Search Techniques and Epimorphisms Between Certain Groups and Fibonacci Groups

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Abstract. We examine the Fibonacci lengths of all generating pairs for certain centro-polyhedral groups. The problem requires a variety of approaches both exhaustive and random search.

1. Introduction

The *Fibonacci group* $F(r,n)$ is the group defined by the presentation

$$\begin{aligned}\langle\, a_1, a_2, \ldots, a_n \mid\ & a_1 a_2 \cdots a_r = a_{r+1},\\ & a_2 a_3 \cdots a_{r+1} = a_{r+2},\\ & \cdots\cdots,\\ & a_{n-1} a_n a_1 \cdots a_{r-2} = a_{r-1},\\ & a_n a_1 a_2 \cdots a_{r-1} = a_r \,\rangle\end{aligned}$$

where $r > 0, n > 0$ and all subscripts are assumed to be reduced modulo n. For a survey on Fibonacci groups see [13]. It is known that all finite and some infinite groups are homomorphic images of some Fibonacci groups, see [10] and [2]. In order to find which Fibonacci groups occur, the concept of Fibonacci length is used.
Let G be a finitely generated group, $G = \langle A \rangle$, where $A = (a_1, \ldots, a_n)$ an ordered n-tuple. Then we have:

Definition 1. The *Fibonacci orbit* of G with respect to the generating n-tuple A, written $F_A(G)$, is the sequence $x_1 = a_1, \ldots, x_n = a_n, x_{i+n} = \prod_{j=1}^{n} x_{i+j-1},\ i \geq 1$.

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Definition 2. If $F_A(G)$ is periodic then the length of the period of the sequence is called the *Fibonacci length* of G with respect to the generating n-tuple A, written $LEN_A(G)$. If $F_A(G)$ is not periodic then we say that the group G has infinite Fibonacci length on the generating n-tuple A, written $LEN_A(G) = \infty$.

We will write $LEN(G)$ when it is clear which generating n-tuple is being used. It is also important to note that the Fibonacci length of a group depends on the chosen generating n-tuple.

From the theory of group presentations and, in particular, von Dyck's Theorem, see [12], it is possible to prove the following theorem:

Theorem 1. *Let G be a group with generating n-tuple $(a_1, a_2, \ldots, a_n)$ and let $LEN_A(G) = m$ for finite m. Then G is an epimorphic image of the Fibonacci group $F(n, m)$.*

For example, a 2-generator presentation for the quaternion group Q_8 is given by

$$\langle\, a, b \mid aba = b, bab = a \,\rangle$$

and $LEN_{\{a,b\}} Q_8 = 3$. Hence Q_8 is an epimorphic image of the Fibonacci group $F(2, 3)$ which is again Q_8. However a 3-generator presentation for Q_8 is

$$\langle\, a, b, c \mid ab = c, bc = a, ca = b \,\rangle$$

and $LEN_{\{a,b,c\}} Q_8 = 8$. Hence Q_8 is an epimomorphic image of the infinite Fibonacci group $F(3, 8)$ which, of course, has the finite Fibonacci group $F(3, 2)$ ($\cong F(2, 3)$) as an epimorphic image. Thus given a group, G say, we can, where appropriate, find a Fibonacci group of which G is an image. We also use von Dyck's Theorem to prove:

Theorem 2. *Let G be a group defined by the presentation $\langle\, X \mid R \,\rangle$. If $LEN_X(G) = n$ and H is a factor group of G on the same set of generating symbols, then $LEN_X(H) \mid LEN_X(G)$.*

In order to compare methods we will use polyhedral groups and the related centro-polyhedral groups. These families of groups have well known connections and properties, see [6]. Polyhedral groups and centro-polyhedral groups are defined as follows:

Definition 3. The *polyhedral group* (ℓ, m, n), for $\ell, m, n \in \mathbb{Z}$ is defined by the presentation

$$\langle\, x, y, z \mid x^{\ell} = y^{m} = z^{n} = xyz = 1 \,\rangle.$$

Definition 4. The *centro-polyhedral group* $\langle \ell, m, n \rangle$, for $\ell, m, n \in \mathbb{Z}$ is defined by the presentation

$$\langle\, x, y, z \mid x^{\ell} = y^{m} = z^{n} = xyz \,\rangle.$$

In this paper we report on our experiments designed to calculate all Fibonacci lengths on all n-tuples when $n = 2$ of certain finite polyhedral and centro-polyhedral groups. The task of calculating all Fibonacci lengths over all generating n-tuples when $n = 2$ and when $n = 3$ of a non-abelian group has only been attempted, to the authors knowledge, in [4] where the Fibonacci lengths of D_{2n}, the dihedral groups of order $2n$, and Q_{2^n} were calculated due to a nice property of their automorphism groups and in [3] where certain centro-polyhedral groups were considered. For related results see also [1] where Fibonacci lengths for p-groups are considered. The present paper completes the work of [3]. It started out as a straightforward generalisation of that paper but immediately ran into problems of computing time and resources requiring the various techniques described in the next section. The final section of the paper raises some open problems.

All computer calculations were carried out using a standard download of the computational algebra system GAP, see [8], together with the coset enumeration package ACE, see [7].

2. Methods

In order to calculate all Fibonacci lengths of a given group on all generating pairs the authors wrote several programs and compared the results and efficiency of each program against the others. One of the main problems that occurred was that there is no known result that will predict, with any great accuracy, the number of distinct Fibonacci lengths that a given group might have. Our programs are denoted by the names full exhausive search, restrictive full exhausive search, restrictive search and random search. A brief description of each method together with its advantages and disadvantages is presented below.

(1) **Full Exhausive Search**: Here we simply choose every pair of elements except for the obvious exceptions. If they generate the group then we calculate the Fibonacci length. This calculation is certain to complete given that the given group is finite but the time to complete can be prohibitively long.
(2) **Restricted Full Exhausive Search**: Here we first calculate the Euler function of the given group $\phi_2(G)$, see [9], and then test all generating pairs, stopping when we have calculated $\phi_2(G)$ distinct pairs. Thus we do not have to calculate all possible Fibonacci lengths; see [4]. This can speed up the running of the program when compared to the Full Exhausive Search method. Unfortunately calculating $\phi_2(G)$ is non-trivial and can take longer to calculate than the Full Exhausive Search method takes to complete.
(3) **Restrictive Search**: Using the results given in [4] it is possible to know from the computing results how many Fibonacci lengths there are. Unfortunately this method requires the calculation of the automorphism group of the group under investigation which is known to be hard in general. In our experiments this method proved to be the longest to complete and was not used much.
(4) **Random Search**: Here we search over a known number of randomly chosen generating pairs and calculate the Fibonacci lengths. This proved to be very fast to compute. The main problem was that nothing is known about the distribution of Fibonacci lengths within a group.

In all the above methods the groups, given by finite presentations, were first converted into the isomorphic permutation groups.

3. Results

In deciding which groups to examine we chose to use the centropolyhedral groups of the form $\langle \pm 2, X\rangle$ or $(\pm 2, X)$, where

$$X = \{\pm 3, \pm 3\}, \{\pm 3, \pm 4\} \quad \text{or} \quad X = \{\pm 3, \pm 5\}.$$

Using the Full Exhausive Search method we were able to find all Fibonacci lengths for the following groups and the distribution of the Fibonacci lengths within each group.

$\mathcal{P}$	$\vert\langle\mathcal{P}\rangle\vert$	$LEN_{(a,b)}(\langle\mathcal{P}\rangle)$
Centro-polyhedral groups		
$\langle x,y,z \vert x^2=y^3=z^3=xyz\rangle$	24	16 (96), 48 (288)
$\langle x,y,z \vert x^{-2}=y^3=z^3=xyz\rangle$	120	16 (384), 48 (1152), 80 (1920), 240 (5760)
$\langle x,y,z \vert x^2=y^{-3}=z^3=xyz\rangle$	72	48 (3456)
$\langle x,y,z \vert x^{-2}=y^{-3}=z^3=xyz\rangle$	216	144 (31104)
$\langle x,y,z \vert x^2=y^{-3}=z^{-3}=xyz\rangle$	168	16 (4608), 48 (13824)
$\langle x,y,z \vert x^{-2}=y^{-3}=z^{-3}=xyz\rangle$	312	112 (16128), 336 (48384)
The polyhedral group $(2,3,3)$		
$\langle x,y,z \vert x^2=y^3=z^3=xyz=1\rangle$	12	16 (96)

$\mathcal{P}$	$\vert\langle\mathcal{P}\rangle\vert$	$LEN_{(a,b)}(\langle\mathcal{P}\rangle)$
Centro-polyhedral groups		
$\langle x,y,z \vert x^2=y^3=z^4=xyz\rangle$	48	18 (864)
$\langle x,y,z \vert x^{-2}=y^3=z^4=xyz\rangle$	528	90 (103680)
$\langle x,y,z \vert x^2=y^{-3}=z^4=xyz\rangle$	336	144 (41472)
$\langle x,y,z \vert x^{-2}=y^{-3}=z^4=xyz\rangle$	912	18 (311040)
$\langle x,y,z \vert x^2=y^3=z^{-4}=xyz\rangle$	240	36 (3456), 108 (17280)
$\langle x,y,z \vert x^{-2}=y^3=z^{-4}=xyz\rangle$	816	36 (248832)
$\langle x,y,z \vert x^2=y^{-3}=z^{-4}=xyz\rangle$	624	252 (145152)
$\langle x,y,z \vert x^{-2}=y^{-3}=z^{-4}=xyz\rangle$	1200	180 (86400), 900 (432000)
The polyhedral group $(2,3,4)$		
$\langle x,y,z \vert x^2=y^3=z^4=xyz=1\rangle$	24	18 (216)

$\mathcal{P}$	$\vert\langle\mathcal{P}\rangle\vert$	$LEN_{(a,b)}(\langle\mathcal{P}\rangle)$
Centro-polyhedral groups		
$\langle x,y,z \vert x^2=y^3=z^5=xyz\rangle$	120	12 (960), 14 (840), 42 (2520), 50 (1200), 150 (3600)
$\langle x,y,z \vert x^2=y^3=z^{-5}=xyz\rangle$	1320	50 (144000), 60 (115200), 70 (100800), 150 (432000), 210 (302400)
$\langle x,y,z \vert x^2=y^{-3}=z^5=xyz\rangle$	2280	36 (345600), 126 (1209600), 450 (1728000)
$\langle x,y,z \vert x^2=y^{-3}=z^{-5}=xyz\rangle$	3720	60 (921600), 150 (4608000), 210 (3225600)
The polyhedral group $(2,3,5)$		
$\langle x,y,z \vert x^2=y^3=z^5=xyz=1\rangle$	60	12 (240), 14 (840), 50 (1200)

The number in brackets indicates the total number of distinct pairs with the stated Fibonacci length.

It is interesting to note that the above groups proved particularly amenable to the random search method. On each run of the random search method a complete list of Fibonacci lengths was calculated, while the full exhausive search method took many times longer to obtain the same results. It is also interesting to note that in Section 3.3 of [4] the random method was able, on one run, to find all Fibonacci lengths on generating pairs.

In this section we have used several of the methods to obtain the results in the tables. For certain of the more manageable groups we have used both the Full Exhausive Search method and the Random Search method in order to illustrate the power of the Random Search method. For example, to obtain the complete results of the first table above (where we considered the groups of the form $\langle \pm 2, \pm 3, \pm 3 \rangle$) the Full Exhausive Search method required several days whereas the Random Search method was completed in tens of minutes. In practice the Restrictive Full Exhausive Search method was not used as the computation of $\phi_2(G)$ was too demanding on computer memory. Likewise the Restrictive Search method was not used as $\mathrm{Aut}(G)$ was too hard to compute in reasonable time. For the other two methods, the Random Search method and Full Exhausive method were used for all groups.

4. Further Work

The area of calculating Fibonacci lengths is not very well developed and requires more attention. Of particular interest one would like to know the following:

(1) Is it possible to predict the distribution of Fibonacci lengths within a particular group? Why is the random search method so efficient in obtaining the complete list of Fibonacci lengths?
(2) What general theories can be obtained regarding the Fibonacci lengths of a general group? For example does there exist a decision process to determine whether, or not, a given group has finite Fibonacci length?
(3) What generalisations of Fibonacci length are possible? It seems likely that a theory may exist for groups defined by a presentation whose relators are given via a recurrence relation.
(4) Resolve Wall's conjecture, see [14] and [5]. The conjecture is that $k(p^2) = pk(p)$, where p is an odd prime. (This holds in all known cases. The other possibility would be $k(p^2) = k(p)$.)

(5) The problem of finding all the Fibonacci lengths on n-tuples of a given group could be parallelised. In order to find the Fibonacci lengths of infinite groups it would be useful to have a program that would use the Knuth-Bendix method to find Fibonacci lengths, see [11].

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