Weyl type Theorems and the Approximate Point Spectrum

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Abstract. It is shown that, if an operator T on a complex Banach space or its adjoint T^* has the single-valued extension property, then the generalized a-Browder's theorem holds for $f(T)$ for every complex-valued analytic function f on a neighborhood of the spectrum of T. We also study the generalized a-Weyl's theorem in connection with the single-valued extension property. Finally, we examine the stability of the generalized a-Weyl's theorem under commutative perturbations by finite rank operators.

1. Introduction

Throughout this paper X will denote an infinite-dimensional complex Banach space and $\mathcal{L}(X)$ the unital (with unit the identity operator, I, on X) Banach algebra of all bounded linear operators acting on X. For an operator $T \in \mathcal{L}(X)$, let T^* denote its adjoint, $N(T)$ its kernel, $R(T)$ its range, $\sigma(T)$ its spectrum, $\sigma_a(T)$ its approximate point spectrum, $\sigma_{su}(T)$ its surjective spectrum and $\sigma_p(T)$ its point spectrum. For a subset K of $\mathbb{C}$ we write $\mathrm{iso}(K)$ for its isolated points and $\mathrm{acc}(K)$ for its accumulation points.

From [14] we recall that for $T \in \mathcal{L}(X)$, the ascent $a(T)$ and the descent $d(T)$ are given by

$$a(T) = \inf\{n \geq 0 : N(T^n) = N(T^{n+1})\}$$

and

$$d(T) = \inf\{n \geq 0 : R(T^n) = R(T^{n+1})\},$$

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respectively; the infimum over the empty set is taken to be ∞. If the ascent and the descent of $T \in \mathcal{L}(X)$ are both finite then $a(T) = d(T) = p$, $X = N(T^p) \oplus R(T^p)$ and $R(T^p)$ is closed.

For $T \in \mathcal{L}(X)$ we will denote by $\alpha(T)$ the nullity of T and by $\beta(T)$ the defect of T. If the range $R(T)$ of T is closed and $\alpha(T) < \infty$ (resp., $\beta(T) < \infty$) then T is called an upper semi-Fredholm (resp., a lower semi-Fredholm) operator. If $T \in \mathcal{L}(X)$ is either upper or lower semi-Fredholm, then T is called a semi-Fredholm operator, and the index of T is defined by $\text{ind}(T) = \alpha(T) - \beta(T)$. If both $\alpha(T)$ and $\beta(T)$ are finite then T is called a Fredholm operator. For a T-invariant closed linear subspace Y of X, let $T \mid Y$ denote the operator given by the restriction of T to Y.

For a bounded linear operator T and for each integer n, define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into itself. If for some integer n the range space $R(T^n)$ is closed and $T_n = T \mid R(T^n)$ is an upper (resp., lower) semi-Fredholm operator then T is called an upper (resp., lower) semi-B-Fredholm operator. Moreover if T_n is a Fredholm operator, then T is called a B-Fredholm operator. In this situation, from [1, Proposition 2.1], T_m is a Fredholm operator and $\text{ind}(T_m) = \text{ind}(T_n)$ for each $m \geq n$ which permits to define the index of a B-Fredholm operator T as the index of the Fredholm operator T_n where n is any integer such that $R(T^n)$ is closed and T_n is a Fredholm operator. Let $BF(X)$ be the class of all B-Fredholm operators and $\rho_{BF}(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \in BF(X)\}$ be the B-Fredholm resolvent of T and let $\sigma_{BF}(T) = \mathbb{C} \setminus \rho_{BF}(T)$ the B-Fredholm spectrum of T. The class $BF(X)$ has been studied by M. Berkani (see [1, Theorem 2.7]), where it was shown that $T \in \mathcal{L}(X)$ is a B-Fredholm operator if and only if $T = T_0 \oplus T_1$ where T_0 is a Fredholm operator and T_1 is a nilpotent one. He also proved that $\sigma_{BF}(T)$ is a closed subset of $\mathbb{C}$ and showed that the spectral mapping theorem holds for $\sigma_{BF}(T)$, that is, $f(\sigma_{BF}(T)) = \sigma_{BF}(f(T))$ for any complex-valued analytic function on a neighborhood of the spectrum $\sigma(T)$.

An operator $T \in \mathcal{L}(X)$ is called a Weyl operator if it is Fredholm of index 0, a Browder operator if it is Fredholm of finite ascent and descent and a B-Weyl operator if it is B-Fredholm of index 0. The Weyl spectrum, the Browder spectrum and the B-Weyl spectrum of T are defined by

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not Weyl}\},$$

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not Browder}\},$$
$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not B-Weyl}\},$$

respectively. We will denote by $E(T)$ (resp. $E^a(T)$) the set of all eigenvalues of T which are isolated in $\sigma(T)$ (resp., $\sigma_a(T)$) and by $E_0(T)$ (resp. $E_0^a(T)$) the set of all eigenvalues of T of finite multiplicity which are isolated in $\sigma(T)$ (resp., $\sigma_a(T)$).

Let $SF(X)$ be the class of all semi-Fredholm operators on X, $SF_+(X)$ the class of all upper semi-Fredholm operators on X and $SF_+^-(X)$ the class of all $T \in SF_+(X)$ such that $\text{ind}(T) \leq 0$. For $T \in \mathcal{L}(X)$, let

$$\sigma_{SF}(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \notin SF(X)\},$$
$$\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \notin SF_+^-(X)\},$$

$\rho_{SF}(T) = \mathbb{C} \setminus \sigma_{SF}(T)$ and $\rho_{SF_+^-}(T) = \mathbb{C} \setminus \sigma_{SF_+^-}(T)$.

Similarly, let $SBF(X)$ be the class of all semi-B-Fredholm operators on X, $SBF_+(X)$ the class of all upper semi-B-Fredholm operators on X and $SBF_+^-(X)$ the class of all $T \in SBF_+(X)$ such that $\text{ind}(T) \leq 0$. For $T \in \mathcal{L}(X)$, the sets $\sigma_{SBF}(T)$, $\rho_{SBF}(T)$, $\sigma_{SBF_+^-}(T)$ and $\rho_{SBF_+^-}(T)$ are defined in an obvious way.

An operator $T \in \mathcal{L}(X)$ is called semi-regular if $R(T)$ is closed and $N(T) \subseteq R(T^n)$ for every $n \in \mathbb{N}$. The semi-regular resolvent set is defined by s-reg$(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I$ is semi- regular$\}$, we note that s-reg$(T) =$ s-reg(T^*) is an open subset of $\mathbb{C}$. As a consequence of [8, Théorème 2.7], we obtain the following result.

Proposition 1.1. *Let $T \in \mathcal{L}(X)$.*

(i) *If T has the SVEP then s-reg$(T) = \rho_a(T) = \mathbb{C} \setminus \sigma_a(T)$.*
(ii) *If T^* has the SVEP then s-reg$(T) = \rho_{su}(T) = \mathbb{C} \setminus \sigma_{su}(T)$.*

We recall that an operator $T \in \mathcal{L}(X)$ has the single-valued extension property, abbreviated SVEP, if, for every open set $U \subset \mathbb{C}$, the only analytic solution $f : \ U \longrightarrow X$ of the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in U$ is the zero function on U. We will denote by $\mathcal{H}(\sigma(T))$ the set of all complex-valued functions which are analytic on an open set containing $\sigma(T)$.

The remainder of the following deals with Riesz points and left poles. A complex number λ is said to be Riesz point of $T \in \mathcal{L}(X)$ if $\lambda \in \text{iso}(\sigma(T))$ and the corresponding spectral projection is of finite-dimensional range. The set of all Riesz points of T will be denoted by

$\Pi_0(T)$. It is known that if $T \in \mathcal{L}(X)$ and $\lambda \in \sigma(T)$, then $\lambda \in \Pi_0(T)$ if and only if $T-\lambda I$ is Fredholm of finite ascent and descent (see [3]). Consequently $\sigma_b(T) = \sigma(T) \setminus \Pi_0(T)$. We will denote by $\Pi(T)$ the set of all poles of the resolvent of T. A complex number $\lambda \in \sigma_a(T)$ is said to be a left pole of T if $a(T-\lambda I) < \infty$ and $R((T-\lambda I)^{a(T-\lambda I)+1})$ is closed, and that it is a left pole of T of finite rank if it is a left pole of T and $\alpha(T-\lambda I) < \infty$. We will denote by $\Pi^a(T)$ the set of all left poles of T, and by $\Pi_0^a(T)$ the set of all left poles of T of finite rank. If $\lambda \in \Pi^a(T)$, then it is easily seen that $T-\lambda I$ is an operator of topological uniform descent, therefore from [4], it follows that λ is isolated in $\sigma_a(T)$ [2, Theorem 2.5]. Let $T \in \mathcal{L}(X)$ and $\lambda \in \mathbb{C}$ be isolated in $\sigma_a(T)$; then $\lambda \in \Pi^a(T)$ if and only if $\lambda \notin \sigma_{SBF_+^-}(T)$, and $\lambda \in \Pi_0^a(T)$ if and only if $\lambda \notin \sigma_{SF_+^-}(T)$.

For $T \in \mathcal{L}(X)$ we will say that:

(i) T satisfies Weyl's theorem if $\sigma_w(T) = \sigma(T) \setminus E_0(T)$;

(ii) T satisfies generalized Weyl's theorem if

$$\sigma_{BW}(T) = \sigma(T) \setminus E(T);$$

(iii) T satisfies a-Weyl's theorem if

$$\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus E_0^a(T);$$

(iv) T satisfies generalized a-Weyl's theorem if

$$\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus E^a(T);$$

(v) T satisfies Browder's theorem if

$$\sigma_w(T) = \sigma(T) \setminus \Pi_0(T);$$

(vi) T satisfies generalized Browder's theorem if

$$\sigma_{BW}(T) = \sigma(T) \setminus \Pi(T);$$

(vii) T satisfies a-Browder's theorem if

$$\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus \Pi_0^a(T);$$

(viii) T satisfies generalized a-Browder's theorem if

$$\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus \Pi^a(T).$$

Before proving our main result we deal with some preliminary results.

Proposition 1.2. *Let $T \in \mathcal{L}(X)$.*

(i) *If T has the SVEP then* $\operatorname{ind}(T - \lambda I) \leq 0$ *for every* $\lambda \in \rho_{SBF}(T)$.
(ii) *If T^* has the SVEP then* $\operatorname{ind}(T - \lambda I) \geq 0$ *for every* $\lambda \in \rho_{SBF}(T)$.

Proof. (i) Let $\lambda \in \rho_{SBF}(T)$, then there exists an integer p such that $(T \mid R(T - \lambda I)^p) - \lambda I = (T - \lambda I) \mid R(T - \lambda I)^p$ is semi-Fredholm. From the Kato decomposition, there exists $\delta > 0$ such that

$$\{\lambda \in \mathbb{C} : \ 0 < |\mu - \lambda| < \delta\} \subseteq \text{ s-reg}(T \mid R(T - \lambda I)^p).$$

Since T has the SVEP, Proposition 1.1 implies that

$$\text{s-reg}(T \mid R(T - \lambda I)^p) = \rho_a(T \mid R(T - \lambda I)^p).$$

Therefore, $N((T \mid R(T - \lambda I)^p) - \mu I) = 0$ and so $\operatorname{ind}(T - \mu I) = \operatorname{ind}((T \mid R(T - \lambda I)^p - \mu I) \leq 0$, holding for $0 < |\mu - \lambda| < \delta$. Thus, by the continuity of the index we obtain $\operatorname{ind}(T - \lambda) \leq 0$.
(ii) Follows by similar reasoning, and may also be derived from the first assertion and the fact that $\operatorname{ind}(T^*) = -\operatorname{ind}(T)$. □

Corollary 1.3. *Let T be a bounded linear operator on X. If T^* has the SVEP, then $\sigma_{SF_+^-}(T) = \sigma_w(T)$.*

Proof. We have only to show that $\sigma_w(T) \subseteq \sigma_{SF_+^-}(T)$, since the other inclusion is always verified. Let λ be given in $\rho_{SF_+^-}(T)$, then $T - \lambda I$ is semi-Fredholm and $\operatorname{ind}(T - \lambda I) \leq 0$. Since T^* has the SVEP, Proposition 1.2 implies that $\operatorname{ind}(T - \lambda I) \geq 0$, and hence $\operatorname{ind}(T - \lambda I) = 0$, which proves that $T - \lambda I$ is Fredholm of index 0 and $\lambda \in \rho_w(T)$. □

The following results relate the generalized a-Weyl's theorem and the generalized a-Browder's theorem to the single-valued extension property. As motivation for the proofs, we use some ideas in [10, 12].

Proposition 1.4. *Let T be a bounded linear operator on X.*

(i) *If T^* has the SVEP, then T satisfies generalized a-Weyl's theorem if and only if it satisfies generalized Weyl's theorem.*
(ii) *If T has the SVEP, then T^* satisfies generalized a-Weyl's theorem if and only if it satisfies generalized Weyl's theorem.*

Proof. (i) Since T^* has the SVEP, [6, Proposition 1.3.2] implies that $\sigma(T) = \sigma_a(T)$ and consequently $E^a(T) = E(T)$. Suppose that T satisfies generalized Weyl's theorem, then $\sigma_{BW}(T) = \sigma(T) \setminus E(T) =$

$\sigma_a(T) \setminus E^a(T)$. Let $\lambda \notin \sigma_{SBF_+^-}(T)$ be given, then $T - \lambda I$ is semi-B-Fredholm and $\operatorname{ind}(T - \lambda I) \leq 0$. Therefore, by Proposition 1.2, it follows that $\operatorname{ind}(T - \lambda I) = 0$ and consequently $T - \lambda I$ is B-Fredholm of index 0. Hence $\lambda \notin \sigma_{BW}(T)$ and $\sigma_{BW}(T) \subset \sigma_{SBF_+^-}(T)$. Since the opposite inclusion is clear, we conclude that indeed $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T) = \sigma_a(T) \setminus E^a(T)$ which proves the equivalence between generalized Weyl's theorem and generalized a-Weyl's theorem for T.

(ii) Similar to the proof of the first assertion. □

Our main result reads now as follows.

Theorem 1.5. *Let T be a bounded linear operator on X. If T or its adjoint T^* satisfies the SVEP, then generalized a-Browder's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.*

Proof. Let us establish that generalized a-Browder's theorem holds for T. If T^* has the SVEP, then by [12, Theorem 2.4], it follows that a-Browder's theorem holds for T, and consequently Browder's theorem holds for T. Thus $\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus \Pi_0^a(T)$ and $\sigma_b(T) = \sigma(T) \setminus \Pi_0(T)$. Moreover, since $\sigma_a(T) = \sigma(T)$, $\Pi_0^a(T) = \Pi_0(T)$, it follows that $\sigma_{SF_+^-}(T) = \sigma(T) \setminus \Pi_0(T)$. Because $\sigma_{SF_+^-}(T) = \sigma_w(T)$, see Corollary 1.3, it follows that $\sigma_{SF_+^-}(T) = \sigma(T) \setminus \Pi_0(T) = \sigma_w(T) = \sigma_b(T)$. Let $\lambda \in \Pi^a(T)$ be given; then λ is isolated in $\sigma_a(T)$ and by [2, Theorem 2.8], it follows that $\lambda \notin \sigma_{SBF_+^-}(T)$ which shows that $\Pi^a(T) \subseteq \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Conversely if $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$, then $T - \lambda I$ is semi-B-Fredholm and $\operatorname{ind}(T - \lambda I) \leq 0$. Then, since T^* has the SVEP, Proposition 1.2 gives $\operatorname{ind}(T - \lambda I) = 0$. Therefore $T - \lambda I$ is Fredholm and $\lambda \notin \sigma_w(T) = \sigma_b(T)$ which shows that $\lambda \in \Pi_0(T)$. Consequently λ is isolated in $\sigma_a(T)$ and hence $\lambda \in \Pi^a(T)$. Thus $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) \subset \Pi^a(T)$ and generalized a-Browder's theorem holds for T. Now if T has the SVEP, let $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$; $\lambda \in \rho_{SBF_+^-}(T)$, then there exists an integer p such that $R(T - \lambda I)^p$ is closed and $(T \mid R(T - \lambda I)^p) - \lambda I = (T - \lambda I) \mid R(T - \lambda I)^p$ is a semi-Fredholm operator. Then, by the Kato decomposition, there exists $\delta > 0$ for which

$$\{\mu \in \mathbb{C} : \; 0 < |\mu - \lambda| < \delta\} \\ \subseteq \text{s-reg}(T \mid R(T - \lambda I)^p) \cap \rho_{SF}(T \mid R(T - \lambda I)^p).$$

Since T has the SVEP, so does $T \mid R(T-\lambda I)^p$. Therefore

$$\text{s-reg}(T \mid R(T-\lambda I)^p) = \rho_a(T \mid R(T-\lambda I)^p)$$

and

$$\{\mu \in \mathbb{C} : \ 0 < |\mu - \lambda| < \delta\} \subseteq \rho_a(T \mid R(T-\lambda I)^p) \cap \rho_{SF}(T \mid R(T-\lambda)^p),$$

hence $\lambda \in \text{iso}(\sigma_a(T) \cap \rho_{SBF}(T))$. By [2, Theorem 2.8], it follows that $\lambda \in \Pi_a(T)$ and $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) \subset \Pi^a(T)$. Since the other inclusion is clear we get $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a(T)$ and thus generalized a-Browder's theorem holds for T. Finally, if $f \in H(\sigma(T))$, by [6, Theorem 3.3.6] $f(T)$ or $f(T^*)$ satisfies the SVEP and the above argument implies that generalized a-Browder's theorem holds for $f(T)$. □

From Theorem 1.5 we obtain the following useful consequence.

Corollary 1.6. *Let T be a bounded linear operator on X. If T or T^* has the SVEP then generalized a-Weyl's theorem holds for T if and only if $E^a(T) = \Pi^a(T)$.*

Proof. We only have to use the fact that an operator T satisfying generalized a-Browder's theorem, satisfies generalized a-Weyl's theorem if and only if $\Pi^a(T) = E^a(T)$. □

In [7] the class of the operators $T \in \mathcal{L}(X)$ for which $K(T) = \{0\}$ was studied and it was shown that for such operators, the spectrum is connected and the single-valued extension property is satisfied.

Proposition 1.7. *Let $T \in \mathcal{L}(X)$. If there exists a complex number λ for which $K(T-\lambda I) = \{0\}$ then $f(T)$ satisfies generalized a-Browder's theorem for every $f \in \mathcal{H}(\sigma(T))$. Moreover, if in addition, $N(T-\lambda I) = \{0\}$, then generalized a-Weyl's theorem holds for $f(T)$ for any $f \in \mathcal{H}(\sigma(T))$.*

Proof. Let f be a non-constant complex-valued analytic function on an open neighborhood of $\sigma(T)$. Since T has the SVEP so does $f(T)$ and by Theorem 1.5 generalized a-Browder's theorem holds for $f(T)$. Now assume that $N(T-\lambda I) = \{0\}$ and $\beta \in \sigma(f(T))$ then $f(z) - \beta I = P(z)g(z)$ where g is complex-valued analytic function on a neighborhood of $\sigma(T)$ without any zeros in $\sigma(T)$ while P is a complex polynomial of the form $P(z) = \prod_{i=1}^{n}(z-\lambda_i)^{p_i}$ with distinct roots $\lambda_1, \ldots, \lambda_n \in \sigma(T)$. Since $g(T)$ is invertible, we have

$$N(f(T)-\beta I) = N(P(T)) = \oplus_{i=1}^{n} N(T-\lambda_i I)^{p_i}.$$

On the other hand, [7, Proposition 2.1] ensures that $\sigma_p(T) \subseteq \{\lambda\}$ and since $T-\lambda I$ is injective, we deduce that $\sigma_p(T) = \emptyset$. Consequently $N(f(T)-\beta I) = \{0\}$ which proves that $\sigma_p(f(T)) = \emptyset$. Thus $E^a(f(T)) = \Pi^a(f(T)) = \emptyset$ and generalized a-Weyl's theorem holds for $f(T)$. □

Proposition 1.8. *Let T be a bounded linear operator on X satisfying the SVEP. If $T-\lambda I$ has finite descent at every $\lambda \in E^a(T)$, then T obeys generalized a-Weyl's theorem.*

Proof. Let $\lambda \in E^a(T)$, then $p = d(T-\lambda I) < \infty$ and since T has the SVEP it follows (see [13, Proposition 3]) that $a(T-\lambda I) = d(T-\lambda I) = p$ and by [5, Satz 101.2], λ is a pole of the resolvent of T or order p, consequently λ is an isolated point in $\sigma_a(T)$. Then $X = K(T-\lambda I) \oplus H_0(T-\lambda I)$, with $K(T-\lambda I) = R(T-\lambda I)^p$ is closed, therefore $\lambda \in \Pi^a(T)$. □

Now let us consider the class $\mathcal{P}(X)$ defined as those operators $T \in \mathcal{L}(X)$ for which for every complex number λ there exists a positive integer p_λ such that $H_0(T-\lambda I) = N(T-\lambda I)^{p_\lambda}$. This class has been introduced and studied in [10, 11], it was shown that it contains every M-hyponormal, log-hyponormal, p-hyponormal and totally paranormal operator. It was also established that the SVEP is shared by all the operators lying in $\mathcal{P}(X)$ and generalized Weyl's theorem holds for $f(T)$ whenever $T \in \mathcal{P}(X)$ and $f \in \mathcal{H}(\sigma(T))$.

Proposition 1.9. *Let $T \in \mathcal{P}(X)$ be such that $\sigma(T) = \sigma_a(T)$ then generalized a-Weyl's theorem holds for $f(T)$ for every $f \in \mathcal{H}(\sigma(T))$.*

Proof. By the spectral mapping theorem for the spectrum and the approximate point spectrum, and the fact that $f(T) \in \mathcal{P}(X)$, it suffices to establish generalized a-Weyl's theorem for T. Since $\sigma(T) = \sigma_a(T)$ it follows that

$$E^a(T) = \sigma_p(T) \cap \text{iso}(\sigma_a(T)) = \sigma_p(T) \cap \text{iso}(\sigma(T)) = E(T).$$

Let $\lambda \in E^a(T) = E(T)$, then $X = H_0(T-\lambda I) \oplus K(T-\lambda I)$ and $K(T-\lambda I)$ is closed. Since $T \in \mathcal{P}(X)$, let p_λ be a positive integer for which $H_0(T-\lambda I) = N(T-\lambda I)^{p_\lambda}$, therefore

$$\begin{aligned} R(T-\lambda I)^{p_\lambda} &= (T-\lambda I)^{p_\lambda}(H_0(T-\lambda I) \oplus K(T-\lambda I)) \\ &= (T-\lambda I)^{p_\lambda}(K(T-\lambda I)) \\ &= K(T-\lambda I), \end{aligned}$$

thus $R(T-\lambda I)^{p_\lambda} = R(T-\lambda I)^{p_\lambda+1}$ which by Proposition 1.8 shows that the operator T obeys generalized a-Weyl's theorem. □

2. Generalized a-Weyl's Theorem and Perturbation

In general, we cannot expect that generalized a-Browder's theorem necessarily holds under finite rank perturbations. However, it does hold under commutative ones, as the following result shows.

Theorem 2.1. [2, Theorem 3.2] *If $T \in \mathcal{L}(X)$ is an operator satisfying generalized a-Browder's theorem and F is a finite rank operator such that $TF = FT$ then $T + F$ satisfies generalized a-Browder's theorem.*

Lemma 2.2. *Let $T \in \mathcal{L}(X)$ be an injective operator. If F is a finite rank operator on X such that $FT = TF$, then $R(F) \subseteq R(T)$.*

Proof. Since F is a finite rank operator on X there exist two systems: a system of linearly independent vectors e_i for $i = 1, \ldots, n$ and a system of non-zero bounded linear functionals f_i for $i = 1, \ldots, n$ on X such that

$$F(x) = \sum_{i=1}^{n} f_i(x)e_i \qquad (x \in X).$$

Moreover, we have

$$\sum_{i=1}^{n} f_i(x)Te_i = TF(x) = FT(x) = \sum_{i=1}^{n} f_i(Tx)e_i \qquad (x \in X).$$

On the other hand, since T is injective, it is clear that the vectors Te_i $(1 \le i \le n)$ are linearly independent. Hence $F(x) \in \mathrm{Vect}(\{e_1, \cdots, e_n\}) = \mathrm{Vect}(\{Te_1, \cdots, Te_n\})$ for all $x \in X$. Thus $R(F) \subseteq R(T)$, as desired. □

Lemma 2.3. *Let $T \in \mathcal{L}(X)$. If F is a finite rank operator on X such that $FT = TF$ then $\lambda \in \mathrm{acc}(\sigma_a(T))$ if and only if $\lambda \in \mathrm{acc}(\sigma_a(T+F))$.*

Proof. Let $\lambda \notin \operatorname{acc}(\sigma_a(T))$ be given, there exists $\delta > 0$ such that if $0 < |\mu - \lambda| < \delta$ then $\alpha(T - \mu I) = 0$ and $R(T - \mu I)$ is closed. This gives us a bounded linear operator $S: R(T - \mu I) \longrightarrow X$ such that $S(T - \mu I) = I$ and $(T - \mu I)S = I \mid R(T - \mu I)$. To see that $\lambda \notin \operatorname{acc}(\sigma_a(T + F))$, suppose that $\mu \in \sigma_a(T + F)$, and choose unit vectors $x_n \in X$ such that $(T+F-\mu I)x_n \to 0$ as $n \to \infty$. Let $(x_{n(k)})_k$ be a subsequence such that $Fx_{n(k)} \to x \in R(F)$ as $k \to \infty$, and since this level of generality is not needed here, we may assume that $Fx_n \to x$ as $n \to \infty$. Therefore $S(T + F - \mu I)x_n = x_n + SFx_n \to 0$ as $n \to \infty$, and since $\lim SFx_n = Sx$ exists, it follows that $\lim x_n = -Sx$ and consequently $x \neq 0$. Next observe that $x = \lim Fx_n = -FSx \in R(F)$, then since Lemma 2.2 asserts that $R(F) \subseteq R(T)$, we obtain $(T - \mu I)x = -(T - \mu I)FSx = -F(T - \mu I)Sx = -Fx$, hence $(T + F - \mu I)x = 0$. Thus $\mu \in \sigma_p(T + F)$. Finally, because eigenvectors corresponding to distinct eigenvalues of an operator are linearly independent, and since all the eigenvectors of $T + F$ belong to the finite dimensional subspace $R(F)$, it follows that $\sigma_a(T + F)$ may contain only finitely many points μ such that $0 < |\mu - \lambda| < \delta$, and consequently $\lambda \notin \operatorname{acc}(\sigma_a(T + F))$. The opposite inclusion is similarly obtained. □

An operator $T \in \mathcal{L}(X)$ is said to be approximate-isoloid if any isolated point of $\sigma_a(T)$ is an eigenvalue of T.

Theorem 2.4. *Let T be an approximate-isoloid operator on X that satisfies generalized a-Weyl's theorem. If F is an operator of finite rank on X such that $FT = TF$ then $T + F$ satisfies generalized a-Weyl's theorem.*

Proof. Since by Theorem 2.1 generalized a-Browder's theorem holds for $T + F$ it suffices, from Corollary 1.5, to prove that $E^a(T + F) = \Pi^a(T + F)$. Let $\lambda \in E^a(T + F)$ be given, then $\lambda \in \operatorname{iso}(\sigma_a(T + F))$ and $\lambda \in \sigma_p(T + F)$, hence $\lambda \notin \operatorname{acc}(\sigma_a(T + F)$ and by Lemma 2.3 $\lambda \notin \operatorname{acc}(\sigma_a(T)$. We distinguish two cases. Firstly if $\lambda \notin \sigma_a(T)$, $T - \lambda I$ is injective with a closed range and $T - \lambda I$ is an upper semi-Fredholm operator on X such that $\operatorname{ind}(T - \lambda I) \leq 0$, and since F is a finite rank operator on X, it follows that $T + F - \lambda I$ is an upper semi-Fredholm operator and $\operatorname{ind}(T + F - \lambda I) = \operatorname{ind}(T - \lambda I) \leq 0$. Then $\lambda \notin \sigma_{SF_+^-}(T + F)$ and $\lambda \in \Pi^a(T + F)$. On the other hand if $\lambda \in \sigma_a(T)$, then $\lambda \in \operatorname{iso}(\sigma_a(T))$ and since T is approximate-isoloid $\lambda \in \sigma_p(T)$. Thus $\lambda \in \operatorname{iso}(\sigma_a(T)) \cap \sigma_p(T) = E^a(T)$. From the fact that T obeys generalized a-Weyl's theorem, it follows that $\lambda \notin \sigma_{SBF_+^-}(T) = \sigma_{SBF_+^-}(T+F)$ and since $\lambda \in \text{iso}(\sigma_a(T+F))$, it follows that $\lambda \in \Pi^a(T+F)$. Finally $E^a(T+F) \subset \Pi^a(T+F)$, and since the reverse inclusion is verified, $T+F$ obeys generalized a-Weyl's theorem. □

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