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Norms of Elementary Operators

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ABSTRACT. We make some simple remarks about the *norm problem* for elementary operators in the contexts of algebras of operators on Banach spaces and C^*-algebras. We illustrate a lack of symmetry in the problem for algebras containing the finite rank operators and also for the Calkin algebra. An isomorphic symmetry is shown for subhomogeneous C^*-algebras.

For A an algebra, an operator $T\colon A \to A$ is called an *elementary operator* if T can be expressed in the form

$$Ta = \sum_{i=1}^{\ell} a_i a b_i \tag{1}$$

with a_i and b_i $(1 \leq i \leq \ell)$ in A. For A a C^*-algebra, one may allow a_i and b_i to be in the multiplier algebra of A (see [3]). Such representations of T may not be unique. We write $\mathcal{E}\ell(A)$ for the class of elementary operators on A. We will always be considering normed algebras A and so there is a natural operator norm on $\mathcal{E}\ell(A)$ (which is contained in the bounded linear operators on A).

The so-called norm problem (see [5] for a survey) asks for $\|T\|$ to be expressed in terms of $\|a_i\|$ and $\|b_i\|$ (or, more realistically, in terms of the norms of some combination of the a_i and some combination of the b_i such as in the Haagerup norm mentioned below). In the case $\ell = 1$ and $Ta = a_1 a b_1$, we know from [4] that if A is ultraprime then there is a constant $c_A > 0$ such that $c_A \|a_1\| \|b_1\| \leq \|T\|$. In fact, we can take the existence of c_A as the definition of A being ultraprime. Of course $\|T\| \leq \|a_1\| \|b_1\|$ for any normed algebra A.

For A ultraprime, the map from the algebraic tensor product $A \otimes A$ to $\mathcal{E}\ell(A)$ which maps $\sum_{j=1}^{\ell} a_j \otimes b_j$ to $T \in \mathcal{E}\ell(A)$ of the form (1) is bijective (see [4, Theorem 5.1]). Hence we can define $T^f \in \mathcal{E}\ell(A)$ (for $T \in \mathcal{E}\ell(A)$ as in (1)) by $T^f a = \sum_{i=1}^{\ell} b_i a a_i$.

In a recent preprint V. Runde [8] considers algebras A of operators on infinite dimensional reflexive Banach spaces E with the approximation property. If A contains the finite rank operators $\mathcal{F}(E)$, he has shown that the map $T \mapsto T^f$ is not norm continuous on $\mathcal{E}\ell(A)$. Following [8], we refer to $T \mapsto T^f$ as the *flip* map on $\mathcal{E}\ell(A)$.

Note that algebras A with $\mathcal{F}(E) \subseteq A \subseteq \mathcal{B}(E)$ (= the bounded operators on a Banach space E) are always ultraprime. Our main result is an elementary proof of the following variant of the result of Runde [8].

Theorem 1. *Assume that E is a Banach space that contains an infinite dimensional complemented subspace with a Schauder basis, and suppose A is an algebra of bounded operators on E with $\mathcal{F}(E) \subseteq A$. (We give A the operator norm.) Then the map $T \mapsto T^f$ is not norm continuous on $\mathcal{E}\ell(A)$.*

Proof. Let $(e_n)_{n=1}^\infty$ be a basic sequence in E with closed linear span complemented in E via a projection P. Then the projections $\pi_n\left(\sum_{j=1}^\infty x_j e_j\right) = \sum_{j=1}^n x_j e_j$ are uniformly bounded on the span of the e_j and so $P_n = \pi_n \circ P$ are uniformly bounded on E, and belong to $\mathcal{F}(E)$.

Consider the biorthogonal functionals f_j to the e_n (on the span of the e_n) given by $f_j\left(\sum_{n=1}^\infty x_n e_n\right) = x_j$ and then $F_j = f_j \circ P$. Define $E_{ij} \in \mathcal{F}(E)$ by $E_{ij}(x) = F_j(x)e_i$. Consider $T_n \in \mathcal{E}\ell(A)$ given by

$$T_n a = \sum_{i=1}^n E_{i1} a E_{1i} \qquad (a \in A).$$

For $x \in E$, we have

$$\begin{aligned}
(T_n a)(x) &= \sum_{i=1}^n E_{i1}(a(E_{1i}(x))) \\
&= \sum_{i=1}^n E_{i1}(a(F_i(x)e_1)) = \sum_{i=1}^n F_i(x) E_{i1}(a(e_1)) \\
&= \sum_{i=1}^n F_i(x) F_1(a(e_1)) e_i = F_1(ae_1) \sum_{i=1}^n F_i(x) e_i \\
&= F_1(ae_1) P_n(x)
\end{aligned}$$

Thus $\|T_n a\| \leq \|F_1\| \|e_1\| \|P_n\| \|a\|$ or $\|T_n\| \leq \|F_1\| \|e_1\| \|P_n\|$ and consequently $\sup_n \|T_n\| < \infty$.

Working with T_n^f, we get from a similar calculation

$$(T_n^f a)(x) = \sum_{i=1}^{n} E_{1i}(a(E_{i1}(x))) = F_1(x) \sum_{i=1}^{n} F_i(a(e_i))e_1$$

Taking $a = P_n$, we have $(T_n^f P_n)(x) = F_1(x) \sum_{i=1}^n e_1 = nF_1(x)e_1$. Thus $\|T_n^f P_n\| = n\|F_1\|\|e_1\| \to \infty$ as $n \to \infty$. As $\sup_n \|P_n\| < \infty$, it follows that $\|T_n^f\| \to \infty$ as $n \to \infty$. $\square$

Remark 1. We consider now the case where $E = \ell^2$ is a (separable) Hilbert space and we assume that A is a C^*-algebra of operators on ℓ^2. We are then restricted to the two examples $A = \mathcal{K}(\ell^2)$ = the compact operators and $A = \mathcal{B}(\ell^2)$, if A is to contain all the finite rank operators (as is required to apply the theorem).

We could also consider the completely bounded norm of operators $T \in \mathcal{E}\ell(A)$. For $T \in \mathcal{E}\ell(A)$ given by (1), the completely bounded norm is $\|T\|_{cb} = \inf \sqrt{\left\|\sum_{j=1}^\ell a_j a_j^*\right\| \left\|\sum_{k=1}^\ell b_k^* b_k\right\|}$, where the infimum is taken over all possible representations of T. In fact this is a well-known relation between $\|T\|_{cb}$ and the Haagerup norm on the tensor product $A \otimes A$ for these examples (see [2, 9]). The T_n constructed as above from an orthonormal basis $(e_i)_{i=1}^\infty$ have $\sup_n \|T_n\|_{cb} < \infty$ while $\|T_n^f\|$, and hence $\|T_n^f\|_{cb}$, is unbounded.

Further, T_n is actually completely positive (see [6] for the definition) and we could even modify T_n to be unital by adding a term $(\mathrm{id} - \pi_n)a(\mathrm{id} - \pi_n)$.

Remark 2. In the case of general C^*-algebras, there are results on elementary operators where subhomogeneous C^*-algebras behave as though they were finite dimensional. At the opposite extreme, antiliminal C^*-algebras sometimes have the same behaviour. (See for example [1, 10].)

Subhomogeneous C^*-algebras are those where there is a finite bound on the dimensions of irreducible $*$-representations. Write A for the algebra and $\hat{A}$ for the class of all inequivalent irreducible representations $\pi\colon A \to \mathcal{B}(H_\pi)$ of A. Then A subhomogeneous means that $\sup_{\pi \in \hat{A}} \dim H_\pi = k < \infty$. (We call A k-subhomogeneous when we want to refer to an upper bound for the dimensions.) Abelian A are the 1-subhomogeneous C^*-algebras and of course then the flip map is the identity on $\mathcal{E}\ell(A)$.

The atomic representation of A (see [7, 4.3.7]) is given by

$$\pi_a \colon A \to \bigoplus_{\pi \in \hat{A}} \mathcal{B}(H_\pi) : a \mapsto (\pi(a))_{\pi \in \hat{A}}.$$

Here $\bigoplus_{\pi \in \hat{A}} \mathcal{B}(H_\pi)$ means the ℓ^∞ sum and it is a dual C^*-algebra in a natural way. Hence it is a von Neumann or W^*-algebra. The range of π_a is weak*-dense in $\mathcal{A} = \bigoplus_{\pi \in \hat{A}} \mathcal{B}(H_\pi)$, and there is a simple way to extend any $T \in \mathcal{E}\ell(A)$ to an elementary operator on $\mathcal{A}$. This extension is the unique weak*-continuous $T_a \colon \mathcal{A} \to \mathcal{A}$ satisfying $T_a \circ \pi_a = \pi_a \circ T$.

From this perspective, one can see that an elementary operator $T \in \mathcal{E}\ell(A)$ given by (1) is zero if and only if

$$\sum_{j=1}^{\ell} \pi(a_j) \otimes \pi(b_j) = 0 \in \mathcal{B}(H_\pi) \otimes \mathcal{B}(H_\pi)$$

for each $\pi \in \hat{A}$. From this we can deduce that T^f is well-defined (independent of the representation (1) of T) when A is a C^*-algebra, even though A may not be ultraprime.

We can pass to T_a from $T \in \mathcal{E}\ell(A)$ of the form (1) and note that $(T^f)_a = (T_a)^f$. Also $\|T\|$ is the supremum over $\pi \in \hat{A}$ of the norms of the elementary operators

$$x \in \mathcal{B}(H_\pi) \mapsto \sum_{j=1}^{\ell} \pi(a_j) x \pi(b_j).$$

We can apply finite-dimensional estimates (depending on the dimension) to deduce that $\|T^f\| \leq C_k \|T\|$ when A is k-subhomogeneous.

The simplest example of an antiliminal C^*-algebra is the so-called Calkin algebra $A = \mathcal{B}(\ell^2)/\mathcal{K}(\ell^2)$. Our examples T_n are built using finite rank a_i and b_i and so they vanish when quotiented by the compact operators. However, we can replace them with rather similar infinite multiplicity examples that will survive the quotient map. One way to do that is to note that ℓ^2 is unitarily equivalent to the Hilbert space tensor product $\ell^2 \otimes_2 \ell^2$. We can take

$$S_n a = \sum_{i=1}^{n} (E_{i1} \otimes \mathrm{id}_{\ell^2}) a (E_{1i} \otimes \mathrm{id}_{\ell^2})$$

and show via a similar computation to that used for T_n that even on the Calkin algebra, $\sup_n \|S_n\|_{cb} < \infty$ while $\|S_n^f\|$ is unbounded.

Question 1. *For which Banach spaces E does the conclusion of Theorem 1 hold for all algebras of operators on E that contain the finite ranks?*

Theorem 1 covers quite a few common spaces, but not all E covered by the result of [8]. Moreover ℓ^∞ fails to satisfy the hypotheses of Theorem 1, but the proof can be carried through in that case.

Question 2. *Is the flip $T \mapsto T^f$ unbounded on $\mathcal{E}\ell(A)$ for all non-subhomogeneous C^*-algebras A?*

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Editorial Note. An argument similar to Theorem 1 for the case of Hilbert space was provided by A. W. Wickstead (private communication).