Old and New on the Bass Note, the Torsion Function and the Hyperbolic Metric

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1. The torsion function and Brownian motion

A cylindrical beam of uniform cross section is subject to an infinitesimal torsion. One needs to know the resulting stresses. The theory of elasticity reduces this to the solution of the problem

$$\begin{cases} \Delta u = -2 & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases} \tag{0.1}$$

if the beam in three dimensional Cartesian coordinates is $D \times \mathbf{R}$ where the cross section D is a simply connected domain in the xy-plane. The partial derivatives u_y and $-u_x$ give the components in the x and y directions respectively of the stress vector relative to the normal direction to D .

The torsional rigidity of the beam is

$$P \stackrel{\text{def}}{=} \int_D |\operatorname{grad} u|^2 = 2\int_D u,$$

where to obtain the second equality we used Green's theorem. It is the torque required per unit angle of twist per unit length of beam and so is a measure of the resistance of the beam to torsion. A famous problem raised by St. Venant (1856) and solved by Pólya (1948) was to show that among all simply connected domains of given area, a disk of that area has the greatest torsional rigidity (see Pólya and Szegö [18, Page 121]).

This is a good example of the isoperimetric-type problems that are the subject of this survey. The classical isoperimetric inequality is

$$4\pi A \le L^2,$$

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for the area A enclosed by a perimeter of length L. Equality holds only in the case of a disk bounded by a circle. Whereas the prototypical isoperimetric inequality relates two geometric quantities, area and length, those under consideration here relate an analytic quantity (the torsional rigidity, for example) and a geometric quantity (the area, for example).

There is an important probabilistic interpretation of the torsion function, the solution to (0.1). A standard Brownian motion in the plane departs from a point in D and runs until it exits D at a time τ_D that depends on the path. This is a stochastic process in D whose transition probabilities are denoted by $p_D(t, \mathbf{x}, \mathbf{y})$, so that the probability that the process that initially departs from $\mathbf{x}$ lies in the Borel subset A of D at time t is

$$\int_A p_D(t, \mathbf{x}, \mathbf{y})\, d\mathbf{y}.$$

These transition probabilities are the fundamental solutions of the heat equation in D - the heat kernel for D - they satisfy

$$\frac{1}{2}\Delta_y\, p_D(t, \mathbf{x}, \mathbf{y}) = \frac{\partial}{\partial t} p_D(t, \mathbf{x}, \mathbf{y}).$$

The calorific interpretation is that of a plate with shape D, its boundary maintained at zero temperature and one unit of heat put at $\mathbf{x}$ at time $t = 0$: the resulting heat density at the point $\mathbf{y}$ at time t is $p_D(t, \mathbf{x}, \mathbf{y})$. The connection between this heat problem and the Brownian motion in D is, in a sense, obvious – each is a diffusion of a concentration at $\mathbf{x}$ at time 0 that is absorbed on reaching the boundary of the domain.

The exit time τ_D of the diffusion depends on the particular Brownian path and as such it is a random variable, a measurable function, on path space. We can therefore take the expectation, or integral, relative to Wiener measure $\mathrm{P}_{\mathbf{x}}$ on path space and we denote this by $E_{\mathbf{x}}\tau_D$. Taking horizontal approximating rectangles to the area under the graph of $\mathrm{P}_{\mathbf{x}}(\tau_D > t)$ as a function of t on $[0, \infty)$ makes it clear that

$$E_{\mathbf{x}}\tau_D = \int_0^\infty \mathrm{P}_{\mathbf{x}}(\tau_D > t)\, dt,$$

(see Lieb and Loss [13, Theorem 1.13]). Now $\mathrm{P}_{\mathbf{x}}(\tau_D > t)$ is the probability that a Brownian motion that departs from $\mathbf{x}$ has not, at time t, already been absorbed at the boundary of the domain. Thus it is the probability that this Brownian motion is still in D at time t and therefore equals $\int_D p_D(t,\mathbf{x},\mathbf{y})\,d\mathbf{y}$. This gives

$$E_{\mathbf{x}}\tau_D = \int_D \left(\int_0^\infty p_D(t,\mathbf{x},\mathbf{y})\,dt \right) d\mathbf{y}.$$

We notice that

$$f(\mathbf{x},\mathbf{y}) = \int_0^\infty p_D(t,\mathbf{x},\mathbf{y})\,dt$$

is a function of $\mathbf{x}$ and $\mathbf{y}$ alone. For fixed $\mathbf{x}$, the function $f(\mathbf{x},\mathbf{y})$ is positive and harmonic in $D \setminus \{\mathbf{x}\}$ as a function of $\mathbf{y}$ and it vanishes on the boundary of D. Thus (see Chung and Zhao [8], Chapter 2, especially Page 44), $f(\mathbf{x},\mathbf{y})$ is the Green's function $G_D(\mathbf{x},\mathbf{y})$ for D, and the expected lifetime of Brownian motion in D starting from $\mathbf{x}$ is

$$E_{\mathbf{x}}\tau_D = \int_D G_D(\mathbf{x},\mathbf{y})\,d\mathbf{y}.$$

This is the probabilist's normalization of the Green's function for one half of the Dirichlet Laplacian (it is twice the analyst's Green's function): in the unit disk $U = \{\mathbf{y} : |\mathbf{y}| < 1\}$ it is

$$G_U(0,\mathbf{y}) = \frac{1}{\pi} \log \frac{1}{|\mathbf{y}|}.$$

At this point we may conclude that

$$\Delta(E_{\mathbf{x}}\tau_D) = \Delta \left(\int_D G_D(\mathbf{x},\mathbf{y})\,d\mathbf{y} \right) = -2,$$

since the Green's function provides the solution

$$v(\mathbf{x}) = \int_D G_D(\mathbf{x},\mathbf{y}) f(\mathbf{y})\,d\mathbf{y}$$

to the Poisson problem $\frac{1}{2}\Delta v + f = 0$ in D and $v = 0$ on the boundary of D. The expected lifetime of Brownian motion in D and the torsion function from elasticity are one and the same.

The probabilistic interpretation of the torsion function makes it intuitively obvious that if $\mathbf{x}$ is a point in D_1 and if the domain D_1 is contained in the domain D_2 then $u_1(\mathbf{x}) \le u_2(\mathbf{x})$, where u_1 and u_2 are the torsion functions for D_1 and D_2 respectively.

One may compute explicit formulae for the torsion function or the expected lifetime of Brownian motion in certain domains. For example, for the disk $D = D(0,R)$ one has $E_{\mathbf{x}}\tau_D = \frac{1}{2}(R^2 - |\mathbf{x}|^2)$ and for the strip $S = \{(x_1,x_2) : |x_2| < R\}$ one has $E_{\mathbf{x}}\tau_S = R^2 - x_2^2$.

This interplay between analysis and probability is a two-way street. For example, one may solve the Dirichlet problem with boundary data f by running a Brownian motion. The solution is

$$v(\mathbf{x}) = E_{\mathbf{x}}\left(f(B_{\tau_D})\right).$$

2. The hyperbolic metric

Each simply connected domain D may be equipped with a metric $d(z,w;D)$ that is compatible with conformal mapping, in that if f is one-to-one and analytic in D and z and w are points in D then

$$d(z,w;D) = d(f(z), f(w); f(D)).$$

This metric is referred to as the *hyperbolic* or *Poincaré* metric on the domain. A conformal map is an isometry between simply connected domains when each domain is viewed as a metric space endowed with its hyperbolic metric.

The metric is Riemannian: a density $\sigma_D(z)$ scales the lengths of vectors based at a general point z in D. If $\gamma(t)$, $t \in [0,1]$, is a smooth curve in D then its σ_D-length is

$$l_{\sigma_D}(\gamma) = \int_0^1 \sigma_D(\gamma(t))|\gamma'(t)|\,dt.$$

The infinitesimal Euclidean length along the curve at $\gamma(t)$ is $|\gamma'(t)|\,dt$ and this is scaled by the scaling factor $\sigma_D(\gamma(t))$ at $\gamma(t)$ and integrated along the curve to give the 'length' of the curve from the point of view of the density σ_D. In the Riemannian manner, we define the distance between two points z and w in D to be the infimum σ_D-length of all curves in D that join z and w. This gives rise to a metric if σ_D is positive and continuous, though there is no a priori reason to believe that a curve of shortest length exists.

The density in the unit disk $U = \{z : |z| < 1\}$ is

$$\sigma_U(z) = \frac{1}{1-|z|^2}.$$

The scaling factor σ_U becomes unbounded near the unit circle. If $M(z)$ is a conformal map of U onto U, that is an automorphism of U, so that $M(z) = e^{i\theta}(z-\alpha)/(1-\overline{\alpha}z)$ for some real θ and some α in U, then

$$\sigma_U(M(z))|M'(z)| = \sigma_U(z).$$

It follows that, for any smooth curve γ in U,

$$\begin{aligned} l_{\sigma_U}(M\circ\gamma) &= \int_0^1 \sigma_U((M\circ\gamma)(t))|(M\circ\gamma)'(t)|\,dt \\ &= \int_0^1 \sigma_U(M(\gamma(t)))|M'(\gamma(t))|\,|\gamma'(t)|\,dt \\ &= \int_0^1 \sigma_U(\gamma(t))||\gamma'(t)|\,dt \\ &= l_{\sigma_U}(\gamma) \end{aligned}$$

and then $d(z,w;U) = d\,(M(z),M(w);U)$: that is, M is an isometry. It is a worthwhile exercise to show that this argument is reversible and one finds that the only densities for which automorphisms of the unit disk are isometries in the resulting Riemannian metric are multiples of $1/(1-|z|^2)$. It may furthermore be shown that for each point z in U there is a curve of shortest σ_U-length, a geodesic arc, joining the origin to z and that it coincides with the Euclidean geodesic arc, the line segment $[0,z]$. The σ_U-length of this line segment is easily calculated to give

$$d(0,z;U) = \frac{1}{2}\log\left(\frac{1+|z|}{1-|z|}\right).$$

Since each automorphism in the group of automorphisms of the unit disk is an isometry, and since any pair of points z_1 and w_1 may be mapped to a second given pair of points z_2 and w_2 by an appropriate automorphism, all geodesic curves in U are images under automorphisms of a diameter of the disk. The automorphisms are linear fractional transformations in the case of a disk and so preserve the collection of all circles and all straight lines. Moreover, these automorphisms are conformal, even on the unit circle. These observations combine to demonstrate that the geodesic curves in the unit disk are all diameters and all arcs of circles that meet the unit circle at right angles. This is shown in Figure 1. Note that two geodesic arcs pass through the point P, yet neither intersects the diameter L of the disk, another geodesic arc. That is, two distinct 'lines' pass through P and each is parallel to the given 'line' L and the Parallel Postulate fails. Clearly one may draw infinitely many geodesic arcs through P that are parallel to L, just as the theory suggests.

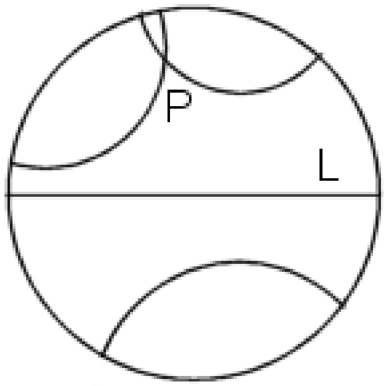

Figure 1: Geodesic Arcs in the Unit Disk

For a general simply connected domain D, we may take a Riemann map of D onto the unit disk and set

$$\sigma_D(z) = \sigma_U(f(z))|f'(z)| = \frac{|f'(z)|}{1-|f(z)|^2}. \tag{0.1}$$

This is independent of the particular Riemann map, since any other such map takes the form $M \circ f$ where M is an automorphism of the unit disk. The Riemannian metric $d(z, w; D)$ is then constructed from the density $\sigma_D(z)$. It is called the hyperbolic metric because, as a Riemannian manifold, D has constant negative curvature.

It transpires that

$$\sigma_D(z) = \sigma_{f(D)}(f(z))|f'(z)| \tag{0.2}$$

whenever f is one-to-one and analytic in D. (One may prove this first for the case $D = U$. For the general case one writes $f = F_2 \circ F_1$

where $F_1 : D \to U$ and $F_2 : U \to f(D)$.) As was the case for automorphisms of the unit disk, $l_{\sigma_D}(\gamma) = l_{\sigma_{f(D)}}(f(\gamma))$ which in turn implies the aforementioned invariance of the resulting Riemannian metric under conformal mapping.

On taking $f(z) = (z-1)/(z+1)$ in (0.1), the hyperbolic density of the half plane $H = \{z : \operatorname{Re} z > 0\}$ is found to be

$$\sigma_H(z) = \frac{1}{2\operatorname{Re} z}.$$

Then taking $D = S$ and $f(z) = \exp z$ in (0.2), the hyperbolic density of the strip $S = \{z : |\operatorname{Im} z| < \pi/2\}$ is found to be

$$\sigma_S(z) = \frac{1}{2\cos(\operatorname{Im} z)}.$$

The hyperbolic density σ_D behaves monotonically: if z lies in the simply connected domain D_1 and, in turn, D_1 is contained in the simply connected domain D_2 then $\sigma_{D_1}(z) \geq \sigma_{D_2}(z)$. One chooses f_1 and f_2 to be conformal mappings of D_1 and D_2, respectively, onto U with $f_1(z) = f_2(z) = 0$. The Schwarz Lemma for the function $g = f_2 \circ f_1^{-1}$ leads to $|g'(0)| = |f_2'(z)|/|f_1'(z)| \leq 1$. This is sufficient since $\sigma_{D_i}(z) = |f_i'(z)|$.

The distance from a point z in D to the complement of D we denote by $\delta_D(z)$. The disk centre z and radius $\delta_D(z)$ is contained in D and, relative to this disk, the hyperbolic density at z is $1/\delta_D(z)$. By monotonicity, $\sigma_D(z) \leq 1/\delta_D(z)$. The Koebe 1/4-theorem may be rephrased as a lower bound on the hyperbolic density, so that

$$\frac{1}{4\delta_D(z)} \leq \sigma_D(z) \leq \frac{1}{\delta_D(z)} \quad \text{for } z \in D.$$

Finally, we note the effect of scaling on the hyperbolic metric. If D is a simply connected domain and r is positive then $f(z) = rz$ is a conformal mapping of D onto the scaled domain rD. Then (0.2) yields

$$\sigma_{rD}(rz) = \frac{1}{r}\sigma_D(z) \quad \text{for } z \in D.$$

In particular, the quantity $\sigma_D(z)\delta_D(z)$ is scale invariant.

3. The bass note of a drum

A drum is a membrane, fixed on its boundary, whose shape is a certain simply connected domain D. The vertical displacement $F(\mathbf{x}, t)$ at a point $\mathbf{x}$ and at time t that results from striking the drum is modelled by the wave equation $\Delta F(\mathbf{x}, t) = \partial^2 F/\partial t^2$, with zero boundary conditions. On seeking a solution of the form $F(\mathbf{x}, t) = u(\mathbf{x})e^{i\omega t}$, one is led to the eigenvalue problem

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } D, \\ u = 0 & \text{on } \partial D. \end{cases}$$

The eigenvalues correspond to the squares of the pure notes that the drum can emit. They are countable in number and form a non-decreasing, unbounded sequence $\{\lambda_n\}_{n=1}^{\infty}$ with $0 < \lambda_1 < \lambda_2$, at least when D is bounded. The eigenfunctions ϕ_n corresponding to the eigenvalues λ_n, $n \geq 1$, may be chosen to form an orthonormal basis for $L^2(D)$, [12, Chapter 10]. The first eigenfunction $\phi_1(\mathbf{x})$ is positive in D. It is important to be aware of a pitfall that awaits those who read the work of both analysts and probabilists: in probability one works with half the Laplacian in the heat equation (and for a good reason – see Kai Lai Chung's book [7] where Einstein's original derivation of the mathematical laws for Brownian motion are discussed) while in analysis one works with the full Laplacian. Thus while the probabilist's eigenfunctions are the same as those of the analyst, the eigenvalues of the probabilist are half those of the analyst.

The eigenvalue problem may be expressed in variational form. The first eigenvalue $\lambda_1(D)$ is

$$\lambda_1(D) = \inf_{f \in C_c^\infty(D)} \frac{\int_D |\operatorname{grad} f|^2}{\int_D |f|^2}.$$

A minimizing f in $L^2(D)$ exists and is, up to normalization, the eigenfunction ϕ_1 for λ_1.

As we will be concerned only with the bass note or fundamental frequency, we sometimes write λ_D for $\lambda_1(D)$. As is clear from the variational formulation, the bass note behaves monotonically – a larger drum has a lower bass note – if the domain D_1 is contained in the domain D_2 then $\lambda_{D_2} \leq \lambda_{D_1}$.

The eigenvalue for a disk of radius R is j_0^2/R^2 where j_0 is the smallest positive zero of the Bessel function J_0. That for a strip of width $2R$ is $\pi^2/(4R^2)$.

The bass note has a number of probabilistic connections. The heat kernel $p_D(t, \mathbf{x}, \mathbf{y})$ is in $L^2(D)$ and it has an expansion in $L^2(D)$ in terms of the eigenvalues and normalized eigenfunctions for one half the Laplacian in D. In terms of the eigenvalues for the full Laplacian, the expansion is

$$p_D(t, \mathbf{x}, \mathbf{y}) = \sum_{n=1}^{\infty} e^{-\lambda_n t/2} \phi_n(\mathbf{x}) \phi_n(\mathbf{y}).$$

Then, for a fixed $\mathbf{x}$ in D,

$$\begin{aligned} P_{\mathbf{x}}(\tau_D > t) &= \int_D p_D(t, \mathbf{x}, \mathbf{y})\, d\mathbf{y} \\ &= \int_D \left(\sum_{n=1}^{\infty} e^{-\lambda_n t/2} \phi_n(\mathbf{x}) \phi_n(\mathbf{y}) \right) d\mathbf{y} \\ &= \sum_{n=1}^{\infty} a_n \phi_n(\mathbf{x}) e^{-\lambda_n t/2}, \end{aligned}$$

where $a_n = \int_D \phi_n(\mathbf{y})\, d\mathbf{y}$. Hence,

$$\begin{aligned} \frac{1}{t} \log \left[\frac{1}{P_{\mathbf{x}}(\tau_D > t)} \right] &= \frac{1}{t} \log \left[1 \Big/ \sum_{n=1}^{\infty} a_n \phi_n(\mathbf{x}) e^{-\lambda_n t/2} \right] \\ &= \frac{1}{t} \log \left[e^{\lambda_1 t/2} \Big/ \sum_{n=1}^{\infty} a_n \phi_n(\mathbf{x}) e^{(\lambda_1 - \lambda_n)t/2} \right] \\ &= \frac{\lambda_1}{2} + \frac{1}{t} \log \left[1 \Big/ \sum_{n=1}^{\infty} a_n \phi_n(\mathbf{x}) e^{(\lambda_1 - \lambda_n)t/2} \right]. \end{aligned}$$

Since $\lambda_n > \lambda_1$ for $n > 1$, the series $\sum_{n=1}^{\infty} a_n \phi_n(\mathbf{x}) e^{(\lambda_1 - \lambda_n)t/2}$ converges to $a_1 \phi_1(\mathbf{x})$ as $t \to \infty$. This gives

$$\lambda_D = 2 \lim_{t \to \infty} \frac{1}{t} \log \left[1 / P_{\mathbf{x}}(\tau_D > t) \right]. \tag{0.1}$$

A further connection between the the bass note λ_D and the first exit time τ_D was found by Graversen and Rao [9]. They showed that

$$\lambda_D = 2 \sup\{c \geq 0 : \sup_{\mathbf{x} \in D} E_{\mathbf{x}} \left[\exp(c\, \tau_D) \right] < \infty\}. \tag{0.2}$$

Next is a connection between the first eigenvalue λ_D and the torsion function $u(\mathbf{x})$. This will be used in the next section when we obtain a lower bound on the bass note of a domain in terms of the inradius of the domain.

$$\begin{aligned}
\int_D \phi_1(\mathbf{x})\,d\mathbf{x} &= -\frac{1}{2}\int_D (\Delta u)(\mathbf{x})\phi_1(\mathbf{x})\,d\mathbf{x} \\
&= -\frac{1}{2}\int_D u(\mathbf{x})\Delta\phi_1(\mathbf{x})\,d\mathbf{x} \qquad \text{(by Green's Theorem)} \\
&= \frac{1}{2}\lambda_D \int_D u(\mathbf{x})\phi_1(\mathbf{x})\,d\mathbf{x} \\
&\le \frac{1}{2}\lambda_D \left(\sup_{\mathbf{x}\in D} u(\mathbf{x})\right)\int_D \phi_1(\mathbf{x})\,dx
\end{aligned}$$

from which it follows that

$$\lambda_D \ge \frac{2}{\sup_{\mathbf{x}\in D} u(\mathbf{x})}. \tag{0.3}$$

4. Isoperimetric-type inequalities

The prime directive is to measure the effect of the geometry of the domain on the analytic quantities that have now been introduced.

4.1 Fixed Area

The most fundamental geometric quantity that one may associate with a domain is its area and the first problems and conjectures of an isoperimetric type are for domains of fixed area. As mentioned in Section 1, St. Venant (1856) conjectured that among all simply connected domains of given area, a disk of that area has the largest torsional rigidity – that a beam of circular cross section is the most resistant to twisting of all beams of prescribed cross sectional area. This was proved by Pólya in (1948). Lord Rayleigh (1877) conjectured that among all simply connected domains of given area, a disk of that area has the lowest fundamental frequency – that a circular drum has the lowest bass note of all drums of prescribed area. This was proved independently by G. Faber and E. Krahn (1923–4). The classic reference in this area is Pólya and Szegö's *Isoperimetric Inequalities in Mathematical Physics* [18]. The book by Catherine Bandle *Isoperimetric Inequalities and Applications* [1] may be viewed as a sequel to Pólya and Szegö's book. Therein she proves a symmetrization result for the Green's function [1, Theorem 2.4]. We set D^* to be the disk with centre 0 and with the same area as D. Then

$$\text{area}\,(\{\mathbf{y} \in D : G_D(\mathbf{x},\mathbf{y}) > t\}) \leq \text{area}\,(\{\mathbf{y} \in D^* : G_{D^*}(0,\mathbf{y}) > t\}) \tag{0.1}$$

for each positive t. From this it follows (again see Lieb and Loss [13, Theorem 1.13]) that for each non-decreasing function ϕ on $[0,\infty)$,

$$\int_D \phi(G_D(\mathbf{x},\mathbf{y}))\,d\mathbf{y} \leq \int_{D^*} \phi(G_{D^*}(0,\mathbf{y}))\,d\mathbf{y}.$$

This result even holds for arbitrary domains of finite volume in $\mathbf{R}^n$.

As a consequence of a rearrangement inequality for multiple integrals proved by Luttinger [14],

$$P_\mathbf{x}(\tau_D > t) \leq P_0(\tau_{D^*} > t) \quad \text{for } t > 0. \tag{0.2}$$

In words, a Brownian motion has a greater probability of being alive at time t if it departs from the center of the disk D^* than from the point $\mathbf{x}$ in D. From this follows

$$E_\mathbf{x}\phi(\tau_D) \leq E_0\phi(\tau_{D^*}),$$

for each non-decreasing function ϕ on $[0,\infty)$.

The distributional inequality for the lifetime (0.2) also gives the Rayleigh-Faber-Krahn Theorem. In fact by (0.1),

$$\begin{aligned}
\lambda_D &= 2\lim_{t\to\infty}\frac{1}{t}\log\left[1\,/P_\mathbf{x}(\tau_D > t)\right] \\
&\geq 2\lim_{t\to\infty}\frac{1}{t}\log\left[1\,/P_0(\tau_{D^*} > t)\right] \\
&= \lambda_{D^*}
\end{aligned}$$

Let us, at this point, outline the approach to isoperimetric inequalities introduced by Luttinger and perfected in the Brascamp-Lieb-Luttinger rearrangement inequality for multiple integrals [6]. We denote by f^* the symmetric decreasing rearrangement of a non-negative measurable function f, so that f^* has the properties

(i) $f^*(\mathbf{x}) = f^*(\mathbf{y})$ if $|\mathbf{x}| = |\mathbf{y}|$

(ii) if $0 < |\mathbf{x}| < |\mathbf{y}|$ then $f^*(\mathbf{x}) \geq f^*(\mathbf{y})$

(iii) $\text{area}\,\{f > t\} = \text{area}\,\{f^* > t\}$ for each $t > 0$.

Implicit in (iii) is the assumption that the sets $\{f > t\}$ have finite area for each positive t, this being the meaning attached to the phrase 'f vanishes at infinity'. In the present situation we will make two distinct choices of function f, the first being that of the characteristic function of a domain D of finite area. The second choice of f is that of the transition probabilities for Brownian motion in the plane, the heat kernel for $\mathbf{R}^2$,

$$p_{\mathbf{R}^2}(t, \mathbf{x}, \mathbf{y}) = \frac{1}{2\pi t} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{2t}},$$

which we view as the function $p_t(\mathbf{x}-\mathbf{y})$ with $p_t(\mathbf{x}) = e^{-|\mathbf{x}|^2/(2t)}/(2\pi t)$. This function is its own symmetric decreasing rearrangement. Here, then, is the Brascamp-Lieb-Luttinger inequality for $\mathbf{R}^2$, there being a corresponding version in n-dimensions.

Theorem *Suppose that $f_i(\mathbf{x})$, $1 \le i \le k$, are measurable, non-negative functions on $\mathbf{R}^2$ that vanish at infinity. Suppose that a_{ij}, $1 \le i \le k$, $1 \le j \le m$ are real numbers. Then*

$$\int_{\mathbf{R}^2} \cdots \int_{\mathbf{R}^2} \prod_{i=1}^{k} f_i \left(\sum_{j=1}^{m} a_{ij} \mathbf{x}_j \right) \, d\mathbf{x}_1 \ldots d\mathbf{x}_m$$
$$\le \int_{\mathbf{R}^2} \cdots \int_{\mathbf{R}^2} \prod_{i=1}^{k} f_i^* \left(\sum_{j=1}^{m} a_{ij} \mathbf{x}_j \right) \, d\mathbf{x}_1 \ldots d\mathbf{x}_m.$$

Note that the number of functions (k) and the number of variables (m) are independent of each other. Thus we may take m functions where

$$f_i \left(\sum_{j=1}^{m} a_{ij} \mathbf{x}_j \right) = 1_D(\mathbf{x}_i) \quad \text{for } 1 \le i \le m,$$

and then k further functions of a general form. Since $(1_D)^* = 1_{D^*}$, we obtain

$$\int_{D} \cdots \int_{D} \prod_{i=1}^{k} f_i \left(\sum_{j=1}^{m} a_{ij} \mathbf{x}_j \right) \, d\mathbf{x}_1 \ldots d\mathbf{x}_m$$
$$\le \int_{D^*} \cdots \int_{D^*} \prod_{i=1}^{k} f_i^* \left(\sum_{j=1}^{m} a_{ij} \mathbf{x}_j \right) \, d\mathbf{x}_1 \ldots d\mathbf{x}_m.$$

Now, $P_{\mathbf{x}}(\tau_D > t)$ is the probability that Brownian motion that departs from $\mathbf{x}$ has not left D by time t. Since the Brownian paths are continuous, it is enough to check very often that Brownian motion in the *plane* that departs from $\mathbf{x}$ is in D, say at each of the times it/m, $1 \le i \le m$, for large m. By the independence of the increments of Brownian motion and the normal distribution of these increments, this last probability is a multiple integral over D of the transition probabilities for Brownian motion in the *plane* and it was Luttinger's key idea to seek a general rearrangement inequality for such multiple integrals. By a translation of D if necessary, we may assume that the Brownian motion departs from 0 and then use the rearrangement inequality to obtain

$$
\begin{aligned}
&P_0(\tau_D > t) \\
&\quad = \lim_{m\to\infty} \left[P_0\left(B_{it/m} \in D,\ i = 1,\ 2, \ldots, m\right)\right] \\
&\quad = \lim_{m\to\infty} \int_D \cdots \int_D p_{t/m}(\mathbf{x}_1) \prod_{i=2}^{m} p_{t/m}(\mathbf{x}_i - \mathbf{x}_{i-1})\, d\mathbf{x}_1 \ldots d\mathbf{x}_m \\
&\quad \le \lim_{m\to\infty} \int_{D^*} \cdots \int_{D^*} p_{t/m}(\mathbf{x}_1) \prod_{i=2}^{m} p_{t/m}(\mathbf{x}_i - \mathbf{x}_{i-1})\, d\mathbf{x}_1 \ldots d\mathbf{x}_m \\
&\quad = \lim_{m\to\infty} \left[P_0\left(B_{it/m} \in D^*,\ i = 1,\ 2, \ldots, m\right)\right] \\
&\quad = P_0(\tau_{D^*} > t).
\end{aligned}
$$

There is a minor technical difficulty with the above argument as it stands, but it seems a shame to complicate such an elegant idea with technicalities. The interested reader may discover the problem and/or its solution in [5]. The Brascamp-Lieb-Luttinger inequality may also be used to resolve St. Venant's problem. The torsional rigidity P is

$$
\begin{aligned}
P &= 2\int_D E_{\mathbf{x}}\tau_D\, d\mathbf{x} \\
&= 2\int_D \int_D G_D(\mathbf{x},\mathbf{y})\, d\mathbf{y}\, d\mathbf{x} \\
&= 2\int_D \int_D \int_0^\infty p_D(t,\mathbf{x},\mathbf{y})\, dt\, d\mathbf{y}\, d\mathbf{x} \\
&= 2\int_0^\infty Q_D(t)\, dt
\end{aligned}
$$

where

$$Q_D(t) = \int_D \int_D p_D(t, \mathbf{x}, \mathbf{y})\, d\mathbf{x}\, d\mathbf{y}.$$

The quantity $Q_D(t)$ is called the *heat content* of D. Imagine that the initial temperature is uniformly 1 throughout D and that the boundary of D is maintained at 0 temperature at all times. Then $Q_D(t)$ represents the total heat in D at time t. Arguing as before,

$$\begin{aligned}
& Q_D(t) \\
&= \int_D P_{\mathbf{x}_0}(\tau_D > t)\, d\mathbf{x}_0 \\
&= \int_D \left(\lim_{m\to\infty} \left(\int_D \cdots \int_D \prod_{i=1}^{m} p_{t/m}(\mathbf{x}_i - \mathbf{x}_{i-1})\, d\mathbf{x}_1 \ldots d\mathbf{x}_m \right) \right) d\mathbf{x}_0 \\
&= \lim_{m\to\infty} \int_D \left(\int_D \cdots \int_D \prod_{i=1}^{m} p_{t/m}(\mathbf{x}_i - \mathbf{x}_{i-1})\, d\mathbf{x}_1 \ldots d\mathbf{x}_m \right) d\mathbf{x}_0 \\
&= \lim_{m\to\infty} \int_D \cdots \int_D \prod_{i=1}^{m} p_{t/m}(\mathbf{x}_i - \mathbf{x}_{i-1})\, d\mathbf{x}_0 \ldots d\mathbf{x}_m \\
&\le \lim_{m\to\infty} \int_{D^*} \cdots \int_{D^*} \prod_{i=1}^{m} p_{t/m}(\mathbf{x}_i - \mathbf{x}_{i-1})\, d\mathbf{x}_0 \ldots d\mathbf{x}_m \\
&= Q_{D^*}(t),
\end{aligned}$$

since all the previous steps are reversible, including that where the bounded convergence theorem was used to interchange the limit and the integral. St. Venant's conjecture is thereby verified.

4.2 Fixed Inradius

A second and, in a sense, more appropriate quantitative indicator of the geometry of a domain D is its *inradius* R_D, the supremum radius of all disks contained in the domain. For example, a strip of width L has inradius $L/2$; the inradius of a halfplane is infinite. It follows from monotonicity that

- $\sup_{\mathbf{x}\in D} E_{\mathbf{x}}(\tau_D) \ge R_D^2/2$,
- $\inf_{z\in D} \sigma_D(z) \le 1/R_D$,
- $\lambda_D \le j_0^2/R_D^2$.

Equality holds in each inequality if and only if D is a disk. Significantly, an analogous inequality in the other direction holds in each case, so that each of the analytical quantities $\sup_{\mathbf{x}\in D} E_{\mathbf{x}}(\tau_D)$, $\inf_{z\in D}\sigma_D(z)$ and λ_D is finite and non-zero if and only if D has finite inradius. Bañuelos and Carroll [2] proved

$$\sup_{\mathbf{x}\in D} E_{\mathbf{x}}(\tau_D) \leq (3.228)R_D^2. \tag{0.3}$$

The idea of the proof was to take the representation of the expected lifetime in terms of the Green's function and to rewrite the Green's function in terms of the hyperbolic distance. Both the Green's function $G_D(z,w)$ and the hyperbolic distance $d(z,w;D)$ are conformal invariants that depend on two interior points – there is only one such conformal invariant, in the sense that any two are functions one of the other. In this case,

$$G_D(z,w) = \frac{1}{\pi}\log[\coth(d(z,w;D))],$$

which is obtained from the explicit formulas for the Green's function and the hyperbolic metric in the disk and holds in general by conformal invariance. A classical problem in complex analysis is to determine the best constant $\mathcal{U}$ in the inequality

$$\sigma_D(z) \geq c/R_D \tag{0.4}$$

as z ranges over all points in each simply connected domain D. That one may take $c = 1/4$ follows from the Koebe 1/4-theorem.

Let us make explicit the equivalence of (0.4) and the schlicht Bloch constant problem. Suppose that $f(z)$ is analytic and one-to-one (univalent) in the unit disk and that $f(0) = 0$ and $f'(0) = 1$, that is f belongs to the class $\mathbf{S}$. Then the image of the unit disk U under f must contain the disk centre 0 and radius 1/4: this is the Koebe 1/4-theorem. The schlicht Bloch constant $\mathcal{U}$ is the supremum of those numbers c such that the image domain $f(U)$ must contain a disk of radius c *somewhere* (not just centred at 0). That is, we want the largest c such that $R_{f(U)} \geq c$ for each $f \in \mathbf{S}$. The number $\mathcal{U}$ was introduced by Landau in 1929 on which occasion he proved $\mathcal{U} \geq 9/16$. Reich improved this to 0.569 in 1956. James Jenkins proved $\mathcal{U} \geq 0.5705$ in 1961. In 1968, Toppila obtained the lower bound of 0.5708, which was improved by Zhang (1989) to 0.57088.

Jenkins gave his own account of this in [11] and he too obtained the lower bound 0.57088. We shall speak of upper bounds for $\mathcal{U}$ in due course. If $f(z)$ is simply analytic and univalent in the unit disk U and no normalization is assumed then $R_{f(U)} \geq \mathcal{U}|f'(0)|$, this because $g(z) = (f(z) - f(0))/f'(0)$ is in the class $\mathbf{S}$. But $|f'(0)|$ is the reciprocal of the hyperbolic density for $f(U)$ at $f(0)$, that is $|f'(0)| = 1/\sigma_{f(U)}(f(0))$ and we are led to $\sigma_{f(U)}(f(0))R_{f(U)} \geq \mathcal{U}$. By the Riemann mapping theorem, if D is simply connected and z belongs to D then we may choose f so that $f(U) = D$ and $f(0) = z$. This is (0.4) and so the best constant c in (0.4) is the schlicht Bloch constant $\mathcal{U}$.

Integration of (0.4) along a geodesic γ in D that joins 0 to z will give

$$d(0, z; D) = \int_\gamma \sigma_D(z)\,|dz| \geq \frac{\mathcal{U}}{R_D}(\text{Euclidean length of } \gamma) \geq \frac{\mathcal{U}}{R_D}|z| \tag{0.5}$$

and then

$$G_D(0, z) \leq \frac{1}{\pi} \log\left[\coth\left(\mathcal{U}|z|\,/R_D\right)\right].$$

This leads to an upper bound for the expected lifetime of Brownian motion,

$$E_0\tau_D = \int_D G_D(0, z)\,dz \leq \int_{\mathbf{C}} \frac{1}{\pi} \log\left[\coth\left(\frac{\mathcal{U}|z|}{R_D}\right)\right]\,dz = \frac{7\zeta(3)}{8\mathcal{U}^2}\,R_D^2.$$

The Jenkins/Zhang estimate $\mathcal{U} \geq 0.57088$ gives (0.3). Honesty compels us to admit that (0.5) is a very crude estimate. Even so, no one has as yet been able to improve on it and it did lead to a much better lower bound on the bass note.

The story of the lower bound for the bass note of a drum is interesting. The Rayleigh-Faber-Krahn theorem is $\lambda_D \geq \pi j_0^2/\text{area}\,(D)$. Pólya and Szegö [18] proved a lower bound in terms of the inradius but only for convex domains, and they raised the problem of finding such a lower bound for general simply connected domains – for a non-convex simply connected domain of finite inradius and infinite area the bounds they had gave no information. Fraenkel mentioned this problem to Hayman who proved (1976)

$$\lambda_D \geq \frac{1}{900R_D^2}.$$

Osserman (1977), who described Hayman's theorem as showing, in effect, 'that in order for a drum to produce an arbitrarily deep note it is necessary that it include an arbitrarily large circular drum', improved the result to

$$\lambda_D \geq \frac{1}{4R_D^2}.$$

Our interest in the bound (0.3) is that together with the estimate (0.3) it gives

$$\lambda_D \geq \frac{0.6197}{R_D^2}.$$

This is currently the best constant available in Hayman's theorem. Now comes the twist in the story. We learned from Mark Ashbaugh (via Richard Laugesen, that Endre Makai [15] had proved $\lambda_D \geq 1/(4R_D^2)$ in 1965, eleven years before Hayman and Osserman. His paper had been missed for years on end. Only now do we know that we are in fact looking for the best constant in Makai's theorem on the fundamental frequency.

The estimate (0.3) also leads to an estimate for the torsional rigidity,

$$P = 2\int_D E_{\mathbf{x}}\tau_D \leq 6.456\, R_D^2 \,\mathrm{area}\,(D).$$

Though it follows from monotonicity of the torsional rigidity that $P \geq \pi R_D^4/2$, a bound of the form $P \leq CR_D^4$ doesn't hold in general since a strip has finite inradius but infinite torsional rigidity. Bañuelos, Carroll and van den Berg [4] have worked on the problem of characterizing in terms of their geometry those domains of infinite area but finite torsional rigidity.

5. Extremal Domains

What are the best constants in the inequalities

$$\begin{aligned} &E_{\mathbf{x}}\tau_D \leq C_1 R_D^2,\\ &\sigma_D(z) \geq c_2/R_D,\\ &\lambda_D \geq c_3/R_D^2, \end{aligned}$$

where D is any simply connected domain? The inequalities hold with $C_1 = 3.228$, $c_2 = 0.57088$ and $c_3 = 0.6197$. This is not simply a numerical problem; the particular values of the constants are not of great interest. The real problem is to determine what the extremal domains are. For example, we would be perfectly happy to know that the schlicht Bloch constant $\mathcal{U}$ is the value of the hyperbolic density at a particular point in a particular domain even if we were unable to compute this exactly.

Among those attacks made on the schlicht Bloch constant that yield upper bounds, the domain constructed by Ruth Goodman (1945), building on previous work by Robinson (1935), was considered for some time to be a candidate for an extremal domain. Here it is, in part.

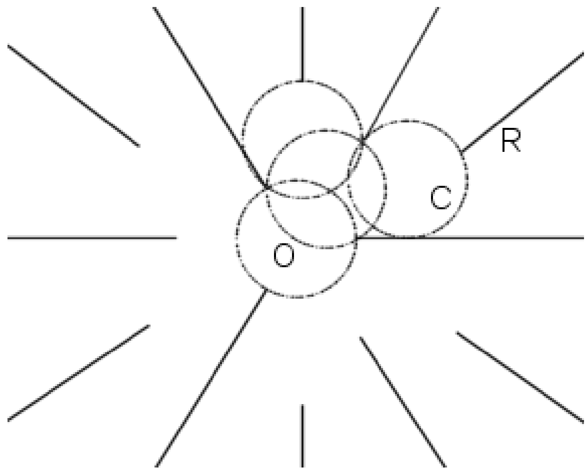

Figure 2: Ruth Goodman's Domain G

Suppose that we wish to make $\sigma_D(0)$ as small as possible. The general inequality $\sigma_D(z) \geq 1/\delta_D(z)$ seems to suggest that we should keep the boundary of the domain as far away as possible from the origin. The counterbalance is that there can be no disk of radius greater than 1, say. Let us denote by $\mathrm{ray}(\rho, t)$ the half-ray $\{re^{it} : r \geq \rho\}$. The first three rays are $\mathrm{ray}(1, 0)$, $\mathrm{ray}(1, 2\pi/3)$ and $\mathrm{ray}(1, -2\pi/3)$. There is a maximal disk of radius 1 centred at 0. The next three rays bisect the three sectors formed by the original three rays and begin on the circle of radius 2, so that there are now 3 more maximal disks, one of which is shown, and 6 sectors.

Consider the first of these sectors, bounded by $\mathrm{ray}(1, 0)$ and $\mathrm{ray}(2, \pi/3)$. This sector contains a maximal disk C centred at $c + i$

where $1 = |(c+i) - 2e^{i\pi/3}|$. This gives $c = 1 + \sqrt{2\sqrt{3} - 3}$. Goodman chose to bisect this sector, inserting ray $(\rho, \pi/6)$, labelled R in Figure 2, so as to touch the disk C and block it from becoming any larger. This process of bisecting each sector as it is formed by a ray designed to touch the maximal disks and block their growth is continued to produce a simply connected domain G of inradius 1. Goodman, by means of explicit conformal mappings, computes

$$0.65646 \leq \sigma_G(0) \leq 0.65647.$$

Since

$$\mathcal{U} = \inf\{\sigma_D(z)R_D : z \in D \text{ and } D \text{ is simply connected}\},$$

this shows that $\mathcal{U} \leq 0.65647$.

Many years later Beller and Hummel (1985) returned to Goodman's example, computer at their side. They focused on the third set of rays – the ray labelled R in Figure 2. They noticed that while the maximal disk C that it touches is tangent to ray$(1, 0)$ it is not tangent to the second ray that defines its sector, namely ray$(2, \pi/3)$. The circle C peeks around the tip of this second ray and so its centre does not have argument $\pi/6$. As a result, when Goodman chose the third generation ray to have argument $\pi/6$, the ray R came a bit closer to the origin that may be strictly necessary. This problem doesn't occur at any later stage in the construction since the later sectors have narrower apertures. We imagine a disk of radius 1 that rolls towards the narrow end of the sector. If the sector has a relatively small opening angle then the disk becomes wedged before reaching the opening. The anomalous, asymmetric, maximal disk C that does not become wedged before reaching the opening of its sector was exploited by Beller and Hummel, who varied the argument of the ray R while making sure that it touched the circle C, and found its optimal position. The optimal argument is about 0.3931 as opposed to Goodman's choice of $\pi/6 \approx 0.5236$. Having estimated the hyperbolic density of their domain at the origin by numerical methods, Beller and Hummel found that

$$\mathcal{U} \leq 0.6564155.$$

This is the best upper bound available for the schlicht Bloch constant.

Beller and Hummel never claimed that their domain could be extremal for the schlicht Bloch constant. On the contrary, they describe a criterion told to them by Jenkins that must be satisfied by an extremal domain and that is not satisfied by their domain. Jenkins subsequently published this criterion separately [11]. Suppose that $\sigma_D(0)R_D = \mathcal{U}$ (a compactness argument shows that extremal domains do exist so that this supposition is not vacuous).

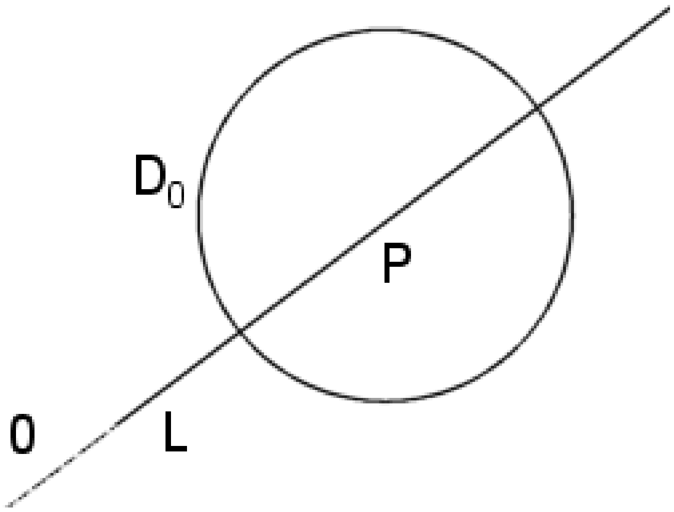

Figure 3: Jenkins' Criterion for an Extremal Domain

We have a point P, a disk D_0 centred at P and a line L through P. The diameter $L \cap D_0$ of the disk belongs to the boundary of D while $D_0 \setminus L$ is part of the domain D. Furthermore, P is not on the boundary of any disk of radius R_D in D. The conclusion is that L passes through the origin and D is symmetric in L. The Beller-Hummel domain fails this extremality test since it is not symmetric in the ray R.

It is currently the case that no candidate for an extremal domain for the schlicht Bloch constant has been put forward. In [2] we suggest an approach that may lead to such a candidate, though we were unable to compute anything for this domain. What did strike us as significant was that, in attempting to imagine an extremal domain for the expected lifetime, that is a simply connected domain D of inradius 1 in which $E_0\tau_D$ is maximal, we came up with something resembling the Goodman domain. This, and the knowledge that the strip is an extremal convex domain in each case, led us to conjecture that the extremal domains for each of the three inequalities listed at the beginning of this section are the same.

6. Convex domains with fixed inradius

In the case of a convex domain, rather than a general simply connected domain, more precise information is available on the influence of geometry on the fundamental frequency, the hyperbolic metric and the torsion function. Among convex domains of fixed inradius the infinite strip is the 'largest' and turns out to be extremal for the isoperimetric type problems we consider. Here are three results for which no sharp counterpart is available for general simply connected domains. If D is a convex domain of finite inradius R_D then

$\sigma_D(z) \geq \pi/(4R_D)$	Szegö (1923)
$E_{\mathbf{x}}\tau_D \leq R_D^2$	Payne (1968)/ Sperb (1981)
$\lambda_D \geq \pi^2/(4R_D^2)$	Hersch (1960)

Equality holds in the first two inequalities if and only if D is an infinite strip and z lies in the centre of the strip. The lower bound on the fundamental frequency is attained, for example, in the case of an infinite strip.

The methods by which these results were originally obtained are quite different. Szegö used the Schwarz-Christoffel formula to map onto a triangle – a simplfying observation is that any convex domain of finite inradius can be enclosed in a strip or a triangle of the same inradius. By monotonicity of the Poincaré density, it is then sufficient to prove the result for triangles. Sperb in his monograph [19] shows that appropriately constructed P–functions ('P' being in honour of Payne, in whose work the method originated) satisfy a maximum principle. A typical result of this type is that, if the domain is convex, the P–function

$$P = |\operatorname{grad} u|^2 + 4u,$$

u being the torsion function, attains its maximum at a point where $\operatorname{grad} u = 0$. The bound $u(\mathbf{x}) \leq R_D^2$ follows from this. In the context of the eigenvalue λ_D, the P–function

$$P = |\operatorname{grad} u|^2 + \lambda_D u^2$$

where u is the eigenfunction for λ_D, has the same property and this leads to a proof of Hersch's result that differs from the original. John O'Donnell includes a readable account of P–functions in the case of the torsion function in his M.Sc. thesis [17]. He also proves a monotonicity result related to the convex Bloch constant and Szegö's work.

An alternative formulation of the above inequalities is as follows. We denote by S^* the infinite horizontal strip of the same inradius as D and symmetric in the real axis. Then, if D is a convex domain of finite inradius,

$$\lambda_D \geq \lambda_{S^*}, \qquad E_{\mathbf{x}}\tau_D \leq E_0\tau_{S^*}, \qquad \sigma_D(z) \geq \sigma_{S^*}(0).$$

Recently, Bañuelos, Latala and Méndez-Hernández [5] proved a Brascamp-Lieb-Luttinger type re-arrangement inequality in which the multiple integral over a convex domain D is found to be majorized by the corresponding multiple integral over the infinite strip S^*. The infinite strip S^* of the same inradius as D in the Bañuelos, Latala, Méndez-Hernández setting replaces the disk D^* of the same area as D in the Brascamp-Lieb-Luttinger setting. Following the line of reasoning described in Section 4.1, Bañuelos, Latala and Méndez-Hernández obtain as a consequence of their inequality that

$$P_{\mathbf{x}}(\tau_D > t) \leq P_0(\tau_{S^*} > t) \quad \text{for } t > 0,$$

for any convex domain D of finite inradius. This estimate for the probability that a Brownian traveller in a convex domain is alive at time t was first proved by Bañuelos and Kroger (1997) by a P–function argument and immediately implies the estimate of Payne and Sperb on the torsion function and the eigenvalue estimate of Hersch.

This approach has been taken even further by Méndez-Hernández, who proves a rearrangement inequality for multiple integrals over convex domains in $\mathbf{R}^n$ and in which the infinite hyperstrip S^* is replaced by a smaller hyper-rectangle. This is an improvement on the known results even in the case of the plane.

Precise results on rearrangements of the Green's function of a convex domain when the pole of the Green's function lies at the centre of the largest disk in the domain were obtained by Bañuelos, Carroll and Housworth [3].

Let us end with an elegant theorem of David Minda on the hyperbolic density in convex domains. Suppose that D is a convex domain of finite inradius R_D and that z is a point in D. We suppose that w is a point on the boundary of D closest to z and construct a comparison strip S of width $2R_D$ that is tangent to D at w and contains z. Then $\sigma_D(z) \geq \sigma_S(z)$. The hyperbolic density for a strip can be computed explicitly and, in any case, its minimum occurs along the main axis of the strip, with value $\pi/(4R_S)$. Thus Szegö's determination of the convex Bloch constant is a simple consequence of Minda's result.

Epilogue

This survey is an expanded version of a talk presented at the Thirteenth September Meeting of the Irish Mathematical Society held at Maynooth in 2000. Technical details have been kept to a minimum in this survey and some may grimace at what has been skated over in places. Certainly, anyone who finds something of use here would do well to consult the original papers or a serious textbook or monograph before taking the matter any further. Not all relevant references are included by any means, my excuse being that there are extensive bibliographies in the books by Bandle, by Pólya and Szegö and by Sperb, and in the paper [2].

It is an honour to acknowledge my debt to my friend and collaborator Rodrigo Bañuelos at Purdue University. We have often discussed the mathematics described here, and much else besides, over the past twelve years. What little I know of this area I have learnt from him.

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