The Decomposition of Lie Derivations

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Abstract. A linear map $D\colon \mathcal{A} \to E$, where $\mathcal{A}$ is an algebra and E an $\mathcal{A}$-bimodule, is a *Lie derivation* if $D([a,b]) = [a, D(b)] + [D(a), b]$ for every $a, b \in \mathcal{A}$. Two examples are a derivation and a centre-valued trace; there are several theorems asserting that, in certain cases, a Lie derivation from $\mathcal{A}$ to E can be written (in "standard form") as a sum $D = d + \tau$, where d is a derivation and τ is a centre-valued trace. These go back to a ring-theoretic theorem of M. Bresar [1]. In the context of Banach algebras, one has a positive result by Miers [5] in the case when $D\colon \mathcal{M} \to \mathcal{M}$ where $\mathcal{M}$ is a von Neumann algebra. M. Mathieu and A. R. Villena [6], [7] have generalised this to a full, positive result in the case of general C^*-algebras. Questions of automatic continuity also arise in the Banach algebra setting, and positive results are achieved by Berenguer and Villena ([2], [3]) when $D\colon \mathcal{A} \to \mathcal{A}$ where $\mathcal{A}$ is a semisimple Banach algebra. In this paper, we start at the other end of this line of enquiry and give some examples of Lie derivations on Banach algebras that cannot be written in standard form. In a related result, we give some examples of discontinuous derivations from semisimple Banach algebras $\mathcal{A}$, such that the separating subspace of the derivation does not lie in the centre of the bimodule.

1. Indecomposable, Continuous Lie Derivations

Definition 1. A Lie derivation D will be said to be *standard* if it can be written as a sum $D = d + \tau$ as in the Abstract; we do not require d or τ to be continuous. Otherwise, D is *nonstandard*.

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In this section we shall exhibit some continuous, nonstandard Lie derivations on Banach algebras. In the radical case the underlying algebra can be finite dimensional, but not in the semisimple case. In both cases, the infinite dimensional version of the example can be varied so that it becomes discontinuous.

Before considering Banach algebras, let us first consider a graded algebra $\mathcal{A} = \bigoplus_{n=0}^{\infty} \mathcal{A}^{(n)}$ (i.e., every $x \in \mathcal{A}$ is uniquely written as a finite sum $\sum_{n=0}^{N} x^{(n)}$, $x^{(n)} \in \mathcal{A}^{(n)}$, and $\mathcal{A}^{(n)} \cdot \mathcal{A}^{(m)} \subset \mathcal{A}^{(n+m)}$). It is a fact that the map $d : \mathcal{A} \to \mathcal{A}$ with $d(\sum_{n=0}^{N} x^{(n)}) = \sum_{n=0}^{N} nx^{(n)}$ is a derivation. In the normed algebra context, it is never continuous unless the subspaces $\mathcal{A}^{(n)}$ are eventually zero — which would mean that, assuming $\mathcal{A}^{(0)} = \mathbb{C}$, $\mathcal{A}$ would necessarily be a nilpotent algebra with unit adjoined. But though the sequence of subspaces must still terminate, the conclusion that $\mathcal{A}$ must be nilpotent is no longer true if we make a similar definition with *Lie* derivations in mind.

Definition 2. A *Lie graded algebra* is an algebra $\mathcal{A} = \bigoplus_{n=0}^{\infty} \mathcal{A}^{(p)}n$, such that every $x \in \mathcal{A}$ is uniquely written as a finite sum $\sum_{n=0}^{N} x^{(p)}n$, $x^{(p)}n \in \mathcal{A}^{(p)}n$, and the commutators $[\mathcal{A}^{(p)}n, \mathcal{A}^{(p)}m] \subset \mathcal{A}^{[n+m]}$ for every n and m. A *Lie graded Banach algebra* is the completion of a Lie graded algebra $\mathcal{A}$ equipped with an algebra norm such that the subspaces $\mathcal{A}^{[n]}$ are closed.

Let's give an example of a Lie graded algebra that is not (with the same grading) a graded algebra. Let $\mathcal{F}$ be the free unital complex algebra on three generators a, b and c. Let $I_1 \subset \mathcal{F}$ be the 2-sided ideal generated by the commutators $[\mathcal{F}, \mathcal{F}]$, and let $I_2 \subset I_1$ be the 2-sided ideal generated by the commutators $[\mathcal{F}, I_1]$. Let $\mathcal{A} = \mathcal{F}/I_2$, so there are natural quotient maps $q_2 : \mathcal{F} \to \mathcal{A}$ and $q_1 : \mathcal{A} \to \mathcal{F}/I_1$.

Now $\mathcal{F}$ is a graded algebra; $\mathcal{F} = \bigoplus_{n=1}^{\infty} \mathcal{F}^{(n)}$, where $\mathcal{F}^{(0)} = \mathbb{C}$, and for $n > 0$, $\mathcal{F}^n$ is the finite dimensional linear subspace of $\mathcal{F}$ spanned by the "words" of length n in the three generators a, b and c. In view of the fact that $[\mathcal{F}^r, \mathcal{F}^s] \subset \mathcal{F}^{r+s}$ it is plain that I_1 is a graded ideal, $I_1 = \bigoplus_{n=1}^{\infty} {I_1}^n$ where $I_1^{(n)} = \mathcal{F}^{(n)} \cap I_1$. Then $[\mathcal{F}^{(r)}, I_1^{(s)}] \subset \mathcal{F}^{(r+s)} \cap I_1 = I_1^{(r+s)}$, so I_2 also is graded, $I_2 = \bigoplus_{n=1}^{\infty} I_2^{(n)}$ where $I_2^{(n)} = \mathcal{F}^{(n)} \cap I_2$. The quotient maps q_2 and q_1 are homomorphisms of graded algebras.

There is a natural inner product on $\mathcal{F}$ such that, if w, w' are "words" in the three generators, we have

$$< w, w' >= \begin{cases} \frac{1}{n!}, & \text{if } w = w' \in \mathcal{F}^{(n)}; \\ 0, & \text{otherwise.} \end{cases}$$

The presence of the factors $\frac{1}{n!}$ ensures that the Euclidean norm $\| \cdot \|_2$ associated with this inner product is an algebra norm, and the decomposition $\mathcal{F} = \bigoplus_{n=1}^{\infty} \mathcal{F}^n$ is of course orthogonal. This is the norm we shall use on $\mathcal{F}$ when we want to get a radical Banach algebra with unit adjoined, indeed, a radical operator algebra.

Now every graded subspace $\mathcal{M} \subset \mathcal{F}$ such as I_1 or I_2 is closed in $(\mathcal{F}, \| \cdot \|_2)$ (after all, the grading is an orthogonal direct sum and the components of the grading are finite dimensional), and there is an orthogonal projection $P : \mathcal{F} \to \mathcal{M}$ because it is none other than $\sum_{i=1}^{\infty} P_i$, where P_i is the orthogonal projection $\mathcal{F}^{(i)} \to \mathcal{M}^{(i)}$. So when we need a continuous projection $P : \mathcal{A} \to I_1/I_2 = \ker q_1$, plainly we can use the orthogonal projection $P : \mathcal{F} \to I_1$.

But when we want a semisimple example we shall go for the l_1 norm on $\mathcal{F}$, that is if $x = \sum_{i=1}^{n} \lambda_i w_i \in \mathcal{F}$ where the w_i are distinct words in the three generators, we define

$$\| x \|_1 = \sum -i = 1^n |\lambda_i|.$$

In this case it is useful to us that in fact P is $\| \cdot \|_1$-continuous as well. The reason is as follows.

If $w = \alpha_1 \alpha_2 \ldots \alpha_n$ is a word of length n (each $\alpha_i \in \{a, b, c\}$), and $\pi \in S_n$, the symmetric group on n elements, then we write $w^\pi = \alpha_{\pi(1)} \alpha_{\pi(2)} \ldots \alpha_{\pi(n)}$; and plainly each difference $w - w^\pi$ lies in the commutator ideal I_1, whereas the average $\frac{1}{n!} \sum_{\pi \in S_n} w^\pi$ is orthogonal to $u_1[u_2, u_3]u_4$ for any 4 words u_1, u_2, u_3 and u_4 (including the trivial word 1), and so is orthogonal to I_1. Thus the orthogonal projection $P : \mathcal{F} \to I_1$ sends each $\| \cdot \|_1$ unit vector w to $w - \frac{1}{n!} \sum_{\pi \in S_n} w^\pi$; hence $\| I - P \|_1 = 1$; so the projection $P : \mathcal{A} = \mathcal{F}/I_2 \to I_1/I_2$ also satisfies $\| I - P \|_1 = 1$. With respect to either norm, we have a decomposition

$$\mathcal{A} = \operatorname{Ker} P \oplus \operatorname{Im} P$$

with a continuous projection P. Writing $\mathcal{A}^{[1]} = \ker P$ and $\mathcal{A}^{[2]} = \operatorname{Im} P$, the decomposition $\mathcal{A} = \mathcal{A}^{[1]} \oplus \mathcal{A}^{[2]}$ is a Lie grading, because if $x, y \in \mathcal{A}^{[1]}$ then $[x, y]$, being a commutator, lies in $\mathcal{A}^{[2]}$, while if $x \in \mathcal{A} = \mathcal{F}/I_2$ and $y \in \mathcal{A}^{[2]} = I_1/I_2$ then $[x, y] \in [\mathcal{F}, I_1]/I_2 =$

$I_2/I_2 = (0)$. So there is a *Lie derivation* $D : \mathcal{A} \to \mathcal{A}$, $D(a^{[1]} + a^{[2]}) = a^{[1]} + 2a^{[2]}$ for any $a^{[i]} \in \mathcal{A}^{[i]}$. D extends continuously to a Lie derivation on the completion of either normed algebra $(\mathcal{A}, \|\cdot\|_1)$ or $(\mathcal{A}, \|\cdot\|_2)$. We claim that in neither case can D be decomposed (continuously or otherwise) into a sum $d + \tau$, with d a derivation and τ a centre-valued trace.

Let's write $\mathcal{A}_i$ for the completion of $(\mathcal{A}, \|\cdot\|_i)$ $(i = 1, 2)$. Now the subspaces $\mathcal{A}^{(n)}$ are finite-dimensional and mutually orthogonal, and with respect to the $\|\cdot\|_1$ norm the natural projection P_n onto $\bigoplus_{i=0}^n \mathcal{A}^{(n)}$ has norm 1; so for any finite-dimensional subspace U of the original, incomplete normed algebra $\mathcal{F}$, the orthogonal projection from $\mathcal{A}$ onto $q_2(U)$ will be continuous with respect to the $\|\cdot\|_1$ norm as well as the Euclidean norm $\|\cdot\|_2$ (for one obtains it by composing P_N, for suitably large N, with a necessarily continuous operator on the finite-dimensional normed space $\operatorname{Im} P_N$). Let us abuse notation slightly by suppressing the quotient map q_2 and regarding a, b etc. as elements of $\mathcal{A}$. With respect to either norm, if $D = d + \tau : \mathcal{A}_i \to \mathcal{A}_i$ we may write

$$d(a) = \lambda a + \alpha,\ \ d(b) = \mu b + \beta, \tag{1}$$

where $\alpha \perp a$ and $\beta \perp b$. Now $\tau(a) = (1 - \lambda)a - \alpha$, hence $\tau(a)b = (1 - \lambda)ab - \alpha b$ (with $\alpha b \perp ab, ba$); and $b\tau(a) = (1 - \lambda)ba - b\alpha$ with $b\alpha \perp ab, ba$. Therefore

$$[\tau(a), b] = (1 - \lambda)[a, b] + (b\alpha - \alpha b), \tag{2}$$

with $b\alpha - \alpha b$ orthogonal to $\operatorname{lin}(ab, ba)$. Now any "double commutator" $x \in [\mathcal{F}, I_1]$ is a linear combination of words of length at least 3, so in our original grading we have $I_2^{(2)} = (0)$. It follows that even in $\mathcal{A} = \mathcal{F}/I_2$, we have $\|\,[\tau(a), b]\,\|_i \geq 2|1 - \lambda|$ (if $i = 1$), or $\sqrt{2}|1 - \lambda|$ (if $i = 2$). Since τ is by hypothesis centre-valued, it follows that $\lambda = 1$, and similarly $\mu = 1$. Given $d(a) = a + \alpha$ and $d(b) = b + \beta$, we obtain

$$d(ab) = (a + \alpha)b + a(b + \beta) = 2ab + \alpha b + a\beta, \tag{3}$$

where $a\beta + \alpha b$ will be orthogonal to both ab and ba.

Now the decomposition of ab into $\mathcal{A}^{[1]} \oplus \mathcal{A}^{[2]}$ is $(I - P)(ab) + P(ab) = \frac{ab+ba}{2} + \frac{ab-ba}{2} + I_2$. Therefore $D(ab) = \frac{ab+ba}{2} + 2(\frac{ab-ba}{2}) = 3ab/2 - ba/2 + I_2$. Accordingly $\tau(ab) = D(ab) - d(ab) = \frac{-ab-ba}{2} - \alpha b - a\beta$, where both αb and $a\beta$ will be orthogonal to $\operatorname{lin}(ab, ba)$. Then

$$[\tau(ab), c] = \frac{cab + cba - abc - bac}{2} + [c, \alpha b + a\beta], \tag{4}$$

where $[c, \alpha b + a\beta]$ will be orthogonal to every permutation $(abc)^\pi$ of abc. Writing $U = \mathrm{lin}\{(abc)^\pi : \pi \in S_3\}$, we recall that I_1 is the linear span of vectors $u_1[u_2, u_3]u_4$ for all words u_i; so I_2 is the linear span of vectors $x = v_1[v_2, u_1[u_2, u_3]u_4]v_3$. The only way such an expression can involve words of length 3 is if v_1, u_1, u_4 and v_3 are all 1, so $x = [v_2, [u_2, u_3]]$; and to get an answer not orthogonal to U the words v_2, u_2, u_3 must be a, b and c in some order. In that case, the vector x lies in U itself; so $I_2 = (I_2 \cap U) \oplus (I_2 \cap U^\perp)$, with $I_2 \cap U = \mathrm{lin}([a, [b, c]], [b, [c, a]]) = \mathrm{lin}(abc - acb - bca + cba, bca - bac - cab + acb)$, which is orthogonal to $\frac{cab+cba-abc-bac}{2}$. So if Q denotes the (continuous!) orthogonal projection from $\mathcal{A}$ onto $q_2(U)$, we find that $Q[\tau(ab), c] = \frac{cab+cba-abc-bac}{2} + I_2$, a nonzero result. This contradicts the assumption that τ is centre-valued, so the Lie derivation D is indeed nonstandard.

Now our argument also shows that the distance of the linear operator $D : \mathcal{A} \to \mathcal{A}$ from the set of all operators $d + \tau$, d a derivation and τ a centre-valued trace, is strictly positive. For if λ and μ are the constants in (1), the equation $\| D - d - \tau \|_i < \varepsilon$ implies $|1 - \lambda| < 2\varepsilon$ and $|1 - \mu| < 2\varepsilon$ because of (2); Equation (3) then becomes $d(ab) = \nu ab + \alpha b + a\beta$, where $|2 - \nu| < 4\varepsilon$ and $a\beta + \alpha b$ will be orthogonal to both ab and ba; using $\| D - d - \tau \| < \varepsilon$ again, we get

$$\left\| [\tau(ab), c] - \frac{\nu(cab + cba - abc - bac)}{2} - [c, \alpha b + a\beta] \right\| < 2\varepsilon \qquad (5)$$

by analogy with (4); and this is a contradiction for small $\varepsilon > 0$.

So the abstract algebra result that not every Lie derivation is standard extends in the fullest possible way to the continuous situation where we have a Banach algebra involved. One can form the Banach space of continuous Lie derivations on $\mathcal{A}_i$, and one can consider the subspace of standard continuous Lie derivations on $\mathcal{A}_i$, where the decomposition may or may not involve continuous d and τ; but even then, relaxing the conditions as much as we can, our subspace is not dense.

Our example $\mathcal{A}_2$ is clearly radical with adjoined unit; the example $\mathcal{A}_1$ is quotient of the semisimple Banach algebra $l_1(S)$, S the free semigroup on 3 generators, and we can define our Lie derivation as a map $D_1 : l_1(S) \to \mathcal{A}_1$, thus obtaining an nonstandard Lie derivation from a semisimple Banach algebra to a bimodule. But it is not quite obvious to the author that $\mathcal{A}_1$ is itself semisimple, though it clearly isn't radical.

2. Discontinuous Examples and Other Variants

One may now vary the Lie grading of $\mathcal{A}_i$ so as to obtain a discontinuous such example, as follows. The original Lie derivation took an element $x \in \mathcal{A}_i$ and decomposed it $x = x^{[1]} + x^{[2]}$, where $x^{[j]}$ lies in the closure $\overline{\mathcal{A}^{[j]}}$ of $\mathcal{A}^{[j]}$ in the completion $\mathcal{A}_i$ of $(\mathcal{A}, \|\cdot\|_i)$. Let us now pick (using the axiom of choice!) an alternative complement $\mathcal{B}^{[1]}$ for $\overline{\mathcal{A}^{[2]}}$ in $\mathcal{A}_i$, so that the projection onto $\overline{\mathcal{A}^{[2]}}$ parallel to $\mathcal{B}^{[1]}$ is no longer continuous. If we decompose x as $x_1 + x_2$ with respect to this Lie grading, we may define $D_2(x) = x_1 + 2x_2$ (the Lie derivation associated with the grading), and D_2 is still nonstandard for similar reasons to the above, but no longer continuous. But because $\overline{\mathcal{A}^{[2]}}$ is closed, the separating subspace of D_2 will be contained in $\overline{\mathcal{A}^{[2]}}$, which is contained in the centre of $\mathcal{A}_i$. We can only perturb our example by a discontinuous, centre-valued trace by this method.

Our example can be made finite dimensional by quotienting $\mathcal{A}$ out further by the ideal $K = \bigoplus_{n=4}^{\infty} \mathcal{A}^{(n)}$. Because our quotient maps all respect the original grading $\mathcal{A} = \oplus \mathcal{A}^{(n)}$, the subspace of words of length 3 or less is untouched by this process; so our argument that the Lie derivation $D : \mathcal{A}/K \to \mathcal{A}/K$ is nonstandard remains valid. That example is a finite dimensional nilpotent algebra with unit adjoined.

3. The Separating Subspace of a Derivation

Now the study of derivations is full of beautiful automatic continuity results. It is of interest whether a Lie derivation has some of the same behaviour; or whether it may fail to be continuous in such a way as to show it is very far from being standard. However, a famous theorem of Johnson and Sinclair asserts that a derivation $d : \mathcal{A} \to \mathcal{A}$ is automatically continuous if $\mathcal{A}$ is a semisimple Banach algebra. This result has been generalised by Berenguer and Villena [2] in the following way: if $\mathcal{A}$ is a semisimple Banach algebra and $D : \mathcal{A} \to \mathcal{A}$ is a Lie derivation, then the separating subspace of D must be contained in the centre of $\mathcal{A}$. Such a map clearly differs from one in standard form only by a continuous perturbation.

Problem 3. *Thus it would be nice to know if one can have any Lie derivation D from a Banach algebra to itself, such that D remains nonstandard even after continuous perturbation (if D' is another Lie derivation such that $D - D'$ is continuous, then D' is nonstandard). As mentioned above the Banach algebra involved cannot be semisimple.*

In this section we give a counterexample to a more optimistic hypothesis, namely that *any* Lie derivation from a semisimple Banach algebra will be nonstandard, provided the separating subspace doesn't lie inside the centre. The reason why this is false is simple; the assertion that derivations are automatically continuous is false for a general derivation $d : \mathcal{A} \to E$ into a bimodule, even if the algebra $\mathcal{A}$ is semisimple. In fact one can have discontinuous derivations $d : \mathcal{A} \to E$ whose separating subspace does not lie inside the centre of the bimodule E; here is an example.

Perhaps the simplest case of a derivation that does not map into the centre of the bimodule is as follows; let $\mathcal{A}$ be the algebra of upper triangular 2×2 matrices over $\mathbb{C}$; and let $E \subset \mathcal{A}$ be its radical, namely the "strictly upper triangular matrices" $E = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in \mathbb{C} \right\}$. $\mathcal{A}$ acts on E in the obvious way, but E is not a commutative bimodule because e.g. $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. So the centre of the $\mathcal{A}$-bimodule E is (0), and $d : \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \to \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$ is a nonzero derivation into E.

We can make the above into an infinite dimensional example in either of the following two ways. Method 1 is as follows:

If E, F are two Banach spaces let's write $\mathcal{B}(E, F)$ for the space of bounded linear maps from E to F; and $\mathcal{K}(E, F)$ for the space of compact operators from E to F; and $\mathcal{W}(E, F)$ for the space of weakly compact operators. When $E = F$ we will write $\mathcal{B}(E)$, $\mathcal{K}(E)$ and $\mathcal{W}(E)$ respectively. Now for $1 \le p < q < \infty$, it is well known that $\mathcal{B}(l^q, l^p) = \mathcal{K}(l^q, l^p)$ (every bounded linear map from l^q to l^p is compact). So let $\mathcal{A}_0 = \mathcal{B}(l^q \oplus l^p)$ and let π be the quotient map from $\mathcal{A}_0$ to $\mathcal{A}_1 = \mathcal{B}(l^q \oplus l^p)/\mathcal{K}(l^q \oplus l^p)$. Every $T \in \mathcal{A}_1$ is naturally represented as a 2 by 2 matrix with operator entries, $T = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}$ with $X \in \mathcal{B}(l^q)/\mathcal{K}(l^q)$, $Y \in \mathcal{B}(l^p, l^q)/\mathcal{K}(l^p, l^q)$ and $Z \in \mathcal{B}(l^p)/\mathcal{K}(l^p)$.

The ideal $E \subset \mathcal{A}_1$ with $E = \{\begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix} : Y \in \mathcal{B}(l^p, l^q)/\mathcal{K}(l^p, l^q)\}$ is a Banach $\mathcal{A}_1$-bimodule, and indeed a Banach $\mathcal{A}_0$-bimodule via the quotient map π. We define a map $d_0 : \mathcal{A}_0 \to E$ with $d_0(a) = \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix}$, where $\pi(a) = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}$; d_0 is a continuous derivation. One can make d_0 discontinuous, and have the separating subspace not in the centre, in the following way. Let $\mathcal{U} \subset \mathcal{A}_1$ be the set of all operators represented as $T = \begin{pmatrix} \lambda I_q & Y \\ 0 & \mu I_p \end{pmatrix}$, with $Y \in \mathcal{B}(l^p, l^q)/\mathcal{K}(l^p, l^q)$, and I_p and I_q the identities of $\mathcal{B}(l^p)/\mathcal{K}(l^p)$ and $\mathcal{B}(l^q)/\mathcal{K}(l^q)$ respectively. Let $\mathcal{A} = \pi^{-1}(U) \subset \mathcal{A}_0$. Now though $\mathcal{A}$ is not the whole of $\mathcal{B}(l^q \oplus l^p)$, it is nonetheless a subalgebra containing the finite rank operators — so $\mathcal{A}$ is semisimple (every Banach operator algebra is semisimple).

To introduce some discontinuity, let

$$\phi : \mathcal{B}(l^p, l^q)/\mathcal{K}(l^p, l^q) \to \mathcal{B}(l^p, l^q)/\mathcal{K}(l^p, l^q)$$

be *any* discontinuous linear map; and let $d : \mathcal{A} \to E$ be the map with $d(a) = \begin{pmatrix} 0 & \phi(Y) \\ 0 & 0 \end{pmatrix}$, where $\pi(a) = \begin{pmatrix} \lambda I_q & Y \\ 0 & \mu I_p \end{pmatrix}$. It is easily checked that d is a *derivation*, for if $a' \in \mathcal{A}$ with $\pi(a') = \begin{pmatrix} \lambda' I_q & Y' \\ 0 & \mu' I_p \end{pmatrix}$ then

$$\pi(aa') = \begin{pmatrix} \lambda\lambda' I_q & \lambda Y' + \mu' Y \\ 0 & \mu\mu' I_p \end{pmatrix},$$

$$d(aa') = \begin{pmatrix} 0 & \lambda\phi(Y') + \mu'\phi(Y) \\ 0 & 0 \end{pmatrix} = a \cdot d(a') + d(a) \cdot a'.$$

Thus we obtain discontinuous derivations from the semisimple Banach algebra $\mathcal{A}$, into a Banach $\mathcal{A}$-bimodule E whose centre is zero. It is an interesting special case when the rank of the discontinuous linear map ϕ is equal to 1; in that case, we obtain discontinuous, noncommutative point derivations (i.e., discontinuous derivations into $\mathbb{C}$, where the left action of $\mathcal{A}$ on $\mathbb{C}$ is not the same as the right action — there are two different characters involved, given by the parameters λ and μ involved in $\pi(a)$ above).

For Method 2, we observe that exactly the same idea can be followed with algebras $\mathcal{B}(X)/\mathcal{W}(X)$, provided we choose the underlying Banach space X appropriately. One choice is to use the James p-spaces J_p (for a definition see e.g. Singer [9] or Read [8]); for if

$1 \leq p < q < \infty$ then $\mathcal{B}(J_q, J_p) = \mathcal{W}(J_q, J_p)$ but not the other way around.

Now the quotient space $\mathcal{B}(J_q)/\mathcal{W}(J_q)$ is famously 1-dimensional, so for our purposes here we let J_p^∞ denote the l^2-direct sum of countably many copies of J_p, and we define a new Banach algebra $\mathcal{A}_1 = \mathcal{B}(J_q^\infty \oplus J_p^\infty)/\mathcal{W}(J_q^\infty \oplus J_p^\infty)$, whose elements are represented $T = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}$ with $X \in \mathcal{B}(J_q^\infty)/\mathcal{W}(J_q^\infty)$, $Y \in \mathcal{B}(J_p^\infty, J_q^\infty)/\mathcal{W}(J_p^\infty, J_q^\infty)$ and $Z \in \mathcal{B}(J_p^\infty)/\mathcal{W}(J_p^\infty)$. If π is the quotient map $\mathcal{B}(J_q^\infty \oplus J_p^\infty) \to \mathcal{A}_1$ we take $\mathcal{A} = \pi^{-1}\{\begin{pmatrix} \lambda I_q & Y \\ 0 & \mu I_p \end{pmatrix}\}$, where I_p and I_q are the identities of $\mathcal{B}(J_p^\infty)/\mathcal{W}(J_p^\infty)$ and $\mathcal{B}(J_q^\infty)/\mathcal{W}(J_q^\infty)$; and one can obtain discontinuous derivations into the bimodule $E = \{\begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix} : Y \in \mathcal{B}(J_p^\infty, J_q^\infty)/\mathcal{W}(J_p^\infty, J_q^\infty)\}$ by the same method as above.

So one can obtain derivations whose separating subspace does not lie in the centre of the bimodule very easily, by arranging that though the algebra involved is semisimple, it has quotient spaces with infinite dimensional radicals; in the case $\mathcal{A} = \mathcal{B}(X)$, one can quotient out by $\mathcal{K}(X)$ or $\mathcal{W}(X)$ for this purpose, provided X is chosen appropriately.

One final problem arises from our study of Lie graded algebras; in our examples the grading tends to terminate (later $\mathcal{A}^{[n]}$ are all zero) after $n = 2$.

Problem 4. *Give further nontrivial examples of Lie graded Banach algebras (where we consider the example trivial if it is a graded Banach algebra with the same grading). Given such an example with all the subspaces $\mathcal{A}^{[n]}$ nonzero, is there a (necessarily discontinuous) Lie derivation D on $\mathcal{A}$ such that $D|_{\mathcal{A}^{[n]}}$ is equal to n times the identity, for each n?*

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