Abstracts of Invited Lectures at BMC2004

Plenary Lectures

Alexander Kechris (CalTech at Pasadena)
Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups

Chair: Garth Dales

Joint work with V. Pestov and S. Todorčevic

It has been recently discovered that the study of topological dynamics of automorphism groups, particularly concerning extreme amenability and the calculation of universal minimal flows, is closely connected to the Fraïssé theory of amalgamation and ultrahomogeneity, and the Ramsey theory for classes of finite structures. In this talk, I will explain these connections and discuss applications to the topological dynamics of automorphism groups and Ramsey theory.

Samuel Patterson (Univ Göttingen)
Squaring circles and circling squares—Gauss' Circle Problem

Chair: Allan Sinclair

Towards the end of Section V of his "Disquisitiones Arithmeticae" Gauss discusses the tables of the arithmetical quantities he has studied and formulated a number of statistical assertions about them. He made two attempts some thirty years later to write these up in detail but this work remained unpublished until it was included in his collected works. The "circle problem" is the most elementary of the questions considered by him. It turns out to be extremely subtle and it is intimately connected with central questions of analytic number theory. Therefore it has become a touchstone for each generation of number-theorists to test recent advances in technique. In the first half of the talk I shall discuss this problem and how one thinks about it.

In the second half I shall turn to the analogue in the case of the hyperbolic plane. This is associated with two distinct questions about quadratic forms (in three or four variables). Here the situation is by no means as well understood. I shall discuss what is known and the several very different techniques used in this connection. Finally I shall address the problem which is the analogue of the central problem in Gauss' circle problem. Here the framework is quite different and the correct conjecture controversial.

Gilles Pisier (Texas A&M University and University Paris VI)
Factorization of completely bounded maps on "exact" C-algebras and operator spaces*

CHAIR: ERLING STØRMER

This talk will be on part of "Operator space theory". "Operator spaces" are also called "non-commutative Banach spaces". Indeed, while Banach spaces can be viewed as subspaces of commmutative C*-algebras (these can be realized as spaces of functions), operator spaces are defined as subspaces of general C*-algebras (realized as spaces of operators). We will review the operator space version of Grothendieck's theorem proved recently in joint work with Shlyakhtenko as well as more recent results related to Junge's remarkable theorem that the operator Hilbert space OH embeds in a non-commutative L_1-space X. We show that Junge's embedding does not exist when the von Neumann algebra X^* is semi-finite.

A sample result characterizes the maps u, from a C*-algebra A to a Hilbert space, for which there are states f, g on A and $C > 0$ such that

$$\|ua\|^2 \leq C f(a^*a)^{1-t} g(aa^*)^t \qquad (a \in A).$$

Claudio Procesi (Universita di Roma La Sapienza)
The role of the Cayley–Hamilton identity in representation theory

CHAIR: ROB CURTIS

The Cayley–Hamilton identity states that a matrix satisfies its characteristic polynomial. There is an abstract theory for general algebras with trace which takes this statement as defining axiom. It turns out that, at least in characteristic 0, this can be developed as a correspondence between such algebras and varieties with an action of the projective group.

This in turn can be used to answer questions in representation theory, as the examples from quantum groups at roots of 1 show. It is thus an effective method of algebraic geometry in non commutative algebra.

Efim Zelmanov (University of California at San Diego)
On Large Pro-p Groups

Chair: Ken Brown

The talk will focus on two families of "large" groups: (i) free pro-p groups and (ii) Golod–Shafarevich groups. I will sketch a recent theorem of mine that these groups are not linear and discuss connections of Golod–Shafarevich groups with Number Theory and 3-manifolds.

Günter Ziegler (Technische Universität Berlin)
On the combinatorics of the 3-sphere

Chair: Martin Mathieu

Triangulations and cell decompositions of the two-dimensional sphere can be understood in terms of three-dimensional polyhedra. The corresponding theory is classical, visually accessible, and quite complete—due to Tutte, Steinitz, and many others.

Triangulations and cell decompositions of the three-dimensional sphere pose much bigger problems to us. In this lecture we shall thus treat questions like:

"How many triangulations are there (with n vertices, say)?"

"Do most of these correspond to convex polytopes?"

"How can the vertex-/ edge-/ face-numbers be characterized?"

Our (partial) answers to such questions involve a nice interplay of combinatorial ideas, new geometric constructions, advanced visualization tools, as well as differential geometric and topological components.

Special Session Lectures

Special Session on Non-commutative Functional Analysis

Mikael Rørdam (University of Southern Denmark)
Purely infinite C-algebras*

The notion of purely infinite C*-algebras was introduced in the late '70s by J. Cuntz for simple C*-algebras; and he showed that his own algebras $\mathcal{O}_n$ have that property. In the simple case, pure infiniteness is equivalent to existence of many projections (real rank zero) together with the requirement that all projections are infinite. Later, in the mid '90s, Kirchberg proved a number of remarkable theorems, one of which says that a simple, separable, nuclear C*-algebra A is purely infinite if and only if the tensor product of A and $\mathcal{O}_\infty$ is isomorphic to A. He and Phillips subsequently proved (independently) a complete classification theorem for this class of C*-algebras (in terms of KK-theory, and in terms of K-theory if the UCT is assumed). Since then, the challenge has been to extend these results to the non-simple case. There is a KK-type classification theorem, due to Kirchberg, for those separable nuclear C*-algebras that tensorially absorb $\mathcal{O}_\infty$ (as above), and there are partial—but to date not complete—results that describe when a (non-simple) C*-algebra absorbs $\mathcal{O}_\infty$. I will describe these results in the talk, and mention some special cases where the theory works out particularly well.

Matthias Neufang (Carleton University, Ottawa)
Quantization in abstract harmonic analysis

Our aim is to give an overview of our recent contributions to various aspects of 'quantization' in abstract harmonic analysis—where this (very fashionable) word has two meanings for us:
1. the use of operator space theory in the study of classical objects in harmonic analysis;
2. the investigation of non-commutative counterparts of these objects arising through the replacement of functions by operators.
We shall illustrate this programme by discussing the following topics (throughout, $\mathcal{G}$ denotes a locally compact group).

1.1 Amenability theory Z.–J. Ruan showed that the canonical operator space structure of the Fourier algebra $A_2(\mathcal{G})$ is crucial for understanding its amenability properties—by proving that $A_2(\mathcal{G})$ is *operator* amenable if and only if the group $\mathcal{G}$ is amenable (this equivalence fails in the realm of classical amenability, as shown by B.E. Johnson). We extend Ruan's theorem to the case of the Figà–Talamanca–Herz algebras $A_p(\mathcal{G})$, for $p \in (1, \infty)$, endowed with an appropriate operator space structure. The construction of the latter requires the concept of non-hilbertian column operator spaces, as developed recently by A. Lambert and G. Wittstock. This is joint work with A. Lambert and V. Runde.

1.2 Representation theory We present a common representation-theoretical framework for various Banach algebras arising in abstract harmonic analysis, such as the measure algebra $M(\mathcal{G})$ and the completely bounded multipliers of the Fourier algebra $M_{cb}A_2(\mathcal{G})$. The image algebras are intrinsically characterized as certain normal completely bounded bimodule maps on $\mathcal{B}(L_2(\mathcal{G}))$. The study of our representations reveals intriguing properties of the algebras, and provides a very simple description of their Kac algebraic duality. Part of this is joint work with Z.–J. Ruan and N. Spronk.

2.1 Harmonic operators In the context of the representation model as described in 1.2, we study an operator version of the classical Choquet–Deny equation (for a fixed probability measure on $\mathcal{G}$), the solutions of which we refer to as *harmonic operators*. We show that the space of harmonic operators is naturally equipped with the structure of a non-commutative von Neumann algebra, and completely describe its structure as a W^*-crossed product over the classical harmonic functions. This is joint work with C. Cuny and W. Jaworski.

2.2 Quantized convolution The space $\mathcal{T}(L_2(\mathcal{G}))$ of trace class operators is usually considered as the non-commutative version of $L_1(\mathcal{G})$ on the level of Banach spaces. We show that this analogy may be extended to the Banach algebra level by introducing a new product on $\mathcal{T}(L_2(\mathcal{G}))$ which parallels convolution of functions.

Marius Junge (University of Illinois at Urbana–Champaign)
Embedding of the operator space OH and related results

The operator space OH is defined by complex interpolation by between the two most elementary operator spaces (i.e., quantized Banach spaces), the space of columns and rows. Pisier introduced this space and shows that this operator space shares many of the interesting properties of Hilbert spaces, a central object in the class of Banach spaces. We show that there is a complete embedding of the operator space OH in the predual of a on Neumann algebra using free probability or alternatively CCR or CAR relations. The embedding using free probability has interesting properties and allows us to calculate the operator space projection constant of the n-dimensional version of OH. It turns that this constant is smaller than in the commutative case. This new surprising logarithmic factor shows that jointly completely bounded sesquilinear forms behave differently than one might have expected from Grothendieck's work in the commutative case.

Special Session on Combinatorics

Graham Brightwell (London School of Economics)
The number of linear extensions of a partially ordered set

Given a partial order $<$ on a finite groundset X, a *linear extension* is a total order $<'$ on X such that, whenever $x < y$, then also $x <' y$. The number $L(X, <)$ of linear extensions is a fundamental parameter of partial orders.

The number $L(X, <)$ is hard to compute exactly, but there are a number of techniques for estimating or approximating it, involving methods from geometry and from the theory of rapidly mixing Markov chains. We look at these, and also investigate methods that apply when the partial order has some special structure.

Oliver Riordan (Trinity College, Cambridge)
Scale-free random graphs

In the last few years there has been a great deal of interest in using random graph models to gain insight into the behaviour of large-scale real-world networks, such as the internet and web graphs, networks of social interactions, and many other examples. Of course, in any one case the real-world network cannot be well approximated by a mathematical model, but simple models incorporating key features of many networks will still be very useful. One such feature is that the degree distributions are often rather skewed, following a 'scale-free' power law. This behaviour is very different from classical random graphs, where the degrees tend to be rather concentrated around their mean. Many new 'scale-free' random graph models have been proposed in the last few years, most having two key features: the graph grows by adding vertices one at a time, and there is some kind of 'preferential attachment': new edges are more likely to join to vertices that already have large degree. So far, much of the work in this area has been by physicists and computer scientists, and is experimental or heuristic. In fact, perhaps the most popular model, the Barabási–Albert or BA model, does not make mathematical sense. I will give a survey of some of the recent mathematical work in this area, starting with a mathematically precise model, the LCD model, fitting the vague BA description.

Alex Scott (University College, London)
Independent sets, lattice gases and the Lovasz Local Lemma

Given a family of independent events in a probability space, the probability that none of the events occurs is of course the product of the probabilities that the individual events do not occur. If there is some dependence between the events, however, then bounding the probability that none occur is a much less trivial matter. The Lovasz Local Lemma is a useful tool in probabilistic combinatorics that enables such estimates to be made, providing the dependencies between events are not too strong.

The lattice gas with repulsive pair interactions is an important model in equilibrium statistical mechanics, and has been studied extensively by mathematical physicists. In the special case of a hard-core nearest-neighbour exclusion (i.e. no pair of adjacent sites can be simultaneously occupied), the partition function of the lattice gas on a graph coincides with the independent-set polynomial; much effort has been devoted to finding regions in the complex plane in which this function is nonvanishing.

In this talk, which presents joint work with Alan Sokal, we examine a connection between these two apparently disparate subjects: in particular, we will discuss closely related results of Shearer in probabilistic combinatorics and of Dobrushin in mathematical physics, and present a 'soft' generalization of the Lovasz Local Lemma.

Morning Lectures

Stephen Buckley (National University of Ireland, Maynooth)
Gromov hyperbolicity for analysts

Gromov hyperbolicity is a concept that was first investigated in the context of finitely generated groups but in recent years has become quite important for analysts. We briefly survey some recent results concerning Gromov hyperbolicity relevant to the quasihyperbolic metric in real analysis and conformally invariant metrics in several complex variables. Some of these results require that the underlying space is bounded with respect to some other metric (such as the inner Euclidean metric). Trying to overcome this restriction leads us to investigate the concepts of *sphericalization* and *flattening*, conformal deformations that take unbounded spaces to bounded ones and vice versa.

Keith Carne (University of Cambridge)
How an analytic function changes the length of a radius

Based on joint work with Alan Beardon.

Let f be an analytic function on the unit disc for which the area of its image, counting multiplicity, is finite. We wish to consider how the hyperbolic length ρ of a radial arc compares with the Euclidean length $\mathcal{E}$ of its image under f. A. Beurling showed that for radii in all directions except for a set of capacity 0, the image of the entire radius has finite length. However, for exceptional directions the length can be infinite. J. E. Littlewood and F. R. Keogh showed that, even for these exceptional directions,

$$\mathcal{E} = o(\rho^{1/2}) \ .$$

I will describe joint work with A. F. Beardon to explore these results and the many variations considered by others.

Tom Carroll (National University of Ireland, Cork)
Harmonic measure in parabola-shaped regions in $\mathbb{R}^n$

We will be concerned with the sharp rate of decay of harmonic measure in a solid of revolution arising from the curve $y = x^a$, $x > 0$. Here $0 < a < 1$ and, more precisely, we seek the rate of decay of the harmonic measure of the exterior of a ball of radius r as r increases. In the case of two dimensions, classical conformal mapping estimates of Ahlfors and Warschawski yield precise results. This method will be described in detail. We will then view the n-dimensional harmonic measure problem as a 2-dimensional problem but for an operator that is no longer the Laplacian, and show how the classical method can be adapted to deal with this more general situation. A number of difficulties arise, whose solutions will be sketched.

The harmonic measure problem may be considered as a distributional inequality for the exit position of Brownian motion from the parabola-shaped region. In closing, we will describe related results by various authors on the distribution of the exit time for these regions.

Sean Dineen (National University of Ireland, Dublin)
Banach-valued spectra

No abstract available.

Graham Ellis (National University of Ireland, Galway)
Polytopes and the cohomology of groups

The cohomology of a group G can be defined as the cohomology of an orbit space X/G where X is any contractible cellular space on which G acts freely. We'll give two methods for constructing X which yield explicit calculations.

Firstly, for an arbitrary finite group G we'll show how to construct X from a faithful representation $G \to GL(\mathbb{R}^n)$ by considering the convex hull of the orbit of a suitable vector $v \in \mathbb{R}^n$. This method can be used to recover results on Coxeter groups due to De Concini and Salvetti.

Then, for certain infinite groups G (such as spherical Artin groups), we'll show how to construct X using polytopes and lattice-like properties of the Cayley graph.

Stephen Huggett (University of Plymouth)
Tutte polynomials of links, graphs, and matroids

We review the relationships between links, graphs, and matroids. In particular, given a planar graph there are two well-established methods of generating an alternating link diagram. Switching from one of these methods to the other corresponds in knot theory to tangle insertion in the link diagrams, and in combinatorics to the tensor product of the cycle matroids of the graphs. We show how the Jones and Tutte polynomials behave under these operations. Then, in a separate piece of work, given a graph G (not necessarily planar) we show how to construct a family of manifolds M_n whose Euler characteristic $x(M_n)$ is the chromatic polynomial $c(G, n)$. The manifolds M_n are simple generalisations of configuration spaces. This is joint work with Michael Eastwood.

Robert Marsh (University of Leicester)
Cluster algebras and tilting theory

Joint work with Aslak Buan (Trondheim and Leicester), Markus Reineke (Münster), Idun Reiten (Trondheim) and Gordana Todorov (Northeastern)

We introduce a new category, which we call the cluster category, obtained as a quotient of the bounded derived category of the module category of a finite-dimensional hereditary algebra H over a field. If H is the path algebra of a simply-laced Dynkin quiver, this category can be regarded as a natural model for the combinatorics of the corresponding Fomin–Zelevinsky cluster algebra. Such algebras have been defined in order to study the dual canonical basis of a quantum group and total positivity in algebraic groups, and have many other interpretations.

As well as giving insight into tilting theory for finite-dimensional algebras and their derived categories, the cluster category admits the generalisation of Auslander–Platzeck–Reiten tilting (so that it is possible to tilt at arbitrary vertices of the quiver).

Sergei Merkulov (Stockholm University)
Infinity constructions of local geometries

We argue that some classical local geometries are of infinity origin, i.e., their smooth formal germs are (homotopy) representations of cofibrant (di)operads in spaces concentrated in degree zero. In particular, they admit natural infinity generalizations when one considers homotopy representations of that (di)operads in generic differential graded spaces. Poisson and Nijenhuis geometries provide us with simplest manifestations of this phenomenon.

Götz Pfeiffer (National University of Ireland, Galway)
Subgroups and cosets in finite Coxeter groups

We present some new quantitative results on the subgroup structure and the Burnside ring of some finite groups. In their complete form, the tables of subgroups of some almost simple groups and some irreducible finite Coxeter groups are made available at a new web site `http://schmidt.nuigalway.ie/subgroups`. In the case of a finite Coxeter group W we use parabolic subgroups to pass from the Burnside ring of W to Solomon's descent algebra of W. There we discuss some recent results in the representation theory of this descent algebra and formulate as a conjecture a generalization of the Gessel–Reutenauer formula for the number of permutations with a given cycle shape and descent set.

Sandra Pott (University of York)
Admissibility of observation operators and vector BMO functions

One fruitful approach to the study of linear systems specified by partial differential equations is by regarding them as abstract ordinary differential equations on an infinite-dimensional Hilbert space. This enables operator-theoretic methods to be employed, such as the theory of strongly continuous semigroups. The notion of *admissibility* of control and observation operators is crucial in this context. Consider the equations

$$\begin{aligned} \dot{x}(t) &= Ax(t), \qquad t \geq 0, \\ y(t) &= Cx(t). \end{aligned} \tag{1}$$

Here $x(t) \in H$, where H is a Hilbert space, is the state of the system at time $t \geq 0$ and $y \in L^2(0, \infty; Y)$ is the output of the system, where Y is another Hilbert space. Both A and C are possibly unbounded operators. We assume that A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on H, and we assume C to be a bounded linear operator from $(D(A), \|.\|_g)$ to Y, where $D(A)$, the domain of A, is equipped with the graph norm. C will be called an *observation operator for* $(T(t))_{t\geq 0}$.

By a solution of (1) with initial condition $x(0) = x_0 \in H$ we then mean the continuous function $x(t) = T(t)x_0$, $t \geq 0$. These assumptions are not sufficient to guarantee that the output of the system is in $L^2(0, \infty; Y)$. This gives rise to the following definition:

Let $C \in \mathcal{L}(D(A), Y)$. Then C is called an *(infinite-time) admissible observation operator for* $(T(t))_{t\geq 0}$, if the map

$$D(A) \to L^2((0,\infty);Y), \qquad x \mapsto CT(\cdot)x,$$

extends to a bounded linear map $\Delta_C : H \to L^2((0,\infty);Y)$.

Admissibility easily implies the following resolvent condition, which is often easier to test. We write $\mathbb{C}_+ = \{s \in \mathbb{C} : \Re s > 0\}$.

(A1) There exists a constant $m > 0$ such that

$$\|C(sI - A)^{-1}x\| \leq \frac{m}{\sqrt{\Re s}}\|x\|, \qquad x \in H,\ s \in \mathbb{C}_+.$$

It was conjectured by G. Weiss that the converse implication would hold. This can be seen as an example of the so-called *reproducing kernel thesis*, namely the question whether an operator (of a certain type) is bounded on a reproducing kernel Hilbert space, if and only if it is bounded on the reproducing kernels.

In the case of a contractive semigroup $(T(t))_{t\geq 0}$, the setting above can, via Nagy–Foias model theory, essentially be reduced to the case of the right shift semigroup, where $H = L^2((0,\infty);K)$, K Hilbert space, and $T_t x = x(\cdot - t)$ for $x \in L^2((0,\infty);K)$, $t \geq 0$.

The operator Δ_C then turns into a (vector) Hankel operator. The resolvent condition (A1) says that the (vector) symbol of this Hankel operator has to belong to a certain space of functions of bounded mean oscillation (BMO).

In the case where Y or K are infinite-dimensional Hilbert spaces, the study of admissibility of C thus leads into the theory of vector BMO functions. We will give examples of situations where the Weiss conjecture holds and of situations where it fails, and we will discuss ways of strengthening the resolvent condition (A1) in such a way that it ensures admissibility of C.

Mary Rees (University of Liverpool)
Dynamics and Automatic Structure of Groups

There are several connections between dynamics and automatic group structure. The simplest connection is through subshifts of finite type. Automatic structure is determined by a finite state automaton, which itself determines a subshift of finite type. The seemingly simple question, of what sort of dynamical subshifts arise with automatic structures of different groups, may be somewhat unexplored. Other connections with dynamics arise in examples. The sort of groups which have automatic structure—free groups, groups of hyperbolic isometries, mapping class groups—themselves have various connections with dynamical systems. They also, of course, have connections with geometry. This sort of geometry and dynamics have always been closely linked.

I shall concentrate on the examples of mapping class groups of finite type surfaces, which were shown by Mosher to have an automatic structure. I shall indicate why the surface of genus two has a biautomatic structure. The proof uses geometry and, in the fine detail, a family of dynamical systems associated to Teichmüller space. I shall also say why I think this is very unlikely to be true for a surface of genus three, and even for a certain subgroup of this mapping class group which is simply a finite extension of $F_2 \times F_2$.

Alastair Wood (CMDE, Dublin City University)
The influence of G. G. Stokes on the modern asymptotic theory of differential equations

In common with many incumbents of the Lucasian Chair of Mathematics in Cambridge, Sir G.G. Stokes, who held it from 1849 to 1903, was primarily a mathematical physicist. Only 7 of the 109 papers in his five-volume Collected Mathematical and Physical Papers are on mathematical analysis as such. It may therefore seem strange to find Stokes celebrated at a BMC (although it is appropriate that he be recognised in Belfast), but the present day ramifications of his mathematical papers, especially those of 1857 and 1864 on asymptotic analysis, are out of all proportion to their relatively small number.

In Stokes phenomenon, an asymptotic expansion of a function of a complex variable can change its form near certain rays in the plane (now known as Stokes lines) through the apparently discontinuous appearance of a further series with an exponentially small prefactor and multiplicative constant (the Stokes multiplier). Although small at the place of its birth, this new term can grow to significantly influence the behaviour of the function in other regions.

Because of the absence of exponentially small terms in the subsequently adopted asymptotic definition of Poincaré (1886), Stokes phenomenon caused controversy, ambiguity and misunderstanding for over a century, until the physicist M. V. Berry published his insightful paper in 1989. His idea has been developed, using rigorous mathematical analysis, by C. J. Howls, A. B. Olde Daalhuis, F. W. J. Olver and R. B. Paris among others, to provide a firm understanding of the phenomenon.

We review the way in which these later authors have made use of two of Stokes's key ideas: the optimal truncation of the dominant series (that is, just before its least term) to give an exponentially small remainder; and secondly the resummation of the divergent tail of the expansion to improve its computational accuracy. We conclude with an outline of a recently established connection between anti-Stokes lines on the real axis and the absolutely continuous spectrum of differential operators (D. G. Gilbert and A. D. Wood, 2004).