Another Proof of Hadamard's Determinantal Inequality

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Abstract. We offer a new proof of Hadamard's celebrated inequality for determinants of positive matrices that is based on a simple identity, which may be of independent interest.

A hermitian $n \times n$ matrix A is said to be positive, if, for all $n \times 1$ vectors x, $x^*Ax > 0$ unless x is the zero vector. Thus, if A is positive, all of its principal sub-matrices are also positive. Moreover, A is positive if and only if the determinants of all these sub-matrices are positive. In particular, if $A = [a_{ij}]$ is positive, then all of its diagonal entries, $a_{11}, a_{22}, \dots, a_{nn}$, and its determinant, $\det A$, are positive. These are well-known facts about positive matrices that can be found in most textbooks on Matrix Analysis, such as, for instance, [1] and [3].

In 1893, Hadamard [2] discovered a fundamental fact about positive matrices, viz., that, for such $A = [a_{ij}]$,

$$\det A \leq a_{11}a_{22}\cdots a_{nn}.$$

Our purpose here is to present another proof of Hadamard's inequality which is based on the following identity.

Lemma 1. *Suppose A is an $n \times n$ matrix, $\tilde{A}$ is its cofactor matrix, and x, y are $n \times 1$ vectors. Then*

$$\det A - \det \begin{bmatrix} A & x \\ y^t & 1 \end{bmatrix} = x^t \tilde{A} y.$$

Proof. Identify $\mathbb{C}^n$ with the space of $n \times 1$ vectors with complex entries, and consider the bilinear form

$$B(x,y) = \det A - \det \begin{bmatrix} A & x \\ y^t & 1 \end{bmatrix}, \; x, y \in \mathbb{C}^n.$$

Denoting the usual orthonormal basis of $\mathbb{C}^n$ by $e_1, e_2, \ldots, e_n$, it's easy to see that

$$B(e_i, e_j) = A_{i,j},$$

the ijth element in $\tilde{A}$. Hence, if

$$x = \sum_{i=1}^{n} x_i e_i, \; y = \sum_{i=1}^{n} y_i e_i \in \mathbb{C}^n,$$

then, by bilinearity,

$$\begin{aligned} B(x,y) &= \sum_{i,j=1}^{n} x_i y_j B(e_i, e_j) = \sum_{i,j=1}^{n} x_i y_j A_{ij} \\ &= \sum_{i=1}^{n} x_i \sum_{j=1}^{n} A_{ij} y_j = x^t \tilde{A} y, \end{aligned}$$

as stated. □

As an easy consequence, we have:

Theorem 1. *Suppose A is an $n \times n$ positive matrix. Then*

$$\det \begin{bmatrix} A & x \\ x^* & 1 \end{bmatrix} \leq \det A \qquad (x \in \mathbb{C}^n),$$

with equality if and only if $x = 0$.

Proof. Since A is invertible, and its inverse is also positive, it follows from the lemma that

$$\det A - \det \begin{bmatrix} A & x \\ x^* & 1 \end{bmatrix} = x^* \tilde{A} x = \det A x^* A^{-1} x \geq 0,$$

and the inequality is strict unless x is the zero vector. The result follows. □

Corollary 1. *Denoting by A_k the sub-matrix of A of order $k \times k$ that occupies the top left-hand corner of $A = [a_{ij}]$, then*

$$\det A \leq a_{nn} \det A_{n-1},$$

and the inequality is strict unless all the entries in the last column of A, save the last one, are zero.

Hadamard's classical inequality is an immediate consequence of this, viz.,

Theorem 2 (Hadamard). *If $A = [a_{ij}]$ is an $n \times n$ positive matrix, then*

$$\det A \leq \prod_{i=1}^{n} a_{ii},$$

with equality if and only if A is a diagonal matrix.

Coupling this with the fact that the determinant of A is the product of its eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_n$, say, we can affirm that

$$\prod_{i=1}^{n} \lambda_i \leq \prod_{i=1}^{n} a_{ii},$$

with equality if and only if A is a diagonal matrix. But, also, the sum of the eigenvalues of A is its trace, i.e.,

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii}.$$

In other words, denoting by $\sigma_r(x_1, x_2, \ldots, x_n)$ the rth symmetric function of n variables, $x_1, x_2, \ldots, x_n$, we have that

$$\sigma_r(\lambda_1, \lambda_2, \ldots, \lambda_n) \leq \sigma_r(a_{11}, a_{22}, \ldots, a_{nn}),$$

if $r = 1$ or $r = n$. It's of interest to observe that this remains true if $1 < r < n$. For completeness, we sketch a proof of this statement

Indeed, $\sigma_r(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the coefficient a_r of t^{n-r} in the polynomial

$$\prod_{i=1}^{n} (t + \lambda_i) = \det(A + tI).$$

But a_r is equal to the sum of the determinants of all the $r \times r$ principal sub-matrices of A, which are also positive. Hence, applying Hadamard's result to each of them, we deduce that $a_r \leq \sigma_r(a_{11}, a_{22}, \ldots, a_{nn})$ as claimed.

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