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Pell Equation $x^2 - Dy^2 = 2$, II

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Abstract. In this paper solutions of the Pell equation $x^2 - Dy^2 = 2$ are formulated for a positive non-square integer D using the solutions of the Pell equation $x^2 - Dy^2 = 1$. Moreover, a recurrence relation on the solutions of the Pell equation $x^2 - Dy^2 = 2$ is obtained. Furthermore, the solutions of the equation $x_n^2 - Dy_n^2 = 2^n$ are obtained using the solutions of the equation $x^2 - Dy^2 = 2$.

1. Introduction

A real *binary quadratic form* f (or just a form) is a polynomial in two variables of the type

$$f(x,y) = ax^2 + bxy + cy^2$$

with real coefficients a, b, c. The *discriminant* of f is defined by the formula $b^2 - 4ac$ and denoted by D.

Let D be a non-square discriminant. Then the *Pell form* is defined by the formula

$$f_D(x,y) = \begin{cases} x^2 - \frac{D}{4}y^2 & \text{if } D \equiv 0 \pmod 4 \\ x^2 + xy - \frac{D-1}{4}y^2 & \text{if } D \equiv 1 \pmod 4 \end{cases} \tag{1.1}$$

Let

$$Pell(D) = \left\{(x,y) \in \mathbb{Z}^2 : \ f_D(x,y) = 1\right\}$$

and

$$Pell^{\pm}(D) = \left\{(x,y) \in \mathbb{Z}^2 : \ f_D(x,y) = \pm 1\right\}.$$

Then $Pell(D)$ is infinite. The binary operation

$$(x_1, y_1).(x_2, y_2) = (x_1x_2 + Dy_1y_2,\ x_1y_2 + y_1x_2)$$

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is a group law on $Pell(D)$ for which $Pell(D) \simeq \{\pm 1\} \times \mathbb{Z}$.

Let d be a positive non-square discriminant and $\mathbb{K} = \mathbb{Q}(\sqrt{d})$ be the *quadratic number field.* Then every element α of $\mathbb{K}$ can be represented as

$$\alpha = \frac{a + b\sqrt{d}}{c}$$

for $a, b, c \in \mathbb{Z}$. The *conjugate* of α is denoted by

$$\overline{\alpha} = \frac{a - b\sqrt{d}}{c}.$$

The *trace* and *norm* of α are given by

$$Tr(\alpha) = \alpha + \overline{\alpha} = \frac{2a}{c}$$

and

$$N(\alpha) = \alpha\overline{\alpha} = \frac{a^2 - db^2}{c^2},$$

respectively. It is easy to show that for $\alpha, \beta \in \mathbb{K}$,

$$Tr(\alpha + \beta) = Tr(\alpha) + Tr(\beta)$$

and

$$N(\alpha\beta) = N(\alpha)N(\beta).$$

If $d \equiv 1 \pmod 4$ then the elements of $\mathbb{K}$ are of the form

$$\alpha = \frac{a + b\sqrt{d}}{2}, \tag{1.2}$$

where $a, b \in \mathbb{Z}$, and if $d \equiv 2, 3 \pmod 4$ then the elements of $\mathbb{K}$ are of the form

$$\alpha = a + b\sqrt{d}, \tag{1.3}$$

where $a, b \in \mathbb{Z}$. As in the case of rationals, the set of integers of $\mathbb{K}$ forms a ring which we will be denoted as $O_{\mathbb{K}}$, the *maximal order* of $\mathbb{K}$.

If every integer $\alpha \in O_{\mathbb{K}}$ can be uniquely expressed as

$$\alpha = a_1 w_1 + a_2 w_2$$

where $a_i \in \mathbb{Z}$ and $w_i \in O_{\mathbb{K}}$, then we call w_1, w_2 an *integral basis* for $\mathbb{K}$, and we denote $O_{\mathbb{K}}$ by the $\mathbb{Z}$-module $[w_1, w_2] = w_1\mathbb{Z} + w_2\mathbb{Z}$.

Every algebraic number field has an integral basis, and in the quadratic fields it is especially easy to give one.

If $d \equiv 1 \pmod 4$, then from Eq. (1.2) it is seen that

$$w_1 = 1 \text{ and } w_2 = \frac{1+\sqrt{d}}{2}$$

is an integral basis.

If $d \equiv 2, 3 \pmod 4$, then from Eq. (1.3) it is seen that

$$w_1 = 2 \text{ and } w_2 = \sqrt{d}$$

is an integral basis.

The *discriminant* of $\mathbb{K}$ is defined as

$$D(\mathbb{K}) = \begin{vmatrix} w_1 & \overline{w}_1 \\ w_2 & \overline{w}_2 \end{vmatrix}^2,$$

hence, $D(\mathbb{K}) = D$ for $d \equiv 1 \pmod 4$ and $D(\mathbb{K}) = 4d$ for $d \equiv 2, 3 \pmod 4$. Therefore,

$$D(\mathbb{K}) = \frac{4d}{r^2}$$

for

$$r = \begin{cases} 2 & d \equiv 1 \pmod{4} \\ 1 & d \equiv 2, 3 \pmod{4} \end{cases} \tag{1.4}$$

and $\{1, w\}$ is an integral basis of $\mathbb{K}$ for

$$w = \frac{r - 1 + \sqrt{d}}{r}$$

(see [2]).

The D-order O_D is defined to be the ring

$$O_D = \{x + y\rho_D : \ x, y \in \mathbb{Z}\}, \tag{1.5}$$

where

$$\rho_D = \begin{cases} \sqrt{\frac{D}{4}} & \text{if } D \equiv 0 \pmod 4 \\ \frac{1+\sqrt{D}}{2} & \text{if } D \equiv 1 \pmod 4 \end{cases}$$

It is clear from definition that O_D is a subring of $\mathbb{Q}\left(\sqrt{D}\right)$.

Lemma 1.1. [1] *Let $\alpha \in O_D$. Then α is a unit in O_D if and only if $N(\alpha) = \pm 1$.*

The unit group O_D^* is defined to be the group of units of the ring O_D. Let

$$O_{D,1}^* = \{\alpha \in O_D^* : \ N(\alpha) = +1\}$$

and for $D > 0$

$$O_{D,+}^* = \{\alpha \in O_D^* : \alpha > 0\}$$

be the subgroup of positive units.

By Eq. (1.1) and Lemma 1.1, there is a bijection $\psi : Pell^{\pm}(D) \to O_D^*$ given by $\psi(x,y) = x + y\rho_D$. Since it is in bijection with a commutative group, $Pell^{\pm}(D)$ itself is a group for every non-square discriminant D. The mapping ψ is used to transport the group law from O_D^*, so that by definition

$$a.b = \psi^{-1}\left(\psi(a)\psi(b)\right)$$

for every $a, b \in Pell^{\pm}(D)$, i.e. the product $(u,v).(U,V)$ of two elements (u,v), $(U,V) \in Pell^{\pm}(D)$ is defined by the rule

$$(u,v).(U,V) = (x,y), \tag{1.6}$$

where $x + y\rho_D = (u + v\rho_D)(U + V\rho_D)$.

It follows by a calculation that

$$(u,v).(U,V) = \begin{cases} \left(uU + \frac{D}{4}vV,\ uV + vU\right) & \text{if } D \equiv 0 \pmod 4 \\ \left(uU + \frac{D-1}{4}vV,\ uV + vU + vV\right) & \text{if } D \equiv 1 \pmod 4 \end{cases} \tag{1.7}$$

The group structure on $Pell^{\pm}(D)$ has been defined as

$$\psi : Pell^{\pm}(D) \to O_D^*$$

and is an isomorphism of groups. Moreover, Eq. (1.7) is a group law on $Pell^{\pm}(D)$ with identity element $(1,0)$ for all non-zero discriminants D.

Lemma 1.2. [1] $O_{D,+}^* \simeq \mathbb{Z}$ *for every positive non-square discriminant* D.

From Lemma 1.2 we obtain

Lemma 1.3. [1] *Let* D *be a non-square discriminant and let* ε_D *be the smallest unit of* O_D *that is greater than* 1 *and let*

$$\tau_D = \begin{cases} \varepsilon_D & \text{if } N(\varepsilon_D) = +1 \\ \varepsilon_D^2 & \text{if } N(\varepsilon_D) = -1 \end{cases}$$

then

$$Pell^{\pm}(D) \simeq O_D^* = \{\pm\varepsilon_D^n;\ n \in \mathbb{Z}\} \simeq \{\pm 1\} \times \mathbb{Z}$$

and

$$Pell(D) \simeq O_{D,1}^* = \{\pm\tau_D^n;\ n \in \mathbb{Z}\} \simeq \{\pm 1\} \times \mathbb{Z}.$$

The *fundamental unit* ε_D is defined to be the smallest unit of O_D that is greater than 1.

The *Pell equation* is the equation $x^2 - dy^2 = 1$, and the *negative Pell equation* is the equation $x^2 - dy^2 = -1$, where d is a positive non-square integer. The set of all solutions of the Pell equation is infinite. The first solution (x_1, y_1) of the Pell equation is called the *fundamental solution.*

One may rewrite the Pell equation as

$$x^2 - dy^2 = \left(x + y\sqrt{d}\right)\left(x - y\sqrt{d}\right) = 1,$$

so that finding a solution comes down to finding a nontrivial unit in the ring $\mathbb{Z}\left[\sqrt{d}\right]$ of norm 1; here the norm $\mathbb{Z}\left[\sqrt{d}\right]^* \to \mathbb{Z}^* = \{\pm 1\}$ between unit groups multiplies each unit by its conjugate, and the units ± 1 of $\mathbb{Z}\left[\sqrt{d}\right]$ are considered trivial. This reformulation implies that once one knows a solution, fundamental solution, of the Pell equation, one can find infinitely many others. More precisely, if the solutions are ordered by magnitude, then the n th solution (x_n, y_n) can be expressed in terms of the fundamental solution by

$$x_n + y_n\sqrt{d} = \left(x_1 + y_1\sqrt{d}\right)^n. \tag{1.8}$$

for $n \geqslant 2$..

In [3], Jacobson and Williams considered the solutions of the consecutive Pell equations $X_0^2 - (D-1)Y_0^2 = 1$ and $X_1^2 - DY_1^2 = 1$, where D and $D - 1$ are not perfect square. They also proved that $\rho(D) = \frac{\log X_1}{\log X_0}$ could be arbitrary large for integers X_1 and X_2.

In [4], McLaughlin considered the solutions of the multi-variable polynomial Pell equation $C_i^2 - F_i H_i^2 = (-1)^{n-1}$, where n is a positive integer, $\{F_i\}$ a finite collection of multi-variable polynomials, C_i and H_i are multi-variable polynomials with integral coefficients.

In [5], Lenstra gave the solution of the Pell equation using the ring $\mathbb{Z}\left[\sqrt{d}\right]$. He considered the solvability of the Pell equation as a special case of Dirichlet's unit theorem from algebraic number theory which describes the structure of the group of units of a general ring of algebraic integers; for the ring $\mathbb{Z}\left[\sqrt{d}\right]$, it is product of $\{\pm 1\}$ and an infinite cyclic group.

In [6], Li proved that the Pell equation $x^2 - dy^2 = 1$ has infinitely positive solutions. If (x_1, y_1) is the fundamental solution, then for $n = 2, 3, 4, \ldots$, $x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$. The pairs (x_n, y_n) are all the positive solutions of the Pell equation. The x_n's and y_n's are strictly increasing to infinity and satisfy the recurrence relations

$$\begin{aligned} x_{n+2} &= 2x_1x_{n+1} - x_n, \\ y_{n+2} &= 2x_1y_{n+1} - y_n. \end{aligned}$$

Li also proved that the fundamental solution of the Pell equation is obtained by writing $\sqrt{d}$ as a simple continued fraction. It turns out that

$$\sqrt{d} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}}$$

where $a_0 = [\sqrt{d}]$ and $a_1,\ a_2, \dots$ is a periodic positive integer sequence. The continued fraction will be denoted by $\langle a_0, a_1, a_2, \dots \rangle$. The kth convergent of $\langle a_0, a_1, a_2, \dots \rangle$ is the number

$$\frac{p_k}{q_k} = \langle a_0, a_1, a_2, \dots, a_k \rangle$$

with p_k and q_k relatively prime numbers. Let $a_0, a_1, a_2, \dots, a_m$ be the period for $\sqrt{d}$. Then the fundamental solution of the Pell equation $x^2 - dy^2 = 1$ is

$$(x_1, y_1) = \begin{cases} (p_{m-1}, q_{m-1}) & \text{if } m \text{ is even} \\ (p_{2m-1}, q_{2m-1}) & \text{if } m \text{ is odd} \end{cases}$$

and the other solutions are $x_n + y_n\sqrt{d} = \left(x_1 + y_1\sqrt{d}\right)^n$ for $n \geqslant 2$.

In [7], Matthews considered the solutions of the equation $x^2 - Dy^2 = N$ for $D > 0$. He shoved that a necessary condition for the solvability of $x^2 - Dy^2 = N$, with $\gcd(x, y) = 1$, is that the congruence $u^2 \equiv D \pmod{Q_0}$ shall be soluble, where $Q_0 = |N|$.

In [8], Mollin gave a formula for the solutions of the equations both $X^2 - DY^2 = c$ and $x^2 - Dy^2 = -c$ using the ideals $I = [Q,\ P + \sqrt{D}]$, for positive non-square integer D.

In [9], Mollin, Cheng and Goddard considered the solutions of the Diophantine equation $aX^2 - bY^2 = c$ in terms of the simple continued fraction expansion of $\sqrt{a^2b}$, and they explored a criteria for the solvability of $AX^2 - BY^2 = C$ for given integers $A, B, C \in \mathbb{N}$.

In [10], Mollin, Poorten and Williams considered the equation $x^2 - Dy^2 = -3$. They obtained a formula for the solutions of this equation using the continued fraction expansion of $\sqrt{D}$ and using the ambiguous ideals $I = [Q, P + \sqrt{D}]$, i.e., $I = \overline{I}$, where $\overline{I}$ denotes the conjugate of I.

In [11], Stevenhagen considered the solutions of the negative Pell equation $x^2 - Dy^2 = -1$ for a positive non-square integer D. He stated a conjecture for the solutions of the equation $x^2 - Dy^2 = -1$.

In [12], we considered the solutions of the Pell equation $x^2 - Dy^2 = 2$ using the fundamental element of the field $\mathbb{Q}(\sqrt{\Delta})$.

In the present paper, the solutions of the Pell equation $x^2 - Dy^2 = 2$ for positive non-square integer and a recurrence relation for the solutions of the Pell equation $x^2 - Dy^2 = 2$ are obtained using the solutions of the Pell equation $x^2 - Dy^2 = 1$.

2. The Pell Equation $x^2 - Dy^2 = 2$

First, consider the fundamental solution of the Pell equation $x^2 - Dy^2 = 2$.

Theorem 2.1. *If $(x_1, y_1) = (a, 1)$ is the fundamental solution of the Pell equation $x^2 - Dy^2 = 1$, then the fundamental solution of the Pell equation $x^2 - (D-1)y^2 = 2$ is $(X_1, Y_1) = (a, 1)$.*

Proof. Since $(x_1, y_1) = (a, 1)$ is the fundamental solution of the Pell equation $x^2 - Dy^2 = 1$, we have

$$a^2 - D = 1.$$

Hence, by basic calculation, it is easily seen that

$$X_1^2 - (D-1)Y_1^2 = a^2 - (D-1) = a^2 - D + 1 = 2.$$

Therefore, $(X_1, Y_1) = (a, 1)$ is the fundamental solution of the Pell equation $x^2 - Dy^2 = 2$. □

Theorem 2.2. *If $D = k^2 - 2, k \geqslant 2$, then the fundamental solution of the Pell equation $x^2 - Dy^2 = 1$ is $(x_1, y_1) = (a, b) = (k^2 - 1, k)$, and the fundamental solution of the Pell equation $x^2 - Dy^2 = 2$ is $(X_1, Y_1) = (b, 1)$.*

Proof. Note that $(x_1, y_1) = (a, b) = (k^2 - 1, k)$ is the fundamental solution of the Pell equation $x^2 - Dy^2 = 1$ since

$$x_1^2 - Dy_1^2 = a^2 - Db^2 = (k^2 - 1)^2 - (k^2 - 2)k^2 = 1.$$

For $(X_1, Y_1) = (b, 1)$, we get

$$X_1^2 - DY_1^2 = b^2 - D = k^2 - (k^2 - 2) = 2.$$

Therefore, $(X_1, Y_1) = (b, 1)$ is the fundamental solution of the Pell equation $x^2 - Dy^2 = 2$. □

Now we consider the solutions of the Pell equation $x^2 - Dy^2 = 2$. To get this we have the following theorem.

Theorem 2.3. *Let $(X_1, Y_1) = (k, l)$ be the fundamental solution of the Pell equation $x^2 - Dy^2 = 2$. Then the other solutions of the Pell equation $x^2 - Dy^2 = 2$ are (X_n, Y_n), where*

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} = \begin{pmatrix} k & lD \\ l & k \end{pmatrix} \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} \tag{2.1}$$

for $n \geqslant 2$.

Proof. From above equalities we get

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} = \begin{pmatrix} k & lD \\ l & k \end{pmatrix} \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} kx_{n-1} + lDy_{n-1} \\ lx_{n-1} + ky_{n-1} \end{pmatrix}. \tag{2.2}$$

Hence it is easily seen that

$$\begin{aligned} X_n^2 - DY_n^2 &= (kx_{n-1} + lDy_{n-1})^2 - D\,(lx_{n-1} + ky_{n-1})^2 \\ &= k^2x_{n-1}^2 + 2klDx_{n-1}y_{n-1} + l^2D^2y_{n-1}^2 \\ &\qquad - D(l^2x_{n-1}^2 + 2klx_{n-1}y_{n-1} + k^2y_{n-1}^2) \\ &= k^2(x_{n-1}^2 - Dy_{n-1}^2) - Dl^2(x_{n-1}^2 - Dy_{n-1}^2) \\ &= (x_{n-1}^2 - Dy_{n-1}^2)(k^2 - Dl^2) \\ &= 2, \end{aligned}$$

since $x_{n-1}^2 - Dy_{n-1}^2 = 1$, and $(X_1, Y_1) = (k, l)$ be the fundamental solution of the Pell equation $x^2 - Dy^2 = 2$, i.e., $k^2 - Dl^2 = 2$. □

From Theorem 2.3 the following corollary can be obtained.

Corollary 2.4. *The solutions (X_n, Y_n) of the Pell equation $x^2 - Dy^2 = 2$ satisfy the following relations*

$$\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} aX_n + bDY_n \\ bX_n + aY_n \end{pmatrix} = \begin{pmatrix} a & bD \\ b & a \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix}$$

for $n \geqslant 1$.

Proof. We know from Eq. (2.2) that

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} = \begin{pmatrix} kx_{n-1} + lDy_{n-1} \\ lx_{n-1} + ky_{n-1} \end{pmatrix}.$$

Hence

$$x_{n-1} = \frac{klX_n + (2 - k^2)Y_n}{2l} \quad \text{and} \quad y_{n-1} = \frac{-lX_n + kY_n}{2}. \tag{2.3}$$

On the other hand from Eq. (2.1)

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} a & bD \\ b & a \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.4}$$
$$= \begin{pmatrix} a & bD \\ b & a \end{pmatrix} \begin{pmatrix} a & bD \\ b & a \end{pmatrix}^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} a & bD \\ b & a \end{pmatrix} \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}$$
$$= \begin{pmatrix} ax_{n-1} + bDy_{n-1} \\ bx_{n-1} + ay_{n-1} \end{pmatrix}.$$

Using Eq. (2.3) and Eq. (2.4) we find

$$\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} k & lD \\ l & k \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} \tag{2.5}$$
$$= \begin{pmatrix} k & lD \\ l & k \end{pmatrix} \begin{pmatrix} ax_{n-1} + bDy_{n-1} \\ bx_{n-1} + ay_{n-1} \end{pmatrix}$$
$$= \begin{pmatrix} x_{n-1}(ak + blD) + y_{n-1}(bkD + alD) \\ x_{n-1}(al + bk) + y_{n-1}(blD + ak) \end{pmatrix}$$

Applying Eq. (2.3) and Eq. (2.5) it follows that

$$\begin{aligned} X_{n+1} &= x_{n-1}(ak + blD) + y_{n-1}(bkD + alD) \\ &= \left(\frac{klX_n + (2 - k^2)Y_n}{2l}\right)(ak + blD) \\ &\qquad + \left(\frac{-lX_n + kY_n}{2}\right)(bkD + alD) \\ &= \frac{2alX_n + 2bDlY_n}{2l} \\ &= aX_n + bDY_n \end{aligned}$$

and

$$\begin{aligned} Y_{n+1} &= x_{n-1}(al + bk) + y_{n-1}(blD + ak) \\ &= \left(\frac{klX_n + (2 - k^2)Y_n}{2l}\right)(al + bk) \\ &\qquad + \left(\frac{-lX_n + kY_n}{2}\right)(blD + ak) \\ &= \frac{2blX_n + 2alY_n}{2l} \\ &= bX_n + aY_n. \end{aligned}$$

Hence

$$\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} aX_n + bDY_n \\ bX_n + aY_n \end{pmatrix} = \begin{pmatrix} a & bD \\ b & a \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix}.$$

□

Example 2.1. Let $D = 2$. Then the fundamental solution of the Pell equation $x^2 - 2y^2 = 1$ is $(x_1, y_1) = (3, 2)$, and the other solutions are

$$\begin{aligned}
\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} &= \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 17 \\ 12 \end{pmatrix}, \\
\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} &= \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 99 \\ 70 \end{pmatrix}, \\
\begin{pmatrix} x_4 \\ y_4 \end{pmatrix} &= \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}^4 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 577 \\ 408 \end{pmatrix}, \\
\begin{pmatrix} x_5 \\ y_5 \end{pmatrix} &= \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}^5 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3363 \\ 2378 \end{pmatrix}, \\
\begin{pmatrix} x_6 \\ y_6 \end{pmatrix} &= \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}^6 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 19601 \\ 13860 \end{pmatrix}, \\
\begin{pmatrix} x_7 \\ y_7 \end{pmatrix} &= \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}^7 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 114243 \\ 80782 \end{pmatrix}, \\
\begin{pmatrix} x_8 \\ y_8 \end{pmatrix} &= \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}^8 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 665857 \\ 470832 \end{pmatrix}, \\
\begin{pmatrix} x_9 \\ y_9 \end{pmatrix} &= \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}^9 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3880899 \\ 2744210 \end{pmatrix}.
\end{aligned}$$

The fundamental solution of the Pell equation $x^2 - 2y^2 = 2$ is $(X_1, Y_1) = (k, l) = (2, 1)$, and the other solutions are

$$\begin{aligned}
\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} &= \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 10 \\ 7 \end{pmatrix}, \\
\begin{pmatrix} X_3 \\ Y_3 \end{pmatrix} &= \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 17 \\ 12 \end{pmatrix} = \begin{pmatrix} 58 \\ 41 \end{pmatrix}, \\
\begin{pmatrix} X_4 \\ Y_4 \end{pmatrix} &= \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 99 \\ 70 \end{pmatrix} = \begin{pmatrix} 338 \\ 239 \end{pmatrix}, \\
\begin{pmatrix} X_5 \\ Y_5 \end{pmatrix} &= \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 577 \\ 408 \end{pmatrix} = \begin{pmatrix} 1970 \\ 1393 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
\begin{pmatrix} X_6 \\ Y_6 \end{pmatrix} &= \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3363 \\ 2378 \end{pmatrix} = \begin{pmatrix} 11482 \\ 8119 \end{pmatrix}, \\
\begin{pmatrix} X_7 \\ Y_7 \end{pmatrix} &= \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 19601 \\ 13860 \end{pmatrix} = \begin{pmatrix} 66922 \\ 47321 \end{pmatrix}, \\
\begin{pmatrix} X_8 \\ Y_8 \end{pmatrix} &= \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 114243 \\ 80782 \end{pmatrix} = \begin{pmatrix} 390050 \\ 275807 \end{pmatrix}, \\
\begin{pmatrix} X_9 \\ Y_9 \end{pmatrix} &= \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 665857 \\ 470832 \end{pmatrix} = \begin{pmatrix} 2273378 \\ 1607521 \end{pmatrix}, \\
\begin{pmatrix} X_{10} \\ Y_{10} \end{pmatrix} &= \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3880899 \\ 2744210 \end{pmatrix} = \begin{pmatrix} 13250218 \\ 9369319 \end{pmatrix}.
\end{aligned}$$

Now we give a relation between $\begin{pmatrix} a & bD \\ b & a \end{pmatrix}$ and $\begin{pmatrix} k & lD \\ l & k \end{pmatrix}$. To get this, set

$$SL_2(2, \mathbb{R}) = \left\{ \begin{pmatrix} r & s \\ t & u \end{pmatrix} : r, s, t, u \in \mathbb{R},\ ru - st = 2 \right\}.$$

Then we have

Theorem 2.5. *There exists an element $A \in SL_2(2, \mathbb{R})$ such that*

$$\begin{pmatrix} a & bD \\ b & a \end{pmatrix} A = \begin{pmatrix} k & lD \\ l & k \end{pmatrix}.$$

Proof. Let $A = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$ for $r, s, t, u \in \mathbb{R}$. Then we have

$$\begin{pmatrix} a & bD \\ b & a \end{pmatrix} \begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} ar + bDt & as + bDu \\ br + at & bs + au \end{pmatrix} = \begin{pmatrix} k & lD \\ l & k \end{pmatrix}.$$

Hence from the two equations

$$\begin{aligned} ar + bDt &= k \\ br + at &= l \end{aligned}$$

and

$$\begin{aligned} as + bDu &= lD \\ bs + au &= k \end{aligned}$$

we obtain

$$\begin{aligned} r &= \frac{l-a^2l+abk}{b} \\ s &= \frac{k-a^2k+ablD}{b} \\ t &= al-bk \\ u &= ak-blD. \end{aligned}$$

Hence

$$A=\begin{pmatrix} \frac{l-a^2l+abk}{b} & \frac{k-a^2k+ablD}{b} \\ al-bk & ak-blD \end{pmatrix}.$$

It is easily seen that

$$\begin{aligned} \det(A) &= \left(\frac{l-a^2l+abk}{b}\right)(ak-blD) \\ &\quad -\left(\frac{k-a^2k+ablD}{b}\right)(al-bk) \\ &= \frac{b(k^2-Dl^2)}{b} \\ &= k^2-Dl^2 \\ &= 2. \end{aligned}$$

Therefore $A \in SL_2(2,\mathbb{R})$. $\square$

Now we would like to obtain a recurrence relation on the solutions of the Pell equation $x^2-Dy^2=2$. To get this using Eq. (2.1) we obtain

$$\begin{aligned} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} &= \begin{pmatrix} a \\ b \end{pmatrix}, \qquad (2.6) \\ \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} &= \begin{pmatrix} 2a^2-1 \\ 2ab \end{pmatrix}, \\ \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} &= \begin{pmatrix} 4a^3-3a \\ b(4a^2-1) \end{pmatrix}, \\ \begin{pmatrix} x_4 \\ y_4 \end{pmatrix} &= \begin{pmatrix} 8a^4-8a^2+1 \\ b(8a^3-4a) \end{pmatrix}, \end{aligned}$$

$$\begin{pmatrix} x_5 \\ y_5 \end{pmatrix} = \begin{pmatrix} 16a^5 - 20a^3 + 5a \\ b(16a^4 - 12a^2 + 1) \end{pmatrix},$$
$$\begin{pmatrix} x_6 \\ y_6 \end{pmatrix} = \begin{pmatrix} 32a^6 - 48a^4 + 18a^2 - 1 \\ b(32a^5 - 32a^3 + 6a) \end{pmatrix},$$
$$\begin{pmatrix} x_7 \\ y_7 \end{pmatrix} = \begin{pmatrix} 64a^7 - 112a^5 + 56a^3 - 7a \\ b(64a^6 - 80a^4 + 24a^2 - 1) \end{pmatrix},$$
$$\begin{pmatrix} x_8 \\ y_8 \end{pmatrix} = \begin{pmatrix} 128a^8 - 256a^6 + 160a^4 - 32a^2 + 1 \\ b(128a^7 - 192a^5 + 80a^3 - 8a) \end{pmatrix},$$

and hence

$$\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} = \begin{pmatrix} ka + lbD \\ la + kb \end{pmatrix}, \tag{2.7}$$
$$\begin{pmatrix} X_3 \\ Y_3 \end{pmatrix} = \begin{pmatrix} k(2a^2 - 1) + 2ablD \\ l(2a^2 - 1) + 2abk \end{pmatrix},$$
$$\begin{pmatrix} X_4 \\ Y_4 \end{pmatrix} = \begin{pmatrix} k(4a^3 - 3a) + blD(4a^2 - 1) \\ l(4a^3 - 3a) + kb(4a^2 - 1) \end{pmatrix},$$
$$\begin{pmatrix} X_5 \\ Y_5 \end{pmatrix} = \begin{pmatrix} k(8a^4 - 8a^2 + 1) + blD(8a^3 - 4a) \\ l(8a^4 - 8a^2 + 1) + kb(8a^3 - 4a) \end{pmatrix},$$
$$\begin{pmatrix} X_6 \\ Y_6 \end{pmatrix} = \begin{pmatrix} k(16a^5 - 20a^3 + 5a) + blD(16a^4 - 12a^2 + 1) \\ l(16a^5 - 20a^3 + 5a) + kb(16a^4 - 12a^2 + 1) \end{pmatrix},$$
$$\begin{pmatrix} X_7 \\ Y_7 \end{pmatrix} = \begin{pmatrix} k(32a^6 - 48a^4 + 18a^2 - 1) + blD(32a^5 - 32a^3 + 6a) \\ l(32a^6 - 48a^4 + 18a^2 - 1) + kb(32a^5 - 32a^3 + 6a) \end{pmatrix},$$
$$\begin{pmatrix} X_8 \\ Y_8 \end{pmatrix} = \begin{pmatrix} k(64a^7 - 112a^5 + 56a^3 - 7a) + blD(64a^6 - 80a^4 + 24a^2 - 1) \\ l(64a^7 - 112a^5 + 56a^3 - 7a) + kb(64a^6 - 80a^4 + 24a^2 - 1) \end{pmatrix},$$
$$\begin{pmatrix} X_9 \\ Y_9 \end{pmatrix} = \begin{pmatrix} k(128a^8 - 256a^6 + 160a^4 - 32a^2 + 1) + blD(128a^7 - 192a^5 + 80a^3 - 8a) \\ l(128a^8 - 256a^6 + 160a^4 - 32a^2 + 1) + kb(128a^7 - 192a^5 + 80a^3 - 8a) \end{pmatrix}.$$

Using Eq. (2.7) it is easily seen that

$$\begin{aligned} X_4 &= (2k^2-1)(X_3-X_2)+X_1, \\ X_5 &= (2k^2-1)(X_4-X_3)+X_2, \\ X_6 &= (2k^2-1)(X_5-X_4)+X_3, \\ X_7 &= (2k^2-1)(X_6-X_5)+X_4, \\ X_8 &= (2k^2-1)(X_7-X_6)+X_5, \\ X_9 &= (2k^2-1)(X_8-X_7)+X_6. \end{aligned} \tag{2.8}$$

Eq. (2.8) is also satisfied for Y_n. Hence we have

Conjecture 2.6. *The solutions of the Pell equation $x^2 - Dy^2 = 2$ satisfy the following recurrence relations*

$$\begin{aligned} X_n &= (2k^2-1)(X_{n-1}-X_{n-2})+X_{n-3}, \\ Y_n &= (2k^2-1)(Y_{n-1}-Y_{n-2})+Y_{n-3} \end{aligned}$$

for $n \geqslant 4$.

Now we obtain a formula for the solutions of the equation $x_n^2 - Dy_n^2 = 2^n$ using the solutions of the Pell equation $x^2 - Dy^2 = 2$.

Theorem 2.7. *Let $(X_1, Y_1) = (k, l)$ be the fundamental solution of the Pell equation $x^2 - Dy^2 = 2$. Let*

$$\begin{pmatrix} U_n \\ V_n \end{pmatrix} = \begin{pmatrix} k & lD \\ l & k \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.9}$$

for $n \geqslant 1$. Then

$$U_n^2 - DV_n^2 = 2^n$$

for $n \geqslant 1$.

Proof. We prove the Theorem by induction on n. For $n = 1$ we have

$$U_1^2 - DV_1^2 = k^2 - Dl^2 = 2$$

since $(X_1, Y_1) = (k, l)$ is the fundamental solution of the Pell equation $x^2 - Dy^2 = 2$.

Let us assume that the equation $U_n^2 - DV_n^2 = 2^n$ is satisfied for (U_{n-1}, V_{n-1}), i.e. ,

$$U_{n-1}^2 - DV_{n-1}^2 = 2^{n-1}.$$

We want to show that the equation $U_n^2 - DV_n^2 = 2^n$ is also satisfied for (U_n, V_n). From Eq. (2.9) it is easily seen that

$$\begin{aligned}
\begin{pmatrix} U_n \\ V_n \end{pmatrix} &= \begin{pmatrix} k & lD \\ l & k \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} k & lD \\ l & k \end{pmatrix} \begin{pmatrix} k & lD \\ l & k \end{pmatrix}^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} k & lD \\ l & k \end{pmatrix} \begin{pmatrix} U_{n-1} \\ V_{n-1} \end{pmatrix} \\
&= \begin{pmatrix} kU_{n-1} + lDV_{n-1} \\ lU_{n-1} + kV_{n-1} \end{pmatrix}.
\end{aligned}$$

Hence

$$\begin{aligned}
U_n^2 - DV_n^2 &= (kU_{n-1} + lDV_{n-1})^2 - D\,(lU_{n-1} + kV_{n-1})^2 \\
&= U_{n-1}^2(k^2 - Dl^2) + U_{n-1}V_{n-1}(2klD - 2klD) \\
&\qquad + V_{n-1}^2(l^2D^2 - k^2D) \\
&= U_{n-1}^2(k^2 - Dl^2) - D(k^2 - Dl^2)V_{n-1}^2 \\
&= (k^2 - Dl^2)\left(U_{n-1}^2 - DV_{n-1}^2\right) \\
&= 2 \cdot 2^{n-1} = 2^n.
\end{aligned}$$

Thus (U_n, V_n) is also a solution of the equation $U_n^2 - DV_n^2 = 2^n$. $\square$

Example 2.2. Consider the Pell equation $x^2 - 2y^2 = 2$. The fundamental solution is $(X_1, Y_1) = (k, l) = (2, 1)$. Using Eq. (2.9) we obtain

$$\begin{aligned}
\begin{pmatrix} U_1 \\ V_1 \end{pmatrix} &= \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}^1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\
\begin{pmatrix} U_2 \\ V_2 \end{pmatrix} &= \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix} \\
\begin{pmatrix} U_3 \\ V_3 \end{pmatrix} &= \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 20 \\ 14 \end{pmatrix} \\
\begin{pmatrix} U_4 \\ V_4 \end{pmatrix} &= \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}^4 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 68 \\ 48 \end{pmatrix}
\end{aligned}$$

$$\begin{pmatrix} U_5 \\ V_5 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}^5 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 232 \\ 164 \end{pmatrix}$$
$$\begin{pmatrix} U_6 \\ V_6 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}^6 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 792 \\ 560 \end{pmatrix}.$$

Hence it is easily seen that

$$\begin{aligned} U_1^2 - 2V_1^2 &= 2, \\ U_2^2 - 2V_2^2 &= 4, \\ U_3^2 - 2V_3^2 &= 8, \\ U_4^2 - 2V_4^2 &= 16, \\ U_5^2 - 2V_5^2 &= 32, \\ U_6^2 - 2V_6^2 &= 64. \end{aligned}$$

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