this purpose, we have opted to use the Fibonacci numbers $\{U_n(1, -1)\}$.

Theorem 3. *Let p be any prime and $2 \cdot 5 p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorization of R_{p-1}. Let $R_p \nmid \prod_{i=1}^{k} U_{(10p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k})/p_i}(1, -1)$. Then, R_p is prime if and only if $R_p \mid V_{5p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}}(1, -1)$.*

Proof. ($\Rightarrow$) Assume that R_p is prime. Since $P = 1$ and $Q = -1$, we have $\sigma = \left(P^2/R_p\right) = (1/R_p) = 1$ and $\tau = (Q/R_p) = ((-1)/R_p) \equiv (-1)^{((10^p-10)/9)/2} \equiv (-1)^{(5(10^{p-1}-1))/9} \pmod{R_p} = -1$. By Gauss's Reciprocity Law, $(5/R_p)(R_p/5) = (-1)^{R_p - 1} = 1$. Hence, $(5/R_p) = (R_p/5)$. Hence, $\epsilon = (D/R_p) = (5/R_p) = (R_p/5) \equiv R_p^2 \pmod{5} = 1$. Furthermore, since $D = 1$, it follows from the second part of Lemma 1 that $U_{10p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}} \equiv 0 \pmod{R_p}$. However, as $\sigma \neq \tau$, by Lemma 6, $R_p \nmid U_{5p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}}$. Thus, $\omega(R_p)$ is either equal to $10p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ or must divide exactly one of $U_{(10p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k})/p_i}$. But, by hypothesis, the latter is impossible. Therefore, $\omega(R_p) = 10p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, and by Lemma 4, $R_p \mid V_{5p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}}$.

($\Leftarrow$) We now suppose that $R_p \mid V_{5p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}}$. By the identity $U_{2n} = U_n V_n$ and Lemma 5, we have $R_p \mid U_{10p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}}$ but $R_p \nmid U_{5p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}}$. As $R_p \nmid \prod_{i=1}^{k} U_{(10p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k})/p_i}$, it follows that $\omega(R_p) = 10p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Therefore, by Lemma 7, R_p is prime. $\square$

In conclusion, it may be argued that the factorization of R_{p-1} is difficult to obtain due to the large size of the number. Indeed, this is true. Nevertheless, if we are able to factor $p-1$, then it follows from the theory of Lucas that $R_k \mid R_{p-1}$, for all factors k of $p-1$.

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The Centre of Unitary Isotopes of JB^*-Algebras

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Abstract. We identify the centre of unitary isotopes of a JB^*-algebra. We show that the centres of any two unitary isotopes of a JB^*-algebra are isometrically Jordan *-isomorphic to each other. However, there need be no inclusion between centres of the two unitary isotopes.

1. Basics

We begin by recalling (from [3], for instance) the following concepts of homotope and isotope of Jordan algebras.

Let $\mathcal{J}$ be a Jordan algebra, cf. [3], and $x \in \mathcal{J}$. The x-homotope of $\mathcal{J}$, denoted by $\mathcal{J}_{[x]}$, is the Jordan algebra consisting of the same elements and linear algebra structure as $\mathcal{J}$ but a different product, denoted by "$\cdot_x$", defined by

$$a \cdot_x b = \{axb\}$$

for all a, b in $\mathcal{J}_{[x]}$. By $\{pqr\}$ we will always denote the Jordan triple product of p, q, r defined in the Jordan algebra $\mathcal{J}$ as below:

$$\{pqr\} = (p \circ q) \circ r - (p \circ r) \circ q + (q \circ r) \circ p \,,$$

where $\circ$ stands for the original Jordan product in $\mathcal{J}$. An element x of a Jordan algebra $\mathcal{J}$ with unit e is said to be invertible if there exists $x^{-1} \in \mathcal{J}$, called the inverse of x, such that $x \circ x^{-1} = e$ and $x^2 \circ x^{-1} = x$. The set of all invertible elements of $\mathcal{J}$ will be denoted by $\mathcal{J}_{inv}$. In this case, x acts as the unit for the homotope $\mathcal{J}_{[x^{-1}]}$ of $\mathcal{J}$.

If $\mathcal{J}$ is a unital Jordan algebra and $x \in \mathcal{J}_{inv}$ then by x-isotope of $\mathcal{J}$, denoted by $\mathcal{J}^{[x]}$, we mean the x^{-1}-homotope $\mathcal{J}_{[x^{-1}]}$ of $\mathcal{J}$. We denote the multiplication "$\cdot_{x^{-1}}$" of $\mathcal{J}^{[x]}$ by "$\circ_x$".

The following lemma gives the invariance of the set of invertible elements in a unital Jordan algebra on passage to any of its isotopes.

Lemma 1.1. *For any invertible element a in a unital Jordan algebra $\mathcal{J}$, $\mathcal{J}_{inv} = \mathcal{J}_{inv}^{[a]}$.*

Proof. See Lemma 1.5 of [8]. □

Let $\mathcal{J}$ be a Jordan algebra and let $a, b \in \mathcal{J}$. The operators T_b and $U_{a,b}$ are defined on $\mathcal{J}$ by $T_b(x) = b \circ x$ and $U_{a,b}(x) = \{axb\}$. We shall denote $U_{a,a}$ simply by U_a. The elements a and b are said to operator commute if T_a commute with T_b.

Let $\mathcal{J}$ be a complex unital Banach Jordan algebra and let $x \in \mathcal{J}$. As usual, the spectrum of x in $\mathcal{J}$, denoted by $\sigma_{\mathcal{J}}(x)$, is defined by

$$\sigma_{\mathcal{J}}(x) = \{\lambda \notin \mathbb{C} : x - \lambda e \text{ is not invertible in } \mathcal{J}\} .$$

A Jordan algebra $\mathcal{J}$ with product $\circ$ is called a Banach Jordan algebra if there is a norm $\|.\|$ on $\mathcal{J}$ such that $(\mathcal{J}, \|.\|)$ is a Banach space and $\|a \circ b\| \leq \|a\| \, \|b\|$. If, in addition, $\mathcal{J}$ has a unit e with $\|e\| = 1$ then $\mathcal{J}$ is called a unital Banach Jordan algebra. In the sequel, we will only be considering unital Banach Jordan algebras; the norm closure of the Jordan subalgebra $J(x_1, \ldots, x_r)$ generated by $x_1, \ldots, x_r$ of Banach Jordan algebra $\mathcal{J}$ will be denoted by $\mathcal{J}(x_1, \ldots, x_r)$.

The following elementary properties of Banach Jordan algebras are similar to those of Banach algebras and their proofs are a fairly routine modifications of these [1, 2, 7, 9].

Lemma 1.2. *Let $\mathcal{J}$ be a Banach Jordan algebra with unit e and $x_1, \ldots, x_r \in \mathcal{J}$.*

(i) *If $J(x_1, \ldots, x_r)$ is an associative subalgebra of $\mathcal{J}$, then $\mathcal{J}(x_1, \ldots, x_r)$ is a commutative Banach algebra.*
(ii) *T_{x_1} and U_{x_1,x_2} are continuous with $\|T_{x_1}\| \leq \|x_1\|$ and $\|U_{x_1,x_2}\| \leq 3\|x_1\| \, \|x_2\|$.*
(iii) *$\mathcal{J}(x_1, \ldots, x_r)$ is a closed subalgebra of $\mathcal{J}$.*
(iv) *If $\mathcal{J}$ is unital then $\mathcal{J}(e, x_1)$ is a commutative Banach algebra.*
(v) *If $x \in \mathcal{J}$ and $\|x\| < 1$ then $e - x$ is invertible and $(e-x)^{-1} = \sum_{n=0}^{\infty} x^n \in \mathcal{J}(e, x)$.*
(vi) *If K is a closed Jordan subalgebra of $\mathcal{J}$ containing e and $x \in K$ such that $\mathbb{C} \setminus \sigma_{\mathcal{J}}(x)$ is connected then $\sigma_{\mathcal{J}}(x) = \sigma_K(x)$.*

We are interested in a special class of Banach Jordan algebras, called JB^*-algebras. These include all C^*-algebras as a proper subclass (see [10, 13]).

A complex Banach Jordan algebra $\mathcal{J}$ with isometric involution * (see [6], for instance) is called a *JB^*-algebra* if $\|\{xx^*x\}\| = \|x\|^3$ for all $x \in \mathcal{J}$.

The class of JB^*-algebras was introduced by Kaplansky in 1976 (see [10]) around the same time when a related class called JB-algebras was being studied by Alfsen, Shultz and Størmer (see [1]).

A real Banach Jordan algebra $\mathcal{J}$ is called a *JB-algebra* if $\|x\|^2 = \|x^2\| \leq \|x^2 + y^2\|$ for all $x, y \in \mathcal{J}$.

These two classes of algebras are linked as follows (see [10, 13]).

Theorem 1.3. (a) *If $\mathcal{A}$ is a JB^*-algebra then the set of self-adjoint elements of $\mathcal{A}$ is a JB-algebra.*

(b) *If $\mathcal{B}$ is a JB-algebra then under a suitable norm the complexification $\mathcal{C}_{\mathcal{B}}$ of $\mathcal{B}$ is a JB^*-algebra.*

There is an easier subclass of these algebras. Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the full algebra of bounded linear operators on $\mathcal{H}$.

(a) Any closed self-adjoint complex Jordan subalgebra of $\mathcal{B}(\mathcal{H})$ is called a *JC^*-algebra*.

(b) Any closed real Jordan subalgebra of self-adjoint operators of $\mathcal{B}(\mathcal{H})$ is called a *JC-algebra*.

Any JB^*-algebra isometrically *-isomorphic to a JC^*-algebra is also called a JC^*-algebra; similarly, any JB-algebra isometrically isomorphic to a JC-algebra is also called a JC-algebra.

It is easy to verify that a JC^*-algebra is a JB^*-algebra and a JC-algebra is a JB-algebra. It might be expected, conversely, that every JB-algebra is a JC-algebra (with a corresponding statement for JB^*-algebras and JC^*-algebras) but unfortunately this is not true (for details see [1]).

2. Unitary Isotopes of a JB^*-algebra

In [8], we presented a study of unitary isotopes of JB^*-algebras. In this section, we recall some facts from [8] which are needed for the sequel.

Let $\mathcal{J}$ be a JB^*-algebra. The element $u \in \mathcal{J}$ is called *unitary* if $u^* = u^{-1}$, the inverse of u. The set of all unitary elements of $\mathcal{J}$

will be denoted by $\mathcal{U}(\mathcal{J})$. If u is a unitary element of JB^*-algebra $\mathcal{J}$ then the isotope $\mathcal{J}^{[u]}$ is called a unitary isotope of $\mathcal{J}$.

Theorem 2.1. *Let u be a unitary element of the JB^*-algebra $\mathcal{J}$. Then the isotope $\mathcal{J}^{[u]}$ is a JB^*-algebra having u as its unit with respect to the original norm and the involution $*_u$ defined by $x^{*_u} = \{ux^*u\}$.*

Proof. See Theorem 2.4 of [8]. □

Recall (from [3], for instance) that a Jordan algebra is said to be *special* if it is isomorphic to a Jordan subalgebra of some associative algebra. We require the following fact.

Lemma 2.2. *If $\mathcal{J}$ is a special Jordan algebra and $a \in \mathcal{J}$, then $\mathcal{J}_{[a]}$ is a special Jordan algebra.*

Proof. See Lemma 1.3 in [8]. □

Theorem 2.3. *The unitary isotope of a JC^*-algebra is again a JC^*-algebra.*

Proof. This follows from Theorem 2.1 and Lemma 2.2 (also see [8, Theorem 2.12]). □

We close this section by noting following facts.

Lemma 2.4. *Let $\mathcal{J}$ be a JB^*-algebra with unit e. Then $u \in \mathcal{U}(\mathcal{J}) \Longrightarrow e \in \mathcal{U}(\mathcal{J}^{[u]})$. Moreover $\mathcal{J}^{[u]^{[e]}} = \mathcal{J}$.*

Proof. See Lemma 2.7 of [8]. □

Next theorem establishes the invariance of unitaries on passage to unitary isotopes of a JB^*-algebra.

Theorem 2.5. *For any unitary element u in the JB^*-algebra $\mathcal{J}$,*

$$\mathcal{U}(\mathcal{J}) = \mathcal{U}(\mathcal{J}^{[u]}) .$$

Proof. See Theorem 2.8 of [8]. □

Corollary 2.6. *Let $\mathcal{J}$ be a JB^*-algebra with unit e and let $u, v \in \mathcal{U}(\mathcal{J})$. Then*

(i) $\mathcal{J}^{[u]^{[v]}} = \mathcal{J}^{[v]}$.

(ii) *The relation of being unitary isotope is an equivalence relation in the class of unital JB^*-algebras.*

Proof. See Corollary 2.9 of [8]. □

3. Centre of Unitary Isotopes

In this section, we identify the centre of unitary isotopes in terms of the centre of the original JB^*-algebra. We recall the following definition from [14].

Definition 3.1. Let $\mathcal{J}$ be a unital JB^*-algebra and let

$$C(\mathcal{J}) = \{x \in \mathcal{J}_{sa} : x \text{ operator commutes with every y} \in \mathcal{J}_{sa}\} .$$

Then the *centre* of $\mathcal{J}$, denoted by $\mathcal{Z}(\mathcal{J})$, is defined by

$$\mathcal{Z}(\mathcal{J}) = C(\mathcal{J}) + iC(\mathcal{J}) .$$

Remark 3.2. It is known from [14] that $\mathcal{Z}(\mathcal{J})$ is a C^*-algebra, and if $\mathcal{J}$ is a JC^*-algebra with $\mathcal{J} \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ then

$$\mathcal{Z}(\mathcal{J}) = \{x \in \mathcal{J} : \ xy = yx \quad \forall y \in \mathcal{J}\} .$$

To investigate further properties of the centre we need the following lemma.

Lemma 3.3. *Let $\mathcal{J}$ be a JB^*-algebra and let $x \in \mathcal{Z}(\mathcal{J})$. Then for all $y \in \mathcal{J}$,*

(i) $T_x T_y = T_y T_x$;
(ii) $T_x U_y = U_y T_x$;
(iii) $U_x U_y = U_y U_x$;
(iv) *if $u \in \mathcal{J}$ is unitary then $(x \circ u^*) \circ u = x$.*

Proof. Let $x = a + ib$ and $y = c + id$ with $a, b \in C(\mathcal{J})$ and $c, d \in \mathcal{J}_{sa}$. Then

$$\begin{aligned} T_x T_y &= (T_a + iT_b)(T_c + iT_d) = T_a T_c + iT_a T_d + iT_b T_c - T_b T_d \\ &= T_c T_a + iT_d T_a + iT_c T_b - T_d T_b = T_y T_x \end{aligned}$$

as $a, b \in C(\mathcal{J})$ which proves (i).
(ii). Since $U_y = 2T_y^2 - T_{y^2}$, we have

$$T_x U_y = T_x(2T_y^2 - T_{y^2}) = 2T_x T_y^2 - T_x T_{y^2} = (2T_y^2 - T_{y^2})T_x = U_y T_x$$

by part (i) (note that the associativity of $\mathcal{B}(\mathcal{J})$ is used here).
(iii). Since $x \in \mathcal{Z}(\mathcal{J})$, $x^2 \in \mathcal{Z}(\mathcal{J})$ by Remark 3.2. Hence by part (ii),

$$\begin{aligned} U_x U_y &= (2T_x^2 - T_{x^2})U_y = 2T_x^2 U_y - T_{x^2} U_y \\ &= 2U_y T_x^2 - U_y T_{x^2} = U_y U_x. \end{aligned}$$

(iv). By part (i), $(x \circ u^*) \circ u = T_u T_x u^* = T_x T_u u^* = T_x e = x$. □

Theorem 3.4. *Let $\mathcal{J}$ be a JB^*-algebra with unit e and let $b \in \mathcal{Z}(\mathcal{J})$. Then for any unitary $u \in \mathcal{U}(\mathcal{J})$ and for any $x \in \mathcal{J}$ we have*

(i) $(u^* \circ x) \circ u = u^* \circ (x \circ u)$;

(ii) $\{(b \circ u)u^* x\} = b \circ x$.

Proof. (i). If $\mathcal{J}$ is special then

$$\begin{aligned}(u^* \circ x) \circ u &= \frac{1}{4}(u(u^*x + xu^*) + (u^*x + xu^*)u) \\ &= \frac{1}{4}(2x + uxu^* + u^*xu) \\ &= \frac{1}{4}(u^*(ux + xu) + (ux + xu)u^*) = u^* \circ (x \circ u).\end{aligned}$$

Hence, by the *Shirshov–Cohn theorem with inverses* [5], we have in the general case $(u^* \circ x) \circ u = u^* \circ (x \circ u)$.

(ii). Since $b \in \mathcal{Z}(\mathcal{J})$ and $u \in \mathcal{U}(\mathcal{J})$, we get by Lemma 3.3 (iv) that

$$(b \circ u) \circ u^* = b. \tag{1}$$

Again by Lemma 3.3 (i),

$$(u^* \circ x) \circ (b \circ u) = T_{(u^* \circ x)} T_b u = T_b T_{(u^* \circ x)} u = b \circ (u \circ (x \circ u^*)) ,$$

and

$$u^* \circ ((b \circ u) \circ x) = T_{u^*} T_x T_b u = T_b T_{u^*} T_x u = b \circ (u^* \circ (x \circ u)) ,$$

so by part (i)

$$(u^* \circ x) \circ (b \circ u) = u^* \circ ((b \circ u) \circ x). \tag{2}$$

Thus by (1) and (2),

$$\begin{aligned}\{(b \circ u)u^* x\} &= ((b \circ u) \circ u^*) \circ x + (u^* \circ x) \circ (b \circ u) - ((b \circ u) \circ x) \circ u^* \\ &= b \circ x .\end{aligned}$$

□

We now need a characterisation of the centre in terms of Hermitian operators. These are defined in terms of the numerical range of operators as follows (see [14], for example).

Definition 3.5. If $\mathcal{J}$ is a complex unital Banach Jordan algebra with unit e and $D(\mathcal{J}) = \{f \in \mathcal{J}^* : f(e) = \|f\| = 1\}$ then, for $a \in \mathcal{J}$, the *numerical range* of a, denoted by $W(a)$, is defined by $W(a) = \{f(a) : f \in D(\mathcal{J})\}$. The element a is called *Hermitian* if $W(a) \subseteq \mathbb{R}$. The set of all Hermitian elements of $\mathcal{J}$ is denoted by $Her\mathcal{J}$.

The Hermitian elements in a unital JB^*-algebra are exactly the self-adjoint elements (see [13]) but we shall need the following characterisation of the Hermitian operators on a JB^*-algebra, given in [14].

Theorem 3.6. *Let $\mathcal{J}$ be a JB^*-algebra with unit e. Then $S \in Her\,\mathcal{B}(\mathcal{J})$ if and only if $S = T_a + \delta$ where δ is a *-derivation and $a = S(e)$ is self-adjoint.*

We can now give a characterisation of the centre of a unitary isotope.

Theorem 3.7. *Let $\mathcal{J}$ be a JB^*-algebra with unit e and let $u \in \mathcal{U}(\mathcal{J})$. Let $\mathcal{A}$ be a JC^*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ with unit e_A and let $w \in \mathcal{U}(\mathcal{A})$.*

(i) *If $x \in \mathcal{Z}(\mathcal{J})$ then $u \circ x \in \mathcal{Z}(\mathcal{J}^{[u]})$.*

(ii) *If $a \in \mathcal{Z}(\mathcal{A}^{[w]})$ then $(a \circ w^*) \circ w = a$.*

(iii) *If $z \in \mathcal{Z}(\mathcal{J}^{[u]})$ then $u \circ (u^* \circ z) = z$.*

(iv) *Define $\psi : \mathcal{Z}(\mathcal{J}) \to \mathcal{Z}(\mathcal{J}^{[u]})$ by $\psi(x) = u \circ x$. Then ψ is an isometric *-isomorphism of $\mathcal{Z}(\mathcal{J})$ onto $\mathcal{Z}(\mathcal{J}^{[u]})$.*

Proof. (i). Let $x = a + ib$ where $a, b \in \mathcal{Z}(\mathcal{J})_{sa}$. Let $S = T_a \in Her\,\mathcal{B}(\mathcal{J})$. Then

$$S(e) = T_a(e) = a \circ e = a \quad \text{and} \quad S(u) = u \circ a\,.$$

As $S \in Her\,\mathcal{B}(\mathcal{J})$, $S(u) \in (\mathcal{J}^{[u]})_{sa}$ by Theorem 3.6. By Theorem 3.4 (ii),

$$S(y) = T_a(y) = a \circ y = \{(a \circ u)u^* y\} = (a \circ u) \circ_u y$$

for all $y \in \mathcal{J}$. Therefore, $S(y) = L^{[u]}_{S(u)}(y)$ for all $y \in \mathcal{J}$, where operator $L^{[u]}_{S(u)}$ stands for the multiplication by $S(u)$ in $\mathcal{J}^{[u]}$. Moreover, as $a \in \mathcal{Z}(\mathcal{J})$ we get by [14, Theorem 14] that $S^2 \in Her\,\mathcal{B}(\mathcal{J}) = Her\,\mathcal{B}(\mathcal{J}^{[u]})$ because $\mathcal{B}(\mathcal{J}^{[u]}) = \mathcal{B}(\mathcal{J})$ (see Theorem 2.1). So again by [14, Theorem 14], $S(u) \in \mathcal{Z}(\mathcal{J}^{[u]})$ as $S = L^{[u]}_{S(u)}$. Therefore, $u \circ a \in \mathcal{Z}(\mathcal{J}^{[u]})_{sa}$. Similarly, $u \circ b \in \mathcal{Z}(\mathcal{J}^{[u]})_{sa}$. Hence $u \circ x = u \circ a + iu \circ b \in \mathcal{Z}(\mathcal{J}^{[u]})$.
(ii). By Remark 3.2,

$$\mathcal{Z}(\mathcal{A}) = \{x \in \mathcal{A} : xy = yx\}. \tag{3}$$

By Theorem 2.3, the isotope $\mathcal{A}^{[w]}$ is a JC^*-algebra and

$$\mathcal{Z}(\mathcal{A}^{[w]}) = \{x \in \mathcal{A} : xw^*y = yw^*x\}. \tag{4}$$

Now, if $a \in \mathcal{Z}(\mathcal{A}^{[w]})$ then (by (4)) $aw^*y = yw^*a$ for all $y \in \mathcal{A}$. In particular,

$$aw^* = w^*a. \tag{5}$$

By part (i), $a \circ w^* = e_A \circ_w a \in \mathcal{Z}(\mathcal{A}^{[w]^{[e_A]}}) = \mathcal{Z}(\mathcal{A})$. So we have by (4) that

$$(a \circ w^*) \circ w = (a \circ w^*)w = \frac{1}{2}(aw^* + w^*a)w$$

hence by (5)

$$(a \circ w^*) \circ w = (aw^*)w = a(w^*w) = a\,,$$

as required.
(iii) Now, let v be any unitary in $\mathcal{Z}(\mathcal{J}^{[u]})$ (the centre of the unitary isotope $\mathcal{J}^{[u]}$ of the JB^*-algebra $\mathcal{J}$). Then v is a unitary in $\mathcal{J}$ by Theorem 2.5. By [8, Corollary 1.14], $\mathcal{J}(e, u, u^*, v, v^*)$ is a JC^*-algebra and $v \in \mathcal{Z}((\mathcal{J}(e, u, u^*, v, v^*))^{[u]})$. Hence, by (ii),

$$u \circ (u^* \circ v) = v. \tag{6}$$

If $z \in \mathcal{Z}(\mathcal{J}^{[u]})$, then by the Russo–Dye Theorem (cf. [11]) for C^*-algebras there exist unitaries $v_j \in \mathcal{Z}(\mathcal{J}^{[u]})$ and scalars $0 \le \lambda_j \le 1$ with $\sum_{j=1}^n \lambda_j = 1$ for some $n \in \mathcal{N}$ such that $\frac{z}{\|z\|+1} = \sum_{j=1}^n \lambda_j v_j$ because $\|\frac{z}{\|z\|+1}\| < 1$ (recall that $\mathcal{Z}(\mathcal{J}^{[u]})$ is a C^*-algebra). Hence, by (6),

$$\begin{aligned} u \circ (u^* \circ z) &= u \circ (u^* \circ (\|z\| + 1) \sum_{j=1}^n \lambda_j v_j\,) \\ &= (\|z\| + 1) \sum_{j=1}^n \lambda_j (u \circ (u^* \circ v_j)) \\ &= (\|z\| + 1) \sum_{j=1}^n \lambda_j v_j = z. \end{aligned}$$

(iv). As $\psi = T_u\,|_{\mathcal{Z}(\mathcal{J})}$, ψ is linear and continuous by Lemma 1.2 (i). Let $z \in \mathcal{Z}(\mathcal{J}^{[u]})$. Applying part (i) to $\mathcal{J}^{[u]}$ we get $e \circ_u z \in \mathcal{Z}(\mathcal{J}^{[u]^{[e]}})$. But $\mathcal{J}^{[u]^{[e]}} = \mathcal{J}$ by Lemma 2.4 and $e \circ_u z = \{eu^*z\} = u^* \circ z$. Hence $u^* \circ z \in \mathcal{Z}(\mathcal{J})$. Moreover, $\psi(u^* \circ z) = u \circ (u^* \circ z) = z$ by part (iii). Thus ψ maps $\mathcal{Z}(\mathcal{J})$ onto $\mathcal{Z}(\mathcal{J}^{[u]})$.

Further, $\|\psi(x)\| \leq \|u\|\,\|x\|$ while, by Lemmas 3.3 (i) and 1.2 (ii),

$$\|x\| = \|T_x T_{u^*} u\| = \|T_{u^*} T_x u\| \leq \|x \circ u\| = \|\psi(x)\| \ .$$

Thus ψ is an isometry.

Finally, as $\psi(e) = u$ and u is the unit of $\mathcal{J}^{[u]}$ it follows from [12, Theorem 6] that ψ is an isometric *-isomorphism. □

Corollary 3.8. *Let $\mathcal{J}$ be a unital JB^*-algebra. Then, for all $u, v \in \mathcal{U}(\mathcal{J})$, $\mathcal{Z}(\mathcal{J}^{[u]})$ is isometrically Jordan *-isomorphic to $\mathcal{Z}(\mathcal{J}^{[v]})$.*

Proof. By Theorem 2.5, $v \in \mathcal{U}(\mathcal{J})$. Hence, by Theorem 3.7, $\mathcal{Z}(\mathcal{J}^{[u]})$ is isometrically *-isomorphic to $\mathcal{Z}(\mathcal{J}^{[u]^{[v]}})$. However, by Corollary 2.6 (i), $\mathcal{J}^{[u]^{[v]}} = \mathcal{J}^{[v]}$. This gives the required result. □

An alternative proof of above Corollary 3.8 can be obtained by noting that $\mathcal{Z}(\mathcal{J}^{[u]})$ is isometrically *-isommorphic to $\mathcal{Z}(\mathcal{J})$ and $\mathcal{Z}(\mathcal{J})$ is isometrically *-isomorphic to $\mathcal{Z}(\mathcal{J}^{[v]})$ by Theorem 3.7 (applied twice). As the next example shows there need be no inclusion between the centre of a unital JB^*-algebra and the centre of its isotopes. In the following discussion $\mathcal{M}_2(\mathbb{C})$ denotes the standard complexification of the real Jordan algebra of all 2×2 symmetric matrices.

Example 3.9. If $u \in \mathcal{U}(\mathcal{M}_2(\mathbb{C})) \setminus \mathcal{Z}(\mathcal{M}_2(\mathbb{C}))$ then the unit $e \notin \mathcal{Z}(\mathcal{M}_2(\mathbb{C})^{[u]})$.

Indeed, $\mathcal{M}_2(\mathbb{C})^{[u]}$ is a 4-dimensional C^*-algebra by Theorem 2.3 with 1-dimensional centre by the above Theorem 3.7. As u does not belong to $\mathcal{Z}(\mathcal{M}_2(\mathbb{C}))$, $u \notin Sp(e)$ where $Sp(e)$ denotes the linear span of e, and hence $e \notin Sp(u)$. This gives that $e \notin \mathcal{Z}(\mathcal{M}_2(\mathbb{C})^{[u]})$.

As a final point on the relationships between the centres it should be noted in the proof of Theorem 3.7 (i) that if $a \in \mathcal{Z}(\mathcal{J})$ and $S = T_a$ then S is left multiplication in any unitary isotope. In order to study the *-derivations it might be hoped that if $T \in Her\,\mathcal{B}(\mathcal{J})$ then there exists a unitary isotope $\mathcal{J}^{[u]}$ such that T is left multiplication operator in $Her\,\mathcal{B}(\mathcal{J}^{[u]})$ since as linear spaces $\mathcal{B}(\mathcal{J}) = \mathcal{B}(\mathcal{J}^{[u]})$ so $Her\,\mathcal{B}(\mathcal{J}) = Her\,\mathcal{B}(\mathcal{J}^{[u]})$. Unfortunately, this fails even when $\mathcal{J} = \mathcal{M}_2(\mathbb{C})$. As all *-derivations are inner in this case, it follows that $T \in Her\,\mathcal{B}(\mathcal{M}_2(\mathbb{C}))$ if and only if $T = l_a + r_b$ where $a, b \in (\mathcal{M}_2(\mathbb{C}))_{sa}$ and $l_a(x) = ax$ and $r_b(x) = xb$.

Corollary 3.10. *If* $a, b \in \mathcal{M}_2(\mathbb{C})$ *are given by* $a = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 6 & 0 \\ 0 & 23 \end{pmatrix}$ *and* $T \in Her\,\mathcal{B}(\mathcal{M}_2(\mathbb{C}))$ *is defined by* $T = l_a + r_b$, *then* T *is not left multiplication in any unitary isotope.*

Proof. It was noted in Example 3.9 that if $u \in \mathcal{U}(\mathcal{M}_2(\mathbb{C}))$ then $\mathcal{M}_2(\mathbb{C})^{[u]}$ is a four-dimensional C^*-algebra with a one-dimensional centre so is isomorphic to $\mathcal{M}_2(\mathbb{C})$. By [4, Theorem 10], $\sigma(T) = \sigma(a) + \sigma(b) = \{7, 8, 24, 25\}$.

On the other hand, if $L_c^{[u]} \in Her\,\mathcal{B}(\mathcal{M}_2(\mathbb{C}))$ with say $\sigma_{\mathcal{M}_2(\mathbb{C})}(c) = \{\lambda_1, \lambda_2\}$ then $\sigma(L_c^{[u]}) = \{\lambda_1, \frac{\lambda_1+\lambda_2}{2}, \lambda_2\}$ again by [4, Theorem 10], so $\sigma(L_c^{[u]})$ contains only three points. Hence $\sigma(T) \neq \sigma(L_c^{[u]})$ for any unitary $u \in \mathcal{U}(\mathcal{M}_2(\mathbb{C}))$. □

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Conformal Symmetries of Regions

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ABSTRACT. We discuss which groups arise as the group of conformal symmetries of a plane region. We also show how to compute the symmetry group of a given region.

1. Conformal Symmetry Groups

A conformal symmetry of a region D in the complex plane is a one-to-one conformal map of D onto itself. In this expository paper we discuss conformal symmetry groups, mostly for plane regions, but also for certain regions in higher dimensional spaces. In particular, we address the following two questions.

Question 1. Which groups arise as conformal symmetry groups?

Question 2. Given a region, what is its conformal symmetry group?

Let us consider Question 2 in a concrete example. Let D be the region exterior to the three circles $C_1 = \{z : |z| = 16\}$, $C_2 = \{z : |z-(-16+24i)| = 4\}$ and $C_3 = \{z : |z-(9+\frac{89}{4}i)| = t\}$, $0 < t < 8$ (this region is illustrated in Figure 1). The condition $0 < t < 8$ ensures that none of the circles intersect; C_3 is tangent to C_1 for $t = 8.0013\cdots$. For each $0 < t < 8$, what is the conformal symmetry group of D? This group is determined up to isomorphism in §5, and explicitly in §6.

Let us briefly review the basic ideas regarding conformal symmetries. As our examples will have symmetries that are Möbius maps we begin by discussing Möbius maps. The *extended complex plane*

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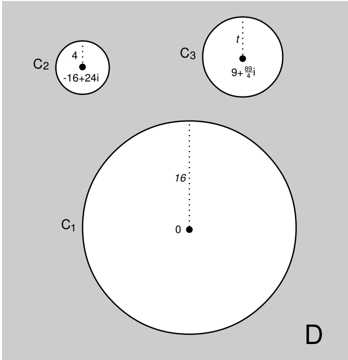

FIGURE 1. The region D is exterior to C_1, C_2 and C_3

is the union of the complex plane $\mathbb{C}$ and the point ∞ at infinity, and is denoted by $\mathbb{C}_\infty$. It is homeomorphic to the unit sphere in $\mathbb{R}^3$ via stereographic projection, and it acquires the structure of a Riemann surface with the introduction of the chart $z \mapsto 1/z$ at ∞. The conformal automorphisms (that is, the holomorphic bijections) of $\mathbb{C}_\infty$ onto itself are the Möbius transformations $z \mapsto (az+b)/(cz+d)$, where $ad - bc \neq 0$. Each Möbius transformation maps circles to circles, where a circle in $\mathbb{C}_\infty$ is either a Euclidean circle or a Euclidean line with ∞ attached. The Möbius transformations that fix ∞ are the maps $z \mapsto az + b$, where $a \neq 0$. These maps form the group of conformal Euclidean similarities, and the subgroup of maps with $|a| = 1$ is the group of conformal Euclidean isometries. The anti-conformal Euclidean isometries are the maps $z \mapsto a\bar{z} + b$ with $|a| = 1$; more generally, the anti-conformal Möbius maps are the maps $z \mapsto (a\bar{z}+b)/(c\bar{z}+d)$, where $ad - bc \neq 0$.

A *region* is a non-empty, open, connected subset of $\mathbb{C}_\infty$. The connectivity of a region is the cardinality of the set of components of its (compact) complement, and this may be finite or infinite. Generally speaking, much is known about regions of finite connectivity; little is known about regions of infinite connectivity.

A *conformal symmetry* (or *automorphism*) of a region D is a bijective holomorphic map of D onto itself. In this paper a conformal symmetry is injective so that we are excluding such maps as $z \mapsto z^2$ which is conformal (in the sense of angle-preserving), but not injective, on $\mathbb{C}\backslash\{0\}$. Because each conformal symmetry of D is a bijection, these symmetries form a group which we denote by $\mathrm{Aut}^+(D)$. An automorphism of a region that fixes three distinct points is the identity map (see [8] and [10]). Also, it is known that if D is a region of finite connectivity which is at least three, then any holomorphic proper self-map of D is a conformal symmetry of D (see [11]).

We end this introduction with a simple example to show that there are regions whose conformal symmetry group is trivial. In fact, the region we construct has a much stronger property: it has no holomorphic self–maps other than the identity map. Let $E = \{1, -1, i, \zeta, \infty\}$, where $\zeta = (1+i)/\sqrt{2}$, and let f be a non–constant holomorphic map of $D = \mathbb{C}_\infty \backslash E$ into itself. Picard's Theorem implies that each point of E is a removable singularity of f, so that f extends to a function that is meromorphic on $\mathbb{C}_\infty$ and hence is a rational map. As f maps D into itself we see that $f^{-1}(E) \subset E$, and as f is rational (and therefore surjective) we conclude that $f(E) = E = f^{-1}(E)$. It follows that f is a bijection of E onto itself, and so some iterate, say f^k of f, acts as the identity on E. In particular, $f^k(z) = \infty$ if and only if $z = \infty$, so that f^k is a polynomial, say of degree d. As $f(z) = 1$ if and only if $z = 1$, and similarly for -1, there must be constants A and B such that $A(z-1)^d + 1 = f(z) = B(z+1)^d - 1$. This can only happen if $d = 1$; thus f^k, and hence f also, is a Möbius map. As f is Möbius, it preserves the concyclic nature of any four points. The circles in $\mathbb{C}_\infty$ that contain ∞ are the Euclidean lines, and each Euclidean line meets E in at most three points (one of which is ∞); therefore f must map the set $\{1, -1, i, \zeta\}$ of four concyclic points onto itself. We deduce that f must fix ∞, so that f is a Euclidean similarity. Clearly, the only Euclidean similarity that maps $\{1, -1, i, \zeta\}$ onto itself is the identity map. For more examples of this type, see [3].

2. Reduction to a Problem in Geometry

In 1851 Riemann gave what is now known as the Riemann Mapping Theorem: *each simply connected proper subregion of* $\mathbb{C}$ *is conformally equivalent to the open unit disc* $\mathbb{D}$. For a discussion of the history of this result, see [12, page 181]. Koebe provided a rigorous proof of Riemann's theorem and, in 1920, he also proved a much stronger result, namely that *each finitely connected region is conformally equivalent to a circular region*, where a *circular region* is a region whose complement is a union of mutually disjoint closed discs.

In 1993, this result was extended to countably connected regions by He and Schramm [6], who also showed that the conformal symmetries of circular regions of countable connectivity are Möbius maps (this was known previously for circular regions of finite connectivity). Although it is not known whether an arbitrary region D is conformally equivalent to a circular region, Maskit [10] has shown that there is a region D' conformally equivalent to D such that $\mathrm{Aut}^+(D')$ consists only of Möbius transformations. In this paper we answer questions 1 and 2 only for circular regions of finite connectivity such that each closed disc in the complement of the region has positive radius. For regions whose complement has closed discs of zero radius, see [3].

Each conformal Möbius transformation acting on $\mathbb{C}_\infty$ is a composition of an even number of inversions in circles in $\mathbb{C}_\infty$, and each anti-conformal Möbius transformation is a composition of an odd number of inversions. We embed $\mathbb{C}$ as the plane $x_3 = 0$ in $\mathbb{R}^3$. If we regard a circle in $\mathbb{C}_\infty$ as the 'equator' of a sphere in $\mathbb{R}^3$, we see that each Möbius transformation can be extended (as the same composition of inversions) so as to act on $\mathbb{R}^3 \cup \{\infty\}$ in such a way that it preserves the upper–half $\mathbb{H}^3 = \{(x_1, x_2, x_3) \,|\, x_3 > 0\}$ of $\mathbb{R}^3$. Now, any *real* Möbius transformations that preserves the upper–half $\mathbb{H}^2 = \{(x_1, x_2) \,|\, x_2 > 0\}$ of the complex plane acts as an isometry of the hyperbolic plane $\mathbb{H}^2$ with the hyperbolic metric $ds^2 = (dx_1^2 + dx_2^2)/x_2^2$. In a similar way, any complex Möbius transformation acts on $\mathbb{H}^3$, which we regard as a model of hyperbolic 3-space with metric $ds^2 = (dx_1^2 + dx_2^2 + dx_3^2)/x_3^2$. In fact, the group of (conformal and anti–conformal) Möbius transformations is precisely the group of isometries of hyperbolic space $\mathbb{H}^3$ (see [1] for details). A corollary of this important fact is that complex analytic questions about conformal symmetries of circular regions can be reduced to geometric questions about the action of the Möbius group of isometries of hyperbolic space $\mathbb{H}^3$. Each circle in $\mathbb{C}_\infty$ is the ideal boundary of a hemisphere in $\mathbb{H}^3$, and these hemispheres are the hyperbolic planes in $\mathbb{H}^3$ (just as semicircles in $\mathbb{H}^2$ orthogonal to $\mathbb{R}$ are geodesics).

We illustrate these ideas in Figure 2. The circular region D lying in $\mathbb{C}_\infty$ is bounded by four circles. One of these circles is C, and this is the ideal boundary of the hyperbolic plane Π. The region enclosed by the four hyperbolic planes is an unbounded hyperbolic polyhedron, P, which has ideal boundary $\overline{D}$. In addition, we can

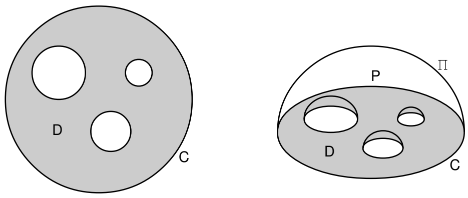

Figure 2

think of P as the hyperbolic convex hull of D; this is the union of all hyperbolic geodesics whose endpoints lie in D.

3. The Solution to Question 1

In general, a conformal symmetry of a region D is defined only on D and it need not extend analytically beyond (or even to) the boundary of D. However, the conformal symmetries of a circular region of countable connectivity are Möbius transformations *and these are defined on the whole of* $\mathbb{C}_\infty$. This is an important point, and this extra information can be used to answer Question 1 in the case of regions of finite connectivity. First, any region of finite connectivity is, by Koebe's result, conformally equivalent to a circular region of finite connectivity. Obviously, conformally equivalent regions have isomorphic symmetry groups so in answering Question 1, we may confine our attention to circular regions of finite connectivity.

Suppose that $D = \mathbb{C}_\infty \setminus (E_1 \cup \cdots \cup E_m)$, where $E_1, \ldots, E_m$ are disjoint closed discs and $m \geqslant 3$. Each conformal symmetry of D, necessarily a Möbius transformation, permutes these discs. This argument provides a homomorphism

$$\Phi : \mathrm{Aut}^+(D) \to S_m, \quad f \mapsto \Phi_f,$$

where $\Phi_f(i) = j$ if and only if $f(E_i) = E_j$, and where S_m is the permutation group on m symbols. If f lies in the kernel of Φ then f fixes each E_i set-wise. By the Brouwer Fixed Point Theorem, f has a fixed point in each E_i. Thus f has at least three fixed points and so is the identity map. Thus Φ is injective, and hence $\mathrm{Aut}^+(D)$ is, up to isomorphism, a subgroup of S_m.

It is well known that each finite Möbius group is isomorphic either to the cyclic or dihedral symmetry group of a 2-dimensional regular polygon or to the rotational symmetry group of a 3-dimensional regular polyhedron (a Platonic solid). See either [1, §5.1] or [7, Chapter VI] for a proof, although we sketch a proof later in this section. There are exactly five Platonic solids: the tetrahedron, cube, octahedron, dodecahedron and icosahedron, and their rotational symmetry groups are, in order, A_4, S_4, S_4, A_5 and A_5. The cube and octahedron have the same rotational symmetry group as they are dual polyhedra; the dodecahedron and icosahedron are also dual polyhedra. These facts show that each finite Möbius group is either cyclic, dihedral or isomorphic to one of A_4, S_4 or A_5. All such groups arise as conformal symmetry groups, and we have now answered Question 1 completely.

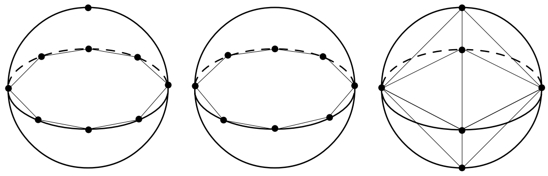

Figure 3

We illustrate these ideas in Figure 3, where each sphere has a number of points removed. The resulting regions have conformal symmetry groups, from left to right, C_8, D_8 and A_4. Two octagons and an octahedron are shown connecting the points to indicate the relationship between subgroups of the Möbius group and the symmetries of the polygons and polyhedra.

Let us sketch a proof that each finite Möbius group G is isomorphic to a finite group of rotations of $\mathbb{R}^3$. Each g in G acts on hyperbolic space $\mathbb{H}^3$. As G is finite, each orbit in $\mathbb{H}^3$ is a finite set. Take any one orbit, say $\mathcal{O}$, and enclose it in the smallest possible closed hyperbolic ball B in $\mathbb{H}^3$, and let ζ be the hyperbolic centre of B. This construction is possible because $\mathcal{O}$ is finite. As $\mathcal{O}$ is G–invariant, so is B, and hence so is ζ. We can now map $\mathbb{H}^3$ bijectively onto the open unit ball $\mathbb{B}^3$ in $\mathbb{R}^3$ (by a Möbius map acting on $\mathbb{R}^3 \cup \{\infty\}$) with ζ mapping to the origin. The elements of G transform into Euclidean rotations, so G is isomorphic (indeed, conjugate in the larger Möbius group) to a finite group of Euclidean rotations of $\mathbb{R}^3$.

We conclude this section with a brief discussion of the higher dimensional case. A (conformal or anti–conformal) Möbius map acting on $\mathbb{R}^m \cup \{\infty\}$ is a composition of inversions across $(m-1)$–dimensional spheres or hyperplanes, and Liouville proved that any conformal or anti–conformal map acting on a subdomain of $\mathbb{R}^m \cup \{\infty\}$ is Möbius. In answer to Question 2 in this case, for each finite group G there is a circular region in some $\mathbb{R}^m \cup \{\infty\}$ such that the group of conformal symmetries of D is isomorphic to G.

4. The Inversive Distance

In general, to understand symmetry groups it is useful to identify group invariants. The inversive distance $\chi(C_1, C_2)$ between two Euclidean circles C_1 and C_2 in $\mathbb{C}$ is given by the equation

$$\chi(C_1, C_2) = \left| \frac{|c_1 - c_2|^2 - r_1^2 - r_2^2}{2r_1r_2} \right|, \tag{4.1}$$

where C_1 has centre c_1 and radius r_1, and C_2 has centre c_2 and radius r_2. There is a similar formula when one or both of C_1 and C_2 are Euclidean lines. The inversive distance is invariant under Möbius transformations. If C_1 and C_2 are disjoint then there is a Möbius transformation f which maps C_1 and C_2 to concentric circles of radii s_1 and s_2 that are centred on the origin, in which case $\chi(C_1, C_2) = \frac{1}{2}(s_1/s_2 + s_2/s_1)$.

The inversive distance has a simple interpretation with hyperbolic geometry. The circles C_1 and C_2 are the ideal boundaries of hyperbolic planes Π_1 and Π_2 in $\mathbb{H}^3$. If C_1 and C_2 are disjoint then Π_1 and Π_2 are separated by a positive hyperbolic distance ρ (see Figure 4), and $\chi(C_1, C_2) = \cosh \rho$. If C_1 and C_2 intersect in an angle θ in $[0, \pi/2]$ then $\chi(C_1, C_2) = \cos \theta$. For more information about the inversive distance, see [1] and [5, page 129].

We can use this invariant of Möbius transformations to calculate conformal symmetry groups. Consider a circular region D with boundary circles $C_1, \dots, C_m$, each of positive radius, and corresponding hyperbolic planes $\Pi_1, \dots, \Pi_m$. If $\phi \in \mathrm{Aut}^+(D)$ then

$$\chi(\phi(C_i), \phi(C_j)) = \chi(C_i, C_j) \qquad (i, j = 1, \dots, m).$$

A converse to this observation, which is discussed in the next section, enables us to answer Question 2.

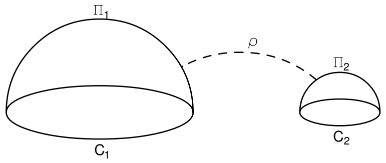

FIGURE 4

5. THE SOLUTION TO QUESTION 2

Let Ω and Ω' be two circular regions in $\mathbb{C}_\infty$ bounded by circles C_j and C_j', $j = 1, \ldots, m$, each of positive radius.

Theorem 5.1. *There exists a hyperbolic isometry ϕ with $\phi(\Omega) = \Omega'$ and $\phi(C_j) = C_j'$ for each $j = 1, \ldots, m$ if and only if $\chi(C_j, C_k) = \chi(C_j', C_k')$ for all $j, k = 1, \ldots, m$.*

We emphasise that the hyperbolic isometry ϕ in Theorem 5.1 may correspond to a conformal or an anti-conformal Möbius transformation.

The circles C_j in Theorem 5.1 are pairwise disjoint. The theorem can be generalised to allow some, but not all, collections of circles. An example of the failure of Theorem 5.1 when circles are allowed to intersect is shown in Figure 5. The equalities $\chi(C_i, C_j) = \chi(C_i', C_j')$ hold for $i, j = 1, 2, 3, 4$; each $\chi(C_i, C_j)$ is either 0 or 1 depending on whether C_i and C_j are orthogonal or parallel. The regions D and D' are conformally equivalent as they are both simply connected. Nevertheless, there is not a conformal map $\phi : D \to D'$ with $\phi(C_i) = C_i'$ for $i = 1, 2, 3, 4$, because the rectangular regions are not similar, in the Euclidean sense.

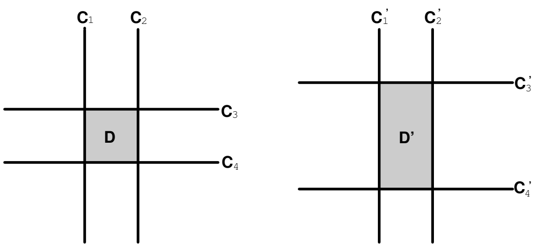

FIGURE 5

A proof of Theorem 5.1 is discussed in the next section.

Question 2 can now be answered with the help of Theorem 5.1. The existence of a conformal symmetry is equivalent to the equality of certain inversive distances, and these inversive distances are easily computed using (4.1). Consider the example in Figure 1. We have

$$\chi(C_1, C_2) = \frac{35}{8}, \quad \chi(C_1, C_3) = \frac{5121 - 16t^2}{512t} \quad \text{and}$$
$$\chi(C_2, C_3) = \frac{9793 - 256t^2}{128t}.$$

It can be checked that $\chi(C_2, C_3) > \chi(C_1, C_3)$ and $\chi(C_2, C_3) > \chi(C_1, C_2)$ for each $t \in (0, 8)$. Also, $\chi(C_1, C_3) = \chi(C_1, C_2)$ if and only if $t = 9/4$. Therefore when $t = 9/4$ there is a Möbius map σ (possibly anti-conformal) that fixes C_1 and interchanges C_2 and C_3. In fact, f is conformal therefore $\mathrm{Aut}^+(D)$ is isomorphic to C_2 when t equals $9/4$, and otherwise $\mathrm{Aut}^+(D)$ is trivial. We show this in §6.

6. Reduction to a Problem in Linear Algebra

There are several models of hyperbolic space and, for calculating symmetries of circular regions (or, equivalently, for calculating hyperbolic isometries that preserve a particular collection of hyperbolic planes), there are many advantages to the hyperboloid model. To construct the hyperboloid model of 3-dimensional hyperbolic space, we endow $\mathbb{R}^4$ with the Lorentz inner-product

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4.$$

The hyperboloid sheet

$$\mathcal{H} = \{x \in \mathbb{R}^4 \,:\, \langle x, x\rangle = -1,\ x_4 > 0\}$$

becomes a model of 3-dimensional hyperbolic space with metric d determined by

$$\cosh d(x, y) = -\langle x, y\rangle.$$

A hyperbolic plane Π in $\mathcal{H}$ is the intersection of $\mathcal{H}$ with a 3-dimensional Euclidean plane P through the origin. Each such plane P has a Lorentz unit normal n such that $P = \{x \,:\, \langle x, n\rangle = 0\}$. The hyperbolic isometries of $\mathcal{H}$ are the positive Lorentz orthogonal maps, namely the linear isomorphisms of $\mathbb{R}^4$ that preserve the Lorentz inner-product and map $\mathcal{H}$ to $\mathcal{H}$. A hyperbolic isometry ϕ maps one hyperbolic plane Π to another, Π', if and only if ϕ maps a unit normal n of Π to a unit normal n' of Π'. The inversive distance between Π and Π' is $|\langle n, n'\rangle|$.

The elements of a proof of Theorem 5.1 are all present in the hyperboloid model. A collection of hyperbolic planes can be represented by associated unit normals $n_1, \dots, n_m$, and the inversive distances are $|\langle n_i, n_j\rangle|$, for $i, j = 1, \dots, m$. Let n_j and n'_j be unit normals associated to C_j and C'_j in Theorem 5.1. By taking some (or all) of the Lorentz normals to the planes as basis vectors, one can construct a linear transformation ϕ that maps n_j to n'_j for each j, and the Lorentz orthogonality of ϕ is ensured by the inversive distance condition.

There are other benefits of the hyperboloid model. The proof of Theorem 5.1 is valid for more general collections of boundary circles C_j; in particular, there may be infinitely many circles. The Lorentz model of hyperbolic space can be constructed in all dimensions, and likewise Theorem 5.1 generalises to all dimensions. Also, the proof of Theorem 5.1 can be adapted to give results analogous to Theorem 5.1 about punctured spheres.

The final advantage of the hyperboloid model that we mention is that it allows one to construct conformal symmetry groups explicitly, not just up to isomorphism. Consider the example of Figure 1. In §5 we showed that the automorphism group is trivial unless $t = 9/4$, in which case the automorphism group is C_2. One can choose Lorentz normals n_1, n_2 and n_3 in $\mathbb{R}^4$ that correspond to the circles C_1, C_2 and C_3 and then construct a Lorentz orthogonal map L that fixes n_1 and interchanges n_2 and n_3. The Möbius map σ corresponding to L is conformal because $\det L = 1$ (the alternative is that if $\det L = -1$ then σ is anti-conformal). In this case, $\mathrm{Aut}^+(D) = \{1, \sigma\}$,

$$L = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{-41}{9} & 0 & \frac{-40}{9} \\ 0 & \frac{425}{12} & \frac{257}{32} & \frac{3485}{96} \\ 0 & \frac{1285}{36} & \frac{255}{32} & \frac{10537}{288} \end{pmatrix}$$

and

$$\sigma(z) = \frac{20iz + 256}{z - 20i}.$$

(One can check that the Lorentz orthogonal map L and the Möbius map σ correspond to the same hyperbolic isometry using [14, Theorem 10].)

7. Regions of Infinite Connectivity

Questions 1 and 2 are more complicated for regions of infinite connectivity, even for regions with a countable number of boundary components. There are many examples of such regions and their automorphisms groups in the literature. For a simple example, $\mathrm{Aut}^+(\mathbb{C} \setminus \mathbb{Z})$ is infinite, and is generated by $z \mapsto -z$ and $z \mapsto z+1$. More complicated examples arise in the theory of discrete Möbius groups where the groups known as *Schottky groups* play a prominent role. Without going into details here, one can construct many examples of a perfect, Cantor-like set L (the *limit set*), and discrete, finitely or infinitely generated, free groups of Möbius conformal symmetries of $\mathbb{C}_\infty \setminus E$.

We shall content ourselves here by discussing just one example (for an extension of these ideas see [2]). Let D be the complement of a sequence $z_1, z_2, \ldots$ of distinct points in $\mathbb{C}$ that converges to ∞. Next, let f be a non–constant holomorphic self-map of D (f may be, but need not be, a conformal symmetry of D, and it need not be injective). By Picard's Theorem, each z_n is a removable singularity for f, and it is easy to see that for each n, $f(z_n) \neq \infty$. Thus f extends to an entire function. As f maps D into itself, so too do all the iterates of f, and Montel's Theorem (on three omitted values) implies that the family of iterates is normal in D. Thus the Julia set of f lies in $\{z_1, z_2, \ldots, \infty\}$. It is well known that the Julia set of an entire function is uncountable unless the function is a linear polynomial (see [4]). We conclude that the entire function f is a linear polynomial; thus *every holomorphic self-map of D is of the form $z \mapsto az + b$, where $a \neq 0$.* In particular, the group of conformal symmetries of D is a group of Mobius maps. Each (Möbius) conformal symmetry of D must fix ∞. Moreover, $\mathrm{Aut}^+(D)$ cannot contain any maps $g(z) = az + b$ with $|a| \neq 1$ as otherwise the set of images of z under the forward and backward iterates of g accumulate at the finite fixed point of g. Therefore $\mathrm{Aut}^+(D)$ consists only of Euclidean rotations and translations. As $E \cap \mathbb{C}$ is discrete, and invariant under the conformal symmetries of D, $\mathrm{Aut}^+(D)$ must be a discrete group of Euclidean isometries. Thus, as is well known, $\mathrm{Aut}^+(D)$ is either a finite cyclic group of Euclidean rotations, or one of the three conformal frieze groups (these are the discrete Euclidean isometry groups whose translations form a cyclic subgroup), or one of the five conformal wallpaper groups (whose translations form an abelian subgroup of rank two). Of course, in the problem we are discussing here the determination of $\mathrm{Aut}^+(D)$ depends on the precise location of the points z_n. In fact, for almost all choices of the points z_n, the group $\mathrm{Aut}^+(D)$ is trivial.

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Another Proof of Hadamard's Determinantal Inequality

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Abstract. We offer a new proof of Hadamard's celebrated inequality for determinants of positive matrices that is based on a simple identity, which may be of independent interest.

A hermitian $n \times n$ matrix A is said to be positive, if, for all $n \times 1$ vectors x, $x^*Ax > 0$ unless x is the zero vector. Thus, if A is positive, all of its principal sub-matrices are also positive. Moreover, A is positive if and only if the determinants of all these sub-matrices are positive. In particular, if $A = [a_{ij}]$ is positive, then all of its diagonal entries, $a_{11}, a_{22}, \ldots, a_{nn}$, and its determinant, $\det A$, are positive. These are well-known facts about positive matrices that can be found in most textbooks on Matrix Analysis, such as, for instance, [1] and [3].

In 1893, Hadamard [2] discovered a fundamental fact about positive matrices, viz., that, for such $A = [a_{ij}]$,

$$\det A \leq a_{11} a_{22} \cdots a_{nn}.$$

Our purpose here is to present another proof of Hadamard's inequality which is based on the following identity.

Lemma 1. *Suppose A is an $n \times n$ matrix, $\tilde{A}$ is its cofactor matrix, and x, y are $n \times 1$ vectors. Then*

$$\det A - \det \begin{bmatrix} A & x \\ y^t & 1 \end{bmatrix} = x^t \tilde{A} y.$$

Proof. Identify $\mathbb{C}^n$ with the space of $n \times 1$ vectors with complex entries, and consider the bilinear form

$$B(x, y) = \det A - \det \begin{bmatrix} A & x \\ y^t & 1 \end{bmatrix}, \; x, y \in \mathbb{C}^n.$$

Denoting the usual orthonormal basis of $\mathbb{C}^n$ by $e_1, e_2, \ldots, e_n$, it's easy to see that

$$B(e_i, e_j) = A_{i,j},$$

the ijth element in $\tilde{A}$. Hence, if

$$x = \sum_{i=1}^{n} x_i e_i, \; y = \sum_{i=1}^{n} y_i e_i \in \mathbb{C}^n,$$

then, by bilinearity,

$$\begin{aligned} B(x, y) &= \sum_{i,j=1}^{n} x_i y_j B(e_i, e_j) = \sum_{i,j=1}^{n} x_i y_j A_{ij} \\ &= \sum_{i=1}^{n} x_i \sum_{j=1}^{n} A_{ij} y_j = x^t \tilde{A} y, \end{aligned}$$

as stated. □

As an easy consequence, we have:

Theorem 1. *Suppose A is an $n \times n$ positive matrix. Then*

$$\det \begin{bmatrix} A & x \\ x^* & 1 \end{bmatrix} \leq \det A \qquad (x \in \mathbb{C}^n),$$

with equality if and only if $x = 0$.

Proof. Since A is invertible, and its inverse is also positive, it follows from the lemma that

$$\det A - \det \begin{bmatrix} A & x \\ x^* & 1 \end{bmatrix} = x^* \tilde{A} x = \det A x^* A^{-1} x \geq 0,$$

and the inequality is strict unless x is the zero vector. The result follows. □

Corollary 1. *Denoting by A_k the sub-matrix of A of order $k \times k$ that occupies the top left-hand corner of $A = [a_{ij}]$, then*

$$\det A \leq a_{nn} \det A_{n-1},$$

and the inequality is strict unless all the entries in the last column of A, save the last one, are zero.

Hadamard's classical inequality is an immediate consequence of this, viz.,

Theorem 2 (Hadamard)**.** *If* $A = [a_{ij}]$ *is an* $n \times n$ *positive matrix, then*

$$\det A \leq \prod_{i=1}^{n} a_{ii},$$

with equality if and only if A *is a diagonal matrix.*

Coupling this with the fact that the determinant of A is the product of its eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_n$, say, we can affirm that

$$\prod_{i=1}^{n} \lambda_i \leq \prod_{i=1}^{n} a_{ii},$$

with equality if and only if A is a diagonal matrix. But, also, the sum of the eigenvalues of A is its trace, i.e.,

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii}.$$

In other words, denoting by $\sigma_r(x_1, x_2, \ldots, x_n)$ the rth symmetric function of n variables, $x_1, x_2, \ldots, x_n$, we have that

$$\sigma_r(\lambda_1, \lambda_2, \ldots, \lambda_n) \leq \sigma_r(a_{11}, a_{22}, \ldots, a_{nn}),$$

if $r = 1$ or $r = n$. It's of interest to observe that this remains true if $1 < r < n$. For completeness, we sketch a proof of this statement

Indeed, $\sigma_r(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the coefficient a_r of t^{n-r} in the polynomial

$$\prod_{i=1}^{n} (t + \lambda_i) = \det(A + tI).$$

But a_r is equal to the sum of the determinants of all the $r \times r$ principal sub-matrices of A, which are also positive. Hence, applying Hadamard's result to each of them, we deduce that $a_r \leq \sigma_r(a_{11}, a_{22}, \ldots, a_{nn})$ as claimed.

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Convergence from Below Suffices

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Abstract. An elementary application of Fatou's lemma gives a strengthened version of the monotone convergence theorem. We call this the *convergence from below theorem*. We make the case that this result should be better known, and deserves a place in any introductory course on measure and integration.

1. The Convergence from Below Theorem

Three famous convergence-related results appear in most introductory courses on measure and integration: the monotone convergence theorem, Fatou's lemma and the dominated convergence theorem. In teaching this material it is common to follow the approach taken in, for example, [1, Chapter 1]. There Rudin begins by proving the monotone convergence theorem and then deduces Fatou's lemma. Finally, he deduces the dominated convergence theorem from Fatou's lemma. The result which we call the *convergence from below theorem* (Theorem 1.2 below) is essentially distilled from this proof of the dominated convergence theorem ([1, pp. 26–27]). We do not claim originality for this result, or for the related Theorem 1.3. They are presumably known, although we know of no explicit references for them. However, we wish to make a case that that they should be better known than they are. In particular, we suggest that Theorem 1.2 deserves a name and a place in the syllabus when this material is taught.

Throughout we discuss results concerning pointwise convergence. In the usual way, there are versions of all these results in terms of almost-everywhere convergence instead.

For convenience, we shall use the following terminology. Let X be a set, let (f_n) be a sequence of functions from X to $[0,\infty]$ and let f be another function from X to $[0,\infty]$. We say that the functions

f_n converge to f *from below on* X if the functions f_n tend to f pointwise on X and $f_n(x) \leq f(x)$ $(n \in \mathbb{N},\ x \in X)$. We say that the functions f_n converge to f *monotonely from below on* X if the functions f_n tend to f pointwise on X and, for all $x \in X$, we have $f_1(x) \leq f_2(x) \leq f_3(x) \leq \cdots$.

We begin by recalling the statement of the monotone convergence theorem.

Theorem 1.1. (Monotone convergence theorem) *Let* $(X, \mathcal{F}, \mu)$ *be a measure space, and let* $f : X \to [0, \infty]$ *be a measurable function. Let* (f_n) *be a sequence of measurable functions from* X *to* $[0, \infty]$ *which converge to* f *monotonely from below on* X. *Then*

$$\int_X f \,\mathrm{d}\mu = \lim_{n\to\infty} \int_X f_n \,\mathrm{d}\mu \,.$$

The measurability assumption on f is, of course, redundant here as it follows from the pointwise convergence of f_n to f. We now observe that an elementary application of Fatou's lemma shows that we may weaken the monotone convergence assumption. We have not found this result stated explicitly in the literature, and it does not appear to have a name. We propose to call it the *convergence from below theorem*.

The concepts involved in the statements and applications of the monotone convergence theorem and the dominated convergence theorem are relatively simple. We suggest that convergence from below is a similarly simple concept, which should appeal to all levels of student. In particular, those students who find the concepts of lim inf and lim sup difficult may be happier applying the convergence from below theorem rather than Fatou's lemma (where possible).

Theorem 1.2. (Convergence from below theorem) *Let* $(X, \mathcal{F}, \mu)$ *be a measure space, and let* $f : X \to [0, \infty]$ *be a measurable function. Let* (f_n) *be a sequence of measurable functions from* X *to* $[0, \infty]$ *which converge to* f *from below on* X. *Then*

$$\int_X f \,\mathrm{d}\mu = \lim_{n\to\infty} \int_X f_n \,\mathrm{d}\mu \,.$$

Proof. Clearly

$$\limsup_{n\to\infty} \int_X f_n \,\mathrm{d}\mu \leq \int_X f \,\mathrm{d}\mu \,.$$

However, by Fatou's lemma,

$$\int_X f \, \mathrm{d}\mu \leq \liminf_{n \to \infty} \int_X f_n \, \mathrm{d}\mu \, .$$

The result follows immediately. □

Remarks.

(1) The monotone convergence theorem is now a special case of our stronger convergence from below theorem.
(2) In the case where $\int_X f \, \mathrm{d}\mu < \infty$, the convergence from below theorem is an immediate consequence of the dominated convergence theorem.
(3) In the case where $\int_X f \, \mathrm{d}\mu = \infty$, the result does not follow directly from either the monotone convergence theorem or the dominated convergence theorem. The following elementary result clarifies the situation in this case.

Theorem 1.3. *Let $(X, \mathcal{F}, \mu)$ be a measure space, and let $f : X \to [0, \infty]$ be a measurable function with $\int_X f \, \mathrm{d}\mu = \infty$. Let (f_n) be a sequence of measurable functions from X to $[0, \infty]$ which converge to f pointwise on X. Then*

$$\lim_{n \to \infty} \int_X f_n \, \mathrm{d}\mu \ = \infty.$$

Proof. By Fatou's lemma,

$$\infty = \int_X f \, \mathrm{d}\mu \leq \liminf_{n \to \infty} \int_X f_n \, \mathrm{d}\mu \, .$$

It follows immediately that $\lim_{n \to \infty} \int_X f_n \, \mathrm{d}\mu \ = \infty$, as required. □

We suggest that the convergence from below theorem deserves a place between Fatou's lemma and the dominated convergence theorem: the dominated convergence theorem may be deduced from the convergence from below theorem as follows. This proof is based on the proof given in [1, pp. 26–27], but applying the convergence from below theorem in the middle.

Theorem 1.4. (Dominated convergence theorem) *Let $(X, \mathcal{F}, \mu)$ be a measure space, let $g : X \to [0, \infty]$ be a measurable function. with $\int_X f \, \mathrm{d}\mu < \infty$ and let f be a measurable function from X to $\mathbb{C}$. Let (f_n) be a sequence of measurable functions from X to*

C which converge to f pointwise on X and such that $|f_n(x)| \leq g(x)$ $(n \in \mathbb{N}, x \in X)$. Then

$$\lim_{n\to\infty} \int_X |f_n - f| \, \mathrm{d}\mu = 0$$

and

$$\int_X f \, \mathrm{d}\mu = \lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu \, .$$

Proof. The second equality follows quickly from the first. To prove the first equality, observe that the non-negative, measurable functions $2g - |f_n - f|$ converge to the function $2g$ from below. Thus, by the convergence from below theorem,

$$\lim_{n\to\infty} \int_X (2g - |f_n - f|) \, \mathrm{d}\mu = \int_X 2g \, \mathrm{d}\mu \, .$$

The result now follows by subtracting $\int_X 2g \, \mathrm{d}\mu$ from both sides and rearranging. □

As discussed above, the convergence from below theorem is more than covered by a combination of the dominated convergence theorem and Theorem 1.3. Also, since the convergence from below theorem is such an elementary consequence of Fatou's lemma, any applications may also be deduced from that lemma. However, the monotone convergence theorem continues to be used in the literature, and any application of the monotone convergence theorem can be replaced directly by an application of the convergence from below theorem. Of course, we then only need to check the weaker conditions of the latter theorem.

Also, the convergence from below theorem can be used to give elegant solutions to simple problems where neither the monotone convergence theorem nor the dominated convergence theorem apply directly. Here is such an application (an elementary undergraduate exercise).

Exercise. Let λ denote Lebesgue measure on $\mathbb{R}$. Prove that, for every Lebesgue measurable subset E of $\mathbb{R}$, we have

$$\int_E x^2 \, \mathrm{d}\lambda(x) = \lim_{n\to\infty} \int_E \left(x^2 - \frac{1}{n} |x \sin nx| \right) \mathrm{d}\lambda(x) \, .$$

Solution. Since $|x \sin nx| \leq nx^2$ $(n \in \mathbb{N},\ x \in \mathbb{R})$, the result is an immediate consequence of the convergence from below theorem.

We may, instead, apply Fatou's lemma directly. This does, of course, lead to a quick solution which essentially proves the convergence from below theorem again along the way.

We may also consider separately the cases where $\int_E x^2 \,\mathrm{d}\lambda(x) < \infty$ and where $\int_E x^2 \,\mathrm{d}\lambda(x) = \infty$. In the first case we may apply the dominated convergence theorem, and in the second case we may use Theorem 1.3. However the use of the convergence from below theorem renders this splitting into two cases unnecessary.

2. Proving the Convergence from Below Theorem Directly

Above we suggested following the usual development of the theory, but inserting the convergence from below theorem between Fatou's lemma and the dominated convergence theorem. There are several alternatives, however. For example, we can prove Fatou's lemma directly first and then deduce the convergence from below theorem. The monotone convergence theorem and the dominated convergence theorem then follow easily.

Another approach is to modify the standard proof of the monotone convergence theorem ([1, 1.26]) in order to give a direct proof of the convergence from below theorem. The monotone convergence theorem, dominated convergence theorem and Fatou's lemma are then corollaries of this. We conclude with such a direct proof.

In this proof we avoid explicit reference to lim inf and lim sup in order to make the proof more accessible to students who have difficulty with these concepts. However, only minor changes are needed to give a direct proof of Fatou's lemma instead.

Direct proof of Theorem 1.2. First note that we have $\int_X f_n \,\mathrm{d}\mu \leq \int_X f \,\mathrm{d}\mu$ $(n \in \mathbb{N})$. Thus it is sufficient to prove that, for all $\alpha < \int_X f \,\mathrm{d}\mu$, $\int_X f_n \,\mathrm{d}\mu$ is eventually greater than α, i.e., there is an $N \in \mathbb{N}$ such that, for all $n \geq N$, we have $\int_X f_n \,\mathrm{d}\mu > \alpha$. Given such an α, the definition of the integral tells us that there is a nonnegative, simple measurable function s with $s(x) \leq f(x)$ $(x \in X)$ and such that $\int_X s \,\mathrm{d}\mu > \alpha$. Choose $c \in (0,1)$ large enough that $\int_X cs \,\mathrm{d}\mu > \alpha$. Set $A_n = \{x \in X : cs(x) \leq f_n(x)\}$ and, for each $k \in \mathbb{N}$, set

$$B_k = \bigcap_{n \geq k} A_n = \{x \in X : cs(x) \leq f_n(x) \text{ for all } n \geq k\}.$$

Clearly, $B_1 \subseteq B_2 \subseteq \cdots$. We claim that $\bigcup_{k=1}^{\infty} B_k = X$. Let $x \in X$. If $s(x) > 0$, then $cs(x) < f(x)$, and so $x \in B_k$ provided that k is large enough. On the other hand, if $s(x) = 0$, then $x \in B_k$ for all $k \in \mathbb{N}$. This proves our claim. By standard continuity properties of measures, we have

$$\int_X cs \, \mathrm{d}\mu = \lim_{k\to\infty} \int_{B_k} cs \, \mathrm{d}\mu \,.$$

Choose $N \in \mathbb{N}$ such that $\int_{B_N} cs \, \mathrm{d}\mu > \alpha$. For all $n \geq N$ and $x \in B_N$ we have $cs(x) \leq f_n(x)$. Thus, for $n \geq N$, we have

$$\int_X f_n \, \mathrm{d}\mu \geq \int_{B_N} f_n \, \mathrm{d}\mu \geq \int_{B_N} cs \, \mathrm{d}\mu > \alpha,$$

as required. □

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MacCool's Proof of Napoleon's Theorem

A sequel to The MacCool/West Point [1]

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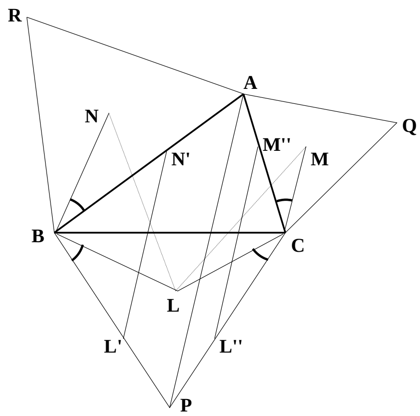
I came across this incredibly short proof in one of MacCool's notebooks. Napoleon's Theorem is one of the most often proved results in mathematics, but having scoured the World Wide Web at some length I have yet to find a proof that comes near to matching this particular one for either brevity or simplicity.

MacCool refers to equilateral triangles as *e-triangles* and he uses κ to denote the distance from a vertex of an e-triangle with unit side to its centroid. Naturally κ is a universal constant. He also treats anti-clockwise rotations as positive and clockwise rotations as negative.

Theorem 1. *If exterior e-triangles are erected on the sides of any triangle then their centroids form a fourth e-triangle.*



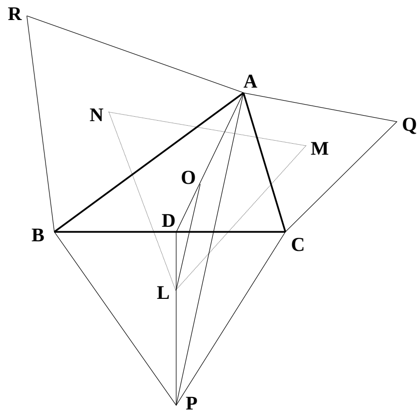
[1] Irish Math. Soc. Bulletin **57** (2006), 93–97

Proof. Let ABC be any triangle and construct the three exterior e-triangles with centroids L, M, N as shown. Rotate LN by $-30°$ about B to give $L'N'$ and LM by $+30°$ about C giving $L''M''$. Since all four marked angles are $30°$ it follows that L', N', L'', M'' will lie on BP, BA, CP, CA respectively and $\kappa = BL' \colon BP = BN' \colon BA = CL'' \colon CP = CM'' \colon CA$. Then by similarity $L'N' = \kappa AP = L''M''$ and $L'N' \parallel AP \parallel L''M''$ so $LN = LM$ and the angle between them is $30° + 30° = 60°$. Hence ΔLMN is an e-triangle. □

Theorem 1 is the classical Napoleon theorem. MacCool refers to the resultant e-triangle as the *outer triangle* to distinguish it from the *inner triangle* whose vertices are the centroids of the internally erected e-triangles.

The proof shows that each side of the outer triangle is equal to κAP. Since it could equally well have used BQ or CR instead this means $AP = BQ = CR$. The common length of these three lines is central to the next result. Also required is the fact that the centroid lies one third of the way along any median. This important property is easily deduced by observing that the medians of any triangle dissect it into six pieces of equal area.

Theorem 2. *The centroids of the outer triangle and the original triangle are coincident.*



Proof. Let D be the mid point of BC, O be the centroid of ΔABC, and L be the centroid of ΔBPC. Then $DA = 3DO$ and $DP = 3DL$ so ΔDLO and ΔDPA are similar, giving $AP \parallel OL$ and $AP = 3\,OL$.

Likewise $BQ = 3\,OM$ and $CR = 3\,ON$. Since $AP = BQ = CR$ the distances from O to the vertices of ΔLMN are equal. As ΔLMN is equilateral O must be its centroid. □

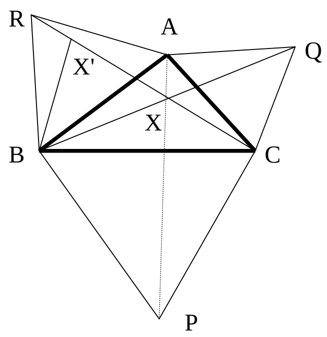
Next MacCool fixes ΔBPC and allows A to vary continuously throughout the plane. He notes that the proofs of these two theorems still apply whenever A drops below the level of BC, in effect making the angle at A reflexive and the angles at B and C negative. Essentially this is because the three e-triangles always retain their original orientation. For the orientation of an e-triangle to change under continuous deformation its area must first become zero which means that it must shrink to a point, but for the e-triangles in question this can only happen at B or C. So long as A avoids those two points no orientational changes to the e-triangles can occur.

However one subtle change does take place as A drops below BC in that the orientation of ΔABC itself changes. When that happens the e-triangles become internal rather than external. This has the following consequence.

Theorem 3. *The inner triangle is an e-triangle whose centroid coincides with the centroid of the original triangle.*

The next result gives an alternative proof that $AP = BQ = CR$. Only the "external" proof is given since the "internal" case is handled by exactly the same proof with the assumption that A lies below rather than above BC.

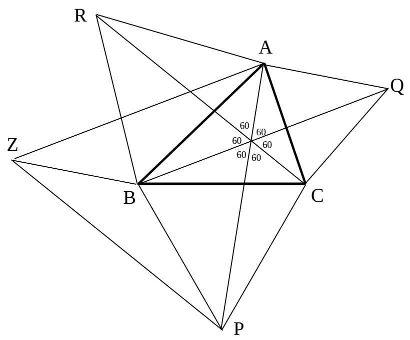
Theorem 4. *Suppose external (internal) e-triangles are erected on the sides of a given triangle. Then the three lines joining each vertex of the given triangle to the remote vertex of the opposite e-triangle are equal in length, concurrent, and cut one another at angles of* $60°$.



Proof. Let ΔABC be given and CBP, ACQ, BAR be the external e-triangles. Clearly ΔABQ is a $+60°$ rotation of ΔARC about A, ΔBCR is a $+60°$ rotation of ΔBPA about B and ΔCAP is a $+60°$ rotation of ΔCQB about C. It follows that $AP = BQ = CR$ and all angles of intersection are 60°. To prove concurrency assume BQ and CR cut at X and construct BX' by rotating BX through $+60°$ about B as shown. Since $\angle BXR = 60°$ and $BX = BX'$ it follows that X' must lie on CR. However a rotation of the line $CX'R$ through $-60°$ about B will map $C \mapsto P$, $R \mapsto A$, and $X' \mapsto X$. Therefore A, X, and P are collinear which means that AP, BQ, CR must be concurrent. □

MacCool next studies the areas of the various triangles. He uses (UVW) to denote the *algebraic* area of ΔUVW. In other words (UVW) is equal to the area of ΔUVW when the orientation of ΔUVW is positive, and minus that value whenever the orientation is negative.

Lemma 5. *In the diagram below BPC, ACQ, and ARB are e-triangles whose mean area is Ω, and Z is constructed so that $AZBQ$ is a parallelogram. Then AZP is also an e-triangle and* $2(AZP) = 3\Omega + 3(ABC)$.



Proof. As $AZBQ$ is a parallelogram $\angle ZAP$ is alternate to an angle of 60° so it too is 60°. Also $AP = BQ = AZ$ so AZP must be an e-triangle. Clearly

$(AZP) = (ABP) + (BZP) + (AZB)$ by tesselation

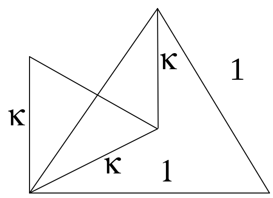
$\quad = (ABP) + (APC) + (ABQ)$

as $(APC) = (BZP)$ and $(ABQ) = (AZB)$.

Now $(BCR) = (ABP)$ and $(BCQ) = (APC)$ and $(ARC) = (ABQ)$ therefore

$$\begin{aligned} 2(AZP) &= (ABP)+(APC)+(ABQ)+(BCR)+(BCQ)+(ARC) \\ &= 3\Omega + 3(ABC). \end{aligned}$$ □

The diagram below shows two e-triangles, one with unit side and the other with side κ. Although I have found no evidence that MacCool was familiar with Pythagoras, he inferred from this diagram that $3\kappa^2 = 1$ and he deduced that the areas of the inner and outer triangles were one third the area of an e-triangle of side AP.



The area of the smaller equilateral triangle is clearly κ^2 that of the larger, from which it follows that κ must satisfy the equation :

$$3\kappa^2 = 1$$

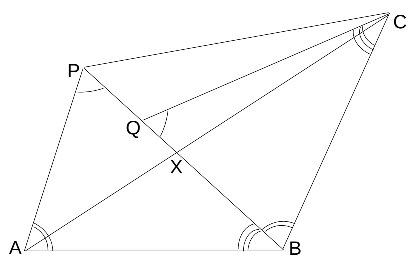
Theorem 6. *The mean area of the three e-triangles plus (minus) the area of the original triangle equals twice the area of the outer (inner) triangle.*

Proof. Let Δ be the area of the outer triangle. As explained on the previous page $(AZP) = 3\Delta$. Applying Lemma 5 now yields $2\Delta = \Omega + (ABC)$. Alternatively, if Δ is the area of the inner triangle this equation still holds, but there is a caveat. The orientations of ΔAZP and the inner triangle don't change as long as A avoids the point P where the latter shrinks to a point, but ΔABC has changed its orientation and so the value of (ABC) is now negative. Hence rewriting the equation in positive terms, $2\Delta = \Omega - (ACB)$. □

Corollary 7. *The area of the outer triangle is that of the inner triangle plus that of the original one.*

Finally MacCool presents a generalisation of Theorem 1.

Lemma 8. *Let A, B, C be non-collinear and X any point between A and C. Construct P and Q on BX such that $\angle PAB = \angle XBC$ and $\angle QCB = \angle XBA$. Then the triangles PAB and QBC are directly* *similar, moreover P and Q coincide if and only if $AX : XC = AB^2 : BC^2$.*



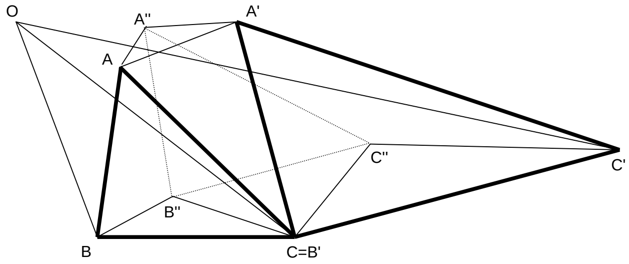
Proof. Clearly ΔPAB and ΔQBC are directly similar. Suppose $BC = \lambda AB$ and $XC = \mu AX$. Then $(QBC) = \lambda^2(PAB)$ whereas $(PBC) = \mu(PAB)$. If P and Q coincide then clearly $\mu = \lambda^2$. Conversely if $\mu = \lambda^2$ then $(PBC) = (QBC)$ so $(PQC) = 0$ which implies $P = Q$. □

Note that if AB and BC have equal length then ΔPAB and ΔPBC are similar (but not directly similar) for all points P on the bisector of $\angle ABC$. Also the lines AB and BC (extended) divide the plane into four zones, and if a point O exists such that ΔOAB and ΔOBC are directly similar then O must lie in the zone that includes the line segment AC. This leads to a key result.

Corollary 9. *If the points A, B, C are non-collinear then there exists a unique point O such that the triangles OAB and OBC are directly similar.*

Theorem 10 (Generalised Napoleon)**.** *Let ABC and $A'B'C'$ be directly similar triangles with a common vertex $C = B'$. Suppose A'', B'', C'' are chosen such that the triangles $AA'A''$, $BB'B''$, $CC'C''$ are directly similar. Then so too are the triangles $A''B''C''$ and ABC.*

Proof. There are 3 separate cases. First if B' is midway between B and C' then $ABB'A'$ is a parallelogram and the result follows easily. Otherwise if B, B', C' are collinear take O to be the point where AA' cuts BB'. Then $\Delta A'B'C'$ is a dilation of ΔABC and it is clear that $\Delta A''B''C''$ may be obtained from ΔABC by a rotation



of $\angle AOA''(= \angle BOB'' = \angle COC'')$ about O followed by a dilation of size OA''/OA. So once again the result holds. Finally if B, B', C' aren't collinear apply Corollary 9 to $\Delta BB'C'$ (aka BCC') giving the point O such that OBB' and OCC' are directly similar. Let $\theta = \angle BOB' = \angle COC'$ and $\lambda = OB' : OB = OC' : OC$. Let τ be the transformation that first rotates through the angle θ about O and then dilates by the scaling factor λ. Clearly τ preserves directly similar figures and maps $B \mapsto B', C \mapsto C'$ so as ABC and $A'B'C'$ are directly similar it must also map $A \mapsto A'$. Thus $\angle AOA' = \theta$ and $OA' : OA = \lambda$ from which it follows that $\Delta OAA'$ is directly similar to both $\Delta OBB'$ and $\Delta OCC'$. Then $OAA''A'$, $OBB''B'$, $OCC''C'$ are directly similar quadrilaterals so OAA'', OBB'', OCC'' are directly similar triangles. Thus $OA'' : OA = OB'' : OB = OC'' : OC = \mu$ and $\angle AOA'' = \angle BOB'' = \angle COC'' = \phi$ for some μ and ϕ. That means the quadrilateral $OA''B''C''$ may be obtained from $OABC$ by rotating it through ϕ about O and dilating the result by the scaling factor μ. Therefore $\Delta A''B''C''$ and ΔABC are directly similar. □

The wheel has come full circle. To derive Napoleon's Theorem from this result take ΔABC to be equilateral and choose A'' so that $\Delta AA'A''$ is isosceles with base AA' and base angles of 30°.

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