Irish Math. Soc. Bulletin **45** *(2000), 135–153*

Polynomial Hulls of Smooth Discs: A Survey

ALEJANDRO SANABRIA-GARCÍA

1. Introduction

Let Ω be an open set in $\mathbb{C}^n$ and denote by $\mathcal{O}(\Omega)$ the space of holomorphic functions on Ω. Given a compact set $K \subset \Omega$, the *holomorphically convex hull* $\hat{K}_{\mathcal{O}(\Omega)}$ in Ω of K is defined by

$$\hat{K}_{\mathcal{O}(\Omega)} = \{z \in \mathbb{C}^n \colon |f(z)| \leq \sup_K |f|, \quad \text{for all } f \in \mathcal{O}(\Omega)\}.$$

The compact set K is called *holomorphically convex relative to* Ω if $\hat{K}_{\mathcal{O}(\Omega)} = K$, and the open set Ω is *holomorphically convex* if for every compact set $K \subset \Omega$, $\hat{K}_{\mathcal{O}(\Omega)}$ is a relatively compact subset of Ω. Holomorphically convex sets play a fundamental role in function theory of several complex variables [15, 20, 28].

Definition 1.1. Let K be a compact set in $\mathbb{C}^n$. The *polynomially convex hull* $\hat{K}$ of K is the set

$$\{z \in \mathbb{C}^n \mid |p(z)| \leq \sup_K |p|, \, p \text{ holomorphic polynomial}\}.$$

The compact set K is called *polynomially convex* if $\hat{K} = K$.

We remark that as a consequence of Taylor's Theorem, polynomially convex sets are the compact sets that are holomorphically convex relative to the ambient space $\mathbb{C}^n$.

The problem of deciding whether a compact set in $\mathbb{C}^n$ is polynomially convex or not is a fundamental and difficult problem in complex analysis.

We note that polynomial convexity is a global condition. A closed subset $E \subset \mathbb{C}^n$ is called *locally polynomially convex* at $z \in E$ if there exists $r > 0$ such that $E \cap \operatorname{clos} \mathbb{B}(z, r)$ is polynomially convex, where

Supported by a grant of the Gobierno Autónomo de Canarias.

$\mathbb{B}(z,r)$ denotes the open ball of center z and radius r. Clearly if the compact set K is polynomially convex then it is locally polynomially convex everywhere, but in general the converse is false.

Remark 1.2. The *rationally convex hull* $\hat{K}_{R(K)}$ of a compact set $K \subset \mathbb{C}^n$ is defined in the same way as $\hat{K}$ but replacing the polynomials by rational functions with poles off K. Again K is said to be *rationally convex* if $\hat{K}_{R(K)} = K$. Both sets $\hat{K}$ and $\hat{K}_{R(K)}$ are compact as well.

One of the motivations for the study of polynomial convexity comes from approximation theory. In the complex plane the well-known Runge's Theorem asserts that whenever the compact set $K \subset \mathbb{C}$ has the property that $\mathbb{C} \setminus K$ is connected, a holomorphic function in a neighbourhood of K is the uniform limit of a sequence of polynomials in the coordinate z, so a topological condition on K is sufficient for the approximation problem. It is known that such analogous topological restriction is not valid in $\mathbb{C}^n$ for $n > 1$. In fact, for $K \subset \mathbb{C}$, $\mathbb{C} \setminus K$ is connected if and only if K is polynomially convex [1, Lemma 7.2, p. 37]. In terms of polynomial convexity Runge's Theorem admits the following generalization to several complex variables.

The Oka-Weil Theorem. [1, 11] *Let X be a compact, polynomially convex set in $\mathbb{C}^n$. Then for every function f holomorphic in some neighbourhood of X, we can find a sequence $\{p_j\}$ of polynomials in $z_1, \ldots, z_n$ with $p_j \longrightarrow f$ uniformly on X.*

Polynomially and rationally convex hulls of compact sets in $\mathbb{C}^n$ appear naturally in the context of uniform algebras. Given a compact set $K \subset \mathbb{C}^n$, denote by $C(K)$ the algebra of continuous functions on K under the uniform norm and consider the subalgebras $P(K)$ (respectively, $R(K)$) of $C(K)$ of uniform limits of polynomials in the coordinates (respectively, rational functions with poles off K). The inclusions

$$P(X) \subseteq R(K) \subseteq C(K)$$

are all evident and one of the major problems in the theory is to determine when equality holds between them. Via Gelfand's Theory of commutative Banach Algebras, the maximal ideal space of $P(K)$ (respectively $R(K)$) can be identified with the polynomially convex hull $\hat{K}$ of K (respectively $\hat{K}_{R(K)}$), so the equality $P(K) = C(K)$ implies the polynomial convexity of K (similarly for $R(K) = C(K)$)

while the converse is not true in general. A consequence of the Oka-Weil Theorem is that $P(K) = R(K)$ if and only if $\hat{K} = \hat{K}_{R(K)}$.

For a compact set $K \subset \mathbb{C}^n$ the chain of inclusions

$$K \subseteq \hat{K}_{R(K)} \subseteq \hat{K}$$

is always true, and in general these inclusions are proper. In fact, the last assertion is exemplified by "topologically trivial" sets in $\mathbb{C}^2$ such as topological discs, i.e. homeomorphic copies of the unit disc $\mathbb{D} \subset \mathbb{C}$ (see below).

The following two general criteria concerning rational and polynomial convexity are due to Oka [29]. The general idea of these criteria is that for a compact set to be rationally convex it is enough to have an algebraic hypersurface avoiding the set, while for polynomial convexity a deformation of this algebraic hypersurface towards infinity is also required. More precisely,

Criterion 1. *The compact set $K \subset \mathbb{C}^n$ is rationally convex if and only if for every $y \in \mathbb{C}^n \setminus K$ there exists a polynomial p, such that $p(y) = 0$ and $p(z) \neq 0$ for all $z \in K$.*

Criterion 2. *The compact set $K \subset \mathbb{C}^n$ is polynomially convex if and only if for every $y \in \mathbb{C}^n \setminus K$ there exists a continuous family of algebraic hypersurfaces $p_t(z) = 0$, $t \in [0, \infty)$ such that:*

i) *$p_0(y) = 0$;*
ii) *$p_t(z) \neq 0$ for every $z \in K$;*
iii) *the distance from a fixed point of $\mathbb{C}^n$ to the curve $\{z \in \mathbb{C}^n : p_t(z) = 0\}$ tends to infinity as t goes to infinity.*

The following result is due to Mergelyan [21].

Theorem 1.3. *Let D be a closed disc in $\mathbb{C}$ and let f be a real-valued function in $C(D)$. If for each a in D, $f^{-1}(f(a))$ has no interior and does not separate $\mathbb{C}$, then $[z, f; D] = C(D)$, where $[z, f; D]$ denotes the algebra generated by the functions z and f with complex coefficients.*

Mergelyan's result has interest in itself and is not directly related to the theory of polynomial convexity, but a corollary of his result is that under the hypothesis of the theorem the topological disc

$$M = \{(z, f(z)) \in \mathbb{C}^2 : z \in D\}$$

is polynomially convex.

By an *analytic disc* with boundary in the compact set $K \subset \mathbb{C}^n$ we mean a mapping ϕ: clos $\mathbb{D} \longrightarrow \mathbb{C}^n$, $\phi \in C(\text{clos } \mathbb{D}) \cap \mathcal{O}(\mathbb{D})$ such that $\phi(\text{bdy } \mathbb{D}) \subset K$. An obstruction to either polynomial convexity is, by virtue of the maximum modulus principle, the existence of analytic discs attached to K, i.e. analytic discs with boundaries in K. During the 1950's the question was asked, whether the failure of polynomial convexity of a compact set in $\mathbb{C}^n$ could be explained by means of analytic discs attached to K, or in general by analytic varieties of positive dimension attached to the set. Stolzenberg gave an example of a non-polynomially convex compact set in $\mathbb{C}^2$ having no analytic varieties contained in its hull (cf. [1, Chapter 24, p. 207] for a modification, due to Wermer, of Stolzenberg's example).

A satisfactory description of polynomial hulls is known for rectifiable curves in $\mathbb{C}^n$ (see [1] and the references therein). Among the class of compact sets in $\mathbb{C}^n$, perhaps the next simplest ones to study are topological discs. In this note we will try to survey some of the results known for topological discs in $\mathbb{C}^2$. These are almost all local results.

2. Totally Real Submanifolds

Let M be a C^k manifold in $\mathbb{C}^n$ with $k \geq 1$, and let $T_p(M)$ be the tangent space to M at the point p viewed as a real-linear subspace of $\mathbb{C}^n$. Associated to the standard complex structure of $\mathbb{C}^n$, i.e. multiplication by i, the notion of a complex tangent space can be defined as follows.

Definition 2.1. Let M be a C^k manifold in $\mathbb{C}^n$ with $k \geq 1$. The *complex tangent space* $T_p^{\mathbb{C}}(M)$ of M at the point p is defined by

$$T_p^{\mathbb{C}}(M) = T_p(M) \cap i\, T_p(M).$$

Note that if the dimension of M (in the real sense) is bigger than n, the complex tangent space to M at the point p is nontrivial.

The manifold M is said to be *totally real* at the point p if the complex tangent space $T_p(M)$ is trivial. If $T_p(M)$ has complex dimension at least 1, the manifold M is said to *have a complex tangent* at the point p.

Let M be a C^k surface in $\mathbb{C}^2$ with $k \geq 1$, and suppose M is totally real at the point p. After a linear change of coordinates we can suppose that p is the origin in $\mathbb{C}^2$ and M is locally parametrized by the graph

$$z_2 = f(z_1)$$

where f is a C^k complex-valued differentiable function defined in a neighbourhood D of $0 \in \mathbb{C}$ such that $f(0) = 0$ and $f_z(0) = 0$. Hence locally M is represented by

$$z_2 = f_{\bar{z}}(0)\bar{z} + g(z)$$

where $g \in C^k(D)$, $g(0) = 0$, $g_z(0) = g_{\bar{z}}(0) = 0$. The condition on M of being totally real at p means that $f_{\bar{z}}(0) \neq 0$ (otherwise the tangent space reduces to the z_1-axis, a complex line), so without loss of generality M is given near p by the graph

$$M = \{(z, f(z)) \in \mathbb{C}^2 \colon z \in D\} \tag{2.1}$$

where $f(z) = \bar{z} + g(z)$ and $g \in C^k(D)$ is as above.

The local polynomial convexity of M near a totally real point is a consequence now of the following theorem of Wermer [31].

Theorem 2.2. *Let D be a closed disc in the complex plane and $f(z) = \bar{z} + R(z)$ where $R \in C(D)$ satisfies the Lipschitz condition $|R(s) - R(t)| \leq k|s - t|$ for all $s, t \in D$, and $k < 1$. Then $[z, \bar{z} + R; D] = C(D)$.*

Remark 2.3. Wermer's result is sharp in the sense that k must be strictly less than one. A counterexample is given by $R(z) = -\bar{z}$. We mention that under the additional hypothesis of $f(z) = \bar{z} + R(z)$ being locally injective, the conclusion of Theorem 2.2 is still valid for $k \leq 1$ [27, Corollary 1.8, p. 451].

Corollary 2.4. *Suppose $f \in C^1(D)$, where D is a neighbourhood of $0 \in \mathbb{C}$, such that $f_{\bar{z}}(0) \neq 0$. Then there exists a closed disc D_0 about 0 for which $[z, f; D_0] = C(D_0)$.*

Clearly as a consequence of Corollary 2.4, the graph (2.1) is polynomially convex providing D is small enough. Therefore the discussion above leads to the following theorem.

Theorem 2.5. *Let $M \subset \mathbb{C}^2$ be a totally real surface. Then M is locally polynomially convex.*

Remark 2.6. Theorem 2.2 admits a generalization to $\mathbb{C}^n$ [14]. By a similar argument it can be proven that a totally real manifold in $\mathbb{C}^n$ is locally polynomially convex.

By the work of Hörmander-Wermer [14] and Harvey-Wells [13] a totally real manifold in $\mathbb{C}^n$ is holomorphically convex relative to some holomorphically convex neighbourhood. The next examples give topological discs in $\mathbb{C}^2$ which are totally real and hence holomorphically convex in this sense but are not polynomially convex (or even rationally convex).

Example 2.7. [14, Example 6.1, p. 20] Let M be the topological disc in $\mathbb{C}^2$

$$M = \{(z, f(z)) \in \mathbb{C}^2 \colon z \in \operatorname{clos} \mathbb{D}\},$$

where $f(z) = -(1+i)\bar{z} + iz\bar{z}^2 + z^2\bar{z}^3$. Then M has a complex tangent if and only if $f_{\bar{z}}(z) = 0$, but $f_{\bar{z}}(z) = -(1+i) + 2i|z|^2 + 3|z|^4$ which never vanishes, so M is totally real. Since $f(\mathrm{e}^{i\theta}) = 0$ for $\theta \in [-\pi, \pi)$, we can attach the analytic disc $\{(z, 0) \colon z \in \operatorname{clos} \mathbb{D}\}$ to M. The maximum modulus principle now implies that M is not polynomially convex.

The following example is a variant of Hörmander's and Wermer's example given by Duval and Sibony [6, Example 4, p. 54]

Example 2.8. Let M be the surface

$$M = \{(z, \bar{z}f(|z|^2)) \in \mathbb{C}^2 \colon z \in \operatorname{clos} \mathbb{D}\}$$

where $f \colon [0,1] \longrightarrow \mathbb{C}$ is a C^1 function. Let $\gamma \colon [0,1] \longrightarrow \mathbb{C}$ be defined by $\gamma(t) = tf(t)$, and suppose that γ has a double point, i.e. $t_1 f(t_1) = t_2 f(t_2) = a$ with $0 < t_1 < t_2 \le 1$ and γ is an immersion. Then M is totally real and the complex variety $\{zw = a\}$ intersects M along two circles which bound a surface on M. Hence M is not polynomially convex.

A smooth surface $M \subset \mathbb{C}^2$ of real dimension two is called *Lagrangian* if the standard symplectic 2-form $\omega = dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2$ vanishes on the second exterior product $\bigwedge_2 T_p(M)$ of the tangent space to M at each point $p \in M$. Equivalently, $T_p(M)$ is $\mathbb{R}$-orthogonal to $i\,T_p(M)$. Duval in [4] (see [6] as well for an explanation based on the concept of currents) gave an example of a topological disc in $\mathbb{C}^2$ which is Lagrangian but not polynomially convex. Since it is Lagrangian, it is totally real, and according to the main result of his paper, such a disc is rationally convex.

2.1. **Global results.** The examples in the last section show that some extra conditions must be imposed on a totally real disc in $\mathbb{C}^2$ to make it polynomially convex. If the function f in Wermer's Theorem 2.2 is of class C^1 then it is locally direction-reversing, i.e. $|f_{\bar{z}}(a)| > |f_z(a)|$ for every a in the domain of f. Preskenis (cf. [27]) conjectured that this condition is sufficient for the polynomial convexity of the totally real disc defined by f. This conjecture was eventually proven, at the end of a chain of results. We give these results chronologically, reformulating them in some cases to expose the polynomially convex aspect of the theorems. We mention that these results are among the few global results for discs.

In the paper mentioned above, Preskenis was able to prove the following result.

Theorem 2.9. *Let f be a C^1 complex-valued differentiable function in a neighbourhood of* clos $\mathbb{D}$, *and assume $|f_{\bar{z}}| > |f_z|$ everywhere. Then the topological disc $M = \{(z, f(z)) \in \mathbb{C}^2 \colon z \in \text{clos } \mathbb{D}\}$ is rationally convex.*

Under the additional hypothesis of f being a diffeomorphism, O'Farrell and Preskenis [24] proved

Theorem 2.10. *Let f be a diffeomorphism of $\mathbb{C}$ into $\mathbb{C}$ having degree -1. Then the topological disc*

$$M = \{(z, f(z)) \in \mathbb{C}^2 \colon z \in \text{clos } \mathbb{D}\}$$

is polynomially convex.

Conditions on the function f were relaxed by Duval [3] and by O'Farrell-Preskenis [25] independently, solving the original conjecture:

Theorem 2.11. *Let f be a C^1 complex-valued function defined in a neighbourhood of* clos $\mathbb{D}$ *and let M be the topological disc $M = \{(z, f(z)) \in \mathbb{C}^2 \colon z \in \text{clos } \mathbb{D}\}$. If f is locally direction-reversing at every point of $\mathbb{D}$, then M is polynomially convex.*

Theorem 2.12. *Suppose $f \colon \mathbb{C} \longrightarrow \mathbb{C}$ is proper. Suppose $E \subset \mathbb{C}$ is closed,* int $E = \emptyset$, *and $\mathbb{C} \setminus E$ has no bounded component. Suppose $0 < r \in \mathbb{Z}$, and for each $a \in \mathbb{C} \setminus E$ we have $f^{-1}f(a) \subset \mathbb{C} \setminus E$ and $\#f^{-1}(a) = r$. Suppose that on $\mathbb{C} \setminus E$, f is C^1, nonsingular, and locally direction-reversing. Then the topological disc $M = \{(z, f(z)) \in \mathbb{C}^2 \colon z \in \text{clos } \mathbb{D}\}$ is polynomially convex.*

A common feature in these results is the systematic use of Oka's criteria 1 and 2 mentioned in the introduction. On the other hand, using a result of Range and Siu (see [24] for example), Theorems 2.10, 2.11 and 2.12 yield as a consequence that the algebra $[f, g; \text{clos}\ \mathbb{D}]$ is dense in $C(\text{clos}\ \mathbb{D})$.

Duval's Theorem 2.11 solves Preskenis' conjecture, while in Theorem 2.12 a small set of singularities for f is allowed. Note that Theorem 2.10 is a particular case of Theorem 2.12 when $r = 1$ and $E = \emptyset$. An open problem in this area is whether the condition of f being a diffeomorphism in Theorem 2.10 could be replaced by just a homeomorphism of degree -1. This problem was first posed by Browder.

3. Complex Tangents

Let $M \subset \mathbb{C}^2$ be a C^k surface, $k \geq 2$. In the previous section we have seen that M is locally polynomially convex whenever M is totally real. We consider now the case where M has a complex tangent. Fundamentally, all results of this kind are of local character.

First we give a *canonical form* for M in a neighbourhood of $p \in M$. By a similar argument as in section 2, M is locally parametrized by the graph

$$z_2 = f(z_1)$$

where f is a C^k complex-valued differentiable function defined in a neighbourhood D of $0 \in \mathbb{C}$ such that $f(0) = 0$ and $f_z(0) = 0$. The surface M has a complex tangent at p if and only if $f_{\bar{z}}(0) = 0$, so under this hypothesis M can be given by

$$z_2 = az_1^2 + b\bar{z}_1^2 + cz_1\bar{z}_1 + H(z_1, \bar{z}_1) \tag{3.1}$$

where H vanishes to third order at $z_1 = 0$. We make the non-degeneracy assumption that $c \neq 0$, so we can suppose without loss of generality that $c = 1$. A rotation in the z_1-plane of the form $z_1 \longmapsto z_1 \mathrm{e}^{i\theta}$ where θ is chosen so that $b\, \mathrm{e}^{-2i\theta} \geq 0$ allows us to write

$$z_2 = \alpha z_1^2 + \gamma\bar{z}_1^2 + z_1\bar{z}_1 + H(z_1, \bar{z}_1)$$

where $\gamma \geq 0$. Finally, under a quadratic coordinate change of the form $(z_1, z_2) \longmapsto (z_1, z_2 + (\alpha - \gamma)z_1^2)$, the surface M can be locally given near 0 by the equation

$$z_2 = \gamma z_1^2 + z_1\bar{z}_1 + \gamma\bar{z}_1^2 + H(z_1, \bar{z}_1)\ . \tag{3.2}$$

The real number $\gamma \geq 0$ is a biholomorphic invariant of M first considered by Bishop [2] and it is known as *Bishop's invariant* of the surface.

Definition 3.1. Let $M \subset \mathbb{C}^2$ be a smooth surface having a complex tangent at the point p. We say that the surface M has an *elliptic complex tangent* at the point p or that p is an *elliptic point* (respectively *parabolic*, *hyperbolic*) if the Bishop's invariant γ for the point p satisfies $0 \leq \gamma < 1/2$ (respectively $\gamma = 1/2$, $\gamma > 1/2$).

Remark 3.2. A similar argument to that given above shows that in the degenerate case $c = 0$ in (3.1) with a and b nonzero, the surface M can be given near the origin by

$$z_2 = z_1^2 + \bar{z}_1^2 + H(z_1, \bar{z}_1)$$

where H vanishes to third order at $z_1 = 0$. This case is sometimes referred to as the hyperbolic case $\gamma = \infty$.

We mention that the quadratic form in (3.2) has two different eigenvalues of the same sign in the elliptic case, one eigenvalue of geometric multiplicity two in the parabolic case and two different eigenvalues of different sign in the hyperbolic case.

Complex tangents of surfaces of $\mathbb{C}^2$ are generically of elliptic or hyperbolic type, which are automatically isolated. The parabolic case (which can be non-isolated) is a non-generic case, but could be thought of as a bifurcation case of elliptic points into hyperbolic or *vice-versa*.

We will see that elliptic points have a non-trivial hull of holomorphy, hyperbolic points have a trivial hull while isolated parabolic points can have both. The situation when a complex tangent is degenerate, i.e. the tangent plane has a contact of order bigger than two with the surface is quite different, as is shown by the following example of Forstnerič [8].

Example 3.3. Let $g\colon [0,\infty) \longrightarrow \mathbb{R}$ be a smooth function with a sequence of simple zeros $a_1 > a_2 > a_3 > \cdots > 0$ converging to 0 and with no other zeros (e.g. $g(t) = \mathrm{e}^{-1/t} \sin(1/t)$). Let

$$h(z) = \bar{z} g(|z|^2) \mathrm{e}^{i|z|^2},$$

and let M be the disc

$$M = \{(z, h(z)) \in \mathbb{C}^2 \colon z \in \operatorname{clos} \mathbb{D}\}.$$

The smooth disc M defined above, which has a contact of infinite order with its tangent plane at the origin satisfies:

i) M is totally real outside the origin,
ii) M is holomorphically convex, and
iii) M has no rationally convex neighbourhood of 0.

As a consequence, M is not locally polynomially convex or even rationally convex near the origin.

Further developments concerning degenerate complex tangents of surfaces in $\mathbb{C}^2$ have been studied by Wiegerinck [32].

3.1. **The elliptic case.** Let $M \subset \mathbb{C}^2$ be a smooth surface having an isolated elliptic complex tangent at the point p. Locally near p, M is given by the equation (3.2) with $0 \leq \gamma < 1/2$. Note that without the perturbation term $H(z, \bar{z})$, the surface M corresponds to the real quadric

$$z_2 = \gamma(z_1^2 + \bar{z}_1^2) + z_1\bar{z}_1,$$

so we see that there exists a one-parameter family of analytic discs attached to M, each of which is bounded by one of the ellipses $\gamma(z_1^2 + \bar{z}_1^2) + z_1\bar{z}_1 = c > 0$.

Bishop introduced a method in [2] to find the desired analytic discs attached to the surface in general. This method reduces to the study of a non-linear functional equation on bdy $\mathbb{D}$ which has the form

$$u(\mathrm{e}^{i\theta}) = c - T(H(c, \theta, u, T(u)))$$

where c is a real parameter, T the harmonic conjugate operator on bdy $\mathbb{D}$ and $H = O(t^2 + u^2 + (T(u))^2)$. The equation above is known as the *Bishop equation*, and fixed points of this equation give the corresponding analytic discs attached to the surface. Bishop, working on the Sobolev space $W^{1,2}(\text{bdy } \mathbb{D})$ found non-trivial solutions to the equation by means of a Picard iteration procedure analogous to the contraction mapping principle. Bishop's result can be summarized in the following theorem.

Theorem 3.4. *Let $M \subset \mathbb{C}^2$ be a smooth surface having an elliptic complex tangent at the point p. Then there exists a one parameter family of analytic discs attached to M.*

As a consequence of the theorem, M has a non-trivial local polynomial hull near the point p. In the same paper he conjectured that these discs give the full local hull of holomorphy and asked as well for the fine structure of the local hull.

Further progress concerning regularity properties of the attached discs was made by Hunt and by Bedford and Gaveau (see [19] and the references therein). By a refinement of the original Bishop equation, Hunt proved that the union of the analytic discs is a 3-manifold of class C^l for any $l < \infty$, while Bedford and Gaveau showed that the local hull of holomorphy is a C^∞ manifold and extends smoothly to points of bdy $M \setminus \{p\}$.

The question of C^∞ regularity near the elliptic point was settled by Kenig and Webster in their deep work [19].

Theorem 3.5. *Let $M \subset \mathbb{C}^2$ be a C^∞ smooth surface with an elliptic complex tangent at a point p. Then there exists a smooth one parameter family of disjoint regularly embedded discs with boundaries on M and converging to p. These discs form a C^∞ 3-manifold $\tilde{M}$ with boundary M in a neighbourhood of p. There are no other analytic discs in $\mathbb{C}^2$ with boundaries on M near p. In particular the corresponding boundary curves are disjoint and fill up a deleted neighbourhood of p in M.*

The discs produced by Kenig and Webster, as in Bishop's approach, are solutions of a non-linear functional equation involving the conjugate operator or Hilbert transform. A fine analysis of the Hilbert transform on variable curves allows them to solve the functional equation via the implicit function theorem. This procedure is not so constructive as the one employed by Bishop.

Theorem 3.5 closed the question of C^∞ regularity of the local hull near an elliptic point. The problem of C^ω regularity was considered by Moser and Webster in the beautiful paper [22]. In this paper much more than the C^ω regularity of the local hull of holomorphy is achieved, namely the local biholomorphic classification of real analytic surfaces in $\mathbb{C}^2$ with an elliptic complex tangent with Bishop's invariant $0 < \gamma < 1/2$.

Theorem 3.6. *Assume M is a real-analytic surface in $\mathbb{C}^2$ with an elliptic complex tangent at a point p with $0 < \gamma < 1/2$. Then there exists a holomorphic coordinate system (z_1, z_2) in which $p = 0$, and M has locally the form*

$$\begin{aligned} x_2 &= z_1\bar{z}_1 + \Gamma(x_2)(z_1^2 + \bar{z}_1^2) \\ y_2 &= 0 \end{aligned} \tag{3.3}$$

where $z_2 = x_2 + iy_2$, $\Gamma(x_2) = \gamma + \delta x_2^s$, $\delta = \pm 1$, $s \in \mathbb{Z}^+$, *or* $\Gamma(x_2) = \gamma$ *in the case of* $s = \infty$. *The quantities* γ, δ, s *form a complete system of biholomorphic invariants for* M *near* p.

Theorem 3.6 admits as an immediate corollary that the local hull of holomorphy of M near p is precisely the real-analytic 3-manifold with boundary

$$\tilde{M} \colon x_2 \geq z_1\bar{z}_1 + \Gamma(x_2)(z_1^2 + \bar{z}_1^2),\ y_2 = 0.$$

$\tilde{M}$ is the union of a one parameter family of ellipses, the boundaries of which are the curves on M obtained by setting $x_2 = c > 0$. Observe as well that $\tilde{M}$ is a Levi-flat 3-manifold.

The idea of the proof of the last theorem differs from the previous ones in that the analytic discs attached to the surface are not solutions of a non-linear functional equation, but are orbits of a complex holomorphic flow.

Remark 3.7. The category of formal surfaces and formal power-series transformations provides a coarser classification of real-analytic surfaces. In this formal sense, the normal form (3.3) is still valid for the hyperbolic case $\gamma > 1/2$ except for a countable set of *exceptional values*. Exceptional values are defined by the following property. Let λ be a root of the quadratic equation $\lambda^2 + (1/\gamma)\lambda + 1 = 0$. Then $|\lambda| = 1$ for $1/2 < \gamma < \infty$, and γ is said to be an exceptional value if λ is a root of unity. In general one expects divergence of the normal form in the hyperbolic case. In [22] the following two examples are given:

i) For $\gamma = 1$, the surface $z_2 = z_1^2 + z_1\bar{z}_1 + \bar{z}_1^2 + z_1\bar{z}_1(z_1 - \bar{z}_1)$ cannot be flattened (even formally) to third order.
ii) If $1/2 < \kappa < \infty$, then the hyperbolic surface $z_2 = z_1\bar{z}_1 + \kappa\bar{z}_1^2 + \kappa z_1^3\bar{z}_1$ cannot be transformed to a real hyperplane by means of a convergent biholomorphic transformation.

We mention that recently, Gong [12] has proved that for each non-exceptional hyperbolic Bishop invariant (in the above sense) there exists a real-analytic hypersurface of the form (3.2) which can be transformed biholomorphically into a subset of a real hyperplane in $\mathbb{C}^2$, but not into the Moser-Webster normal form.

Finally the remaining case $\gamma = 0$ has been studied by Moser [23] and by Huang and Krantz [16]. In [23] Moser shows that under a formal transformation the surface (3.2) with $\gamma = 0$ can be viewed as

$$z_2 = z_1\bar{z}_1 + \varphi(z) + \bar{\varphi}(\bar{z}),$$

where $\varphi(z) = \sum_{k\geq s} c_k z^k$, $c_s \neq 0$ and $s \geq 3$. By definition if $\varphi \equiv 0$ then $s = \infty$. Under the transformation $(z_1, z_2) \longmapsto (\lambda z_1, |\lambda|^2 z_2)$, $\lambda \neq 0$, the surface is given by

$$z_2 = z_1\bar{z}_1 + z_1^s + \bar{z}_1^s + \psi(z) + \bar{\psi}(\bar{z}) \tag{3.4}$$

with ψ containing terms of order bigger than s only, or

$$z_2 = z_1\bar{z}_1 \tag{3.5}$$

in the case $s = \infty$. Note that because s is a biholomorphic invariant, the surface (3.4) cannot be reduced to (3.5) by a biholomorphic transformation. Moser in the paper mentioned above considered only the case $\gamma = 0$ and $s = \infty$, which in general is an exceptional case. By means of a rapidly convergent iteration scheme he proved the following result.

Theorem 3.8. *Let M be a real-analytic surface in $\mathbb{C}^2$ having an elliptic complex tangent at the point p, with $\gamma = 0$ and $s = \infty$. Then locally M is biholomorphically equivalent to $z_2 = z_1\bar{z}_1$.*

As a consequence the local hull of holomorphy is a real-analytic 3-manifold with boundary given by $x_2 \geq z_1\bar{z}_1$, $y_2 = 0$, where $z_2 = x_2 + iy_2$. We point out that if the surface M can be formally transformed into the quadric (3.5) then by virtue of Theorem 3.8 the transformation is holomorphic.

Finally, by a similar method to the one employed by Kenig and Webster, the real-analyticity of the local hull across the boundary in the case where $\gamma = 0$ and $s < \infty$ was sorted out by Huang and Krantz. These results together with the Moser-Webster Theorem can be summarized as follows.

Theorem 3.9. *Let M be a real-analytic surface in $\mathbb{C}^2$ having an elliptic complex tangent at the point p. Then the local hull of holomorphy of M near p is a Levi-flat hypersurface which is real-analytic across the boundary manifold M.*

Moser had already noted that if the local hull was real-analytic then the formal transformation into the normal form (3.4) would be holomorphic. Therefore as a consequence of Theorem 3.9 we have the following corollary.

Corollary 3.10. *Let M be a real-analytic surface in $\mathbb{C}^2$ with an elliptic complex tangent at the point p with $\gamma = 0$. Then there exist a holomorphic coordinate system (z_1, z_2) in which $p = 0$ and M has locally the form*

$$z_2 = z_1\bar{z}_1 + z_1^s + \bar{z}_1^s + \sum_{i+j>s} a_{ij} z_1^i \bar{z}_1^j$$

with $\bar{a}_{ij} = a_{ji}$, and $s \geq 3$.

We note that the set of coefficients $\{a_{ij}\}$ is not a full set of holomorphic invariants of the surface. One does not know the biholomorphic classification in the case $\gamma = 0$ and $s < \infty$.

Remark 3.11. A complete study of the local hull of holomorphy of a manifold of real dimension n in $\mathbb{C}^n$ near an elliptic complex point has been carried out recently by Huang [17].

3.2. **The hyperbolic case.** Let M be a surface in $\mathbb{C}^2$ having a hyperbolic complex tangent at the origin in $\mathbb{C}^2$. We have seen that locally M is given by the graph

$$M = \{(z, f(z)) \in \mathbb{C}^2 \colon z \in D\}$$

where D is a small closed neighbourhood of the origin in $\mathbb{C}$, and $f(z) = q(z) + o(|z|^2)$ with $q(z) = \gamma z^2 + z\bar{z} + \gamma\bar{z}^2$ and $\gamma > 1/2$.

In connection with Mergelyan's Theorem mentioned in the Introduction, M. Freeman [10] was one of the first to study surfaces in $\mathbb{C}^2$ with hyperbolic points under a certain additional hypothesis.

Theorem 3.12. *If f is a twice continuously differentiable complex-valued function on a neighbourhood U of the origin in $\mathbb{C}$ such that $f(z) = q(z)+o(|z|^2)$ where q is a real-valued quadratic form with non-zero eigenvalues of opposite sign, and f has rank ≤ 1 near 0, then there exists a compact neighbourhood D of 0 such that any compact subset of the graph*

$$M = \{(z, f(z)) \in \mathbb{C}^2 \colon z \in D\}$$

is polynomially convex.

The rank condition is understood in relation to f as a map of $\mathbb{R}^2$ into $\mathbb{R}^2$ and simply means that the Jacobian determinant of f vanishes in a neighbourhood of 0. We note that the condition on the eigenvalues is equivalent to saying that 0 is a hyperbolic point in Bishop's sense as we remarked at the beginning of the section. Using a result of Wermer together with Theorem 3.12 we have the corollary:

Corollary 3.13. *Under the hypothesis of Theorem 3.12, there exists a compact neighbourhood D of zero such that $[z, f; D] = C(D)$.*

The general hyperbolic case was studied by Stout in [30] under the regularity hypothesis of a C^3 surface and finally published in [9] where the surface now is supposed to be of class C^2.

Theorem 3.14. *Let M be a C^2 surface in $\mathbb{C}^2$ given by the equation*

$$z_2 = \gamma z_1^2 + z_1\bar{z}_1 + \gamma\bar{z}_1^2 + o(|z|^2)$$

with $\gamma > 1/2$. Then M is locally polynomially convex near the origin in $\mathbb{C}^2$ and $[z_1, z_2; D] = C(D)$, whenever D is a sufficiently small neighbourhood of the origin in $\mathbb{C}$.

Two fundamental ideas are used in the proof of Theorem 3.14. The first is to consider a polynomial map from $\mathbb{C}^2$ into itself and pull-back the surface. In the new setting the pulled-back surface consists of the union of two new surfaces having an intersection at the origin. The second is that for the new surfaces it is much easier to establish their polynomial convexity, so an application of the celebrated Kallin's Lemma yields the polynomial convexity of the union of the surfaces. After that a standard argument leads to the conclusion. We refer to the recent survey of de Paepe [26] for the use of Eva Kallin's Lemma in polynomial convexity. The proof of the density of the algebra $[z_1, z_2; D]$ is proved by an *ad hoc* argument.

A different approach to the local polynomial convexity of surfaces with hyperbolic points is given by Duval in [5].

3.3. **The parabolic case.** An important notion related to exceptional points of totally real manifolds in $\mathbb{C}^n$ is that of the *(Maslov) index of the exceptional point*. In the particular case of a surface in $\mathbb{C}^2$ having isolated complex tangents, the definition of the index can be given as follows.

Definition 3.15. Let M be a smooth surface in $\mathbb{C}^2$ given locally by the graph

$$M = \{(z, f(z)) \in \mathbb{C}^2 \colon z \in D\}$$

where D is a small neighbourhood of the origin in $\mathbb{C}$. Suppose M has an isolated complex tangent at $0 \in \mathbb{C}^2$. Then the *index* of the complex tangent point (i.e. the origin) is defined as the winding number of $f_{\bar{z}}(z)$ around 0.

We note that several different equivalent definitions of the index of an isolated exceptional point can be found in [7]. It is easy to see that elliptic points of surfaces in $\mathbb{C}^2$ have index $+1$ while hyperbolic points have index -1.

In what follows suppose M is a C^2 surface in $\mathbb{C}^2$ having an isolated complex point at the origin of parabolic type, i.e. M is locally given by

$$z_2 = \tfrac{1}{2}(z_1^2 + \bar{z}_1^2) + z_1\bar{z}_1 + o(|z_1|^2).$$

Under the transformation $(z_1, z_2) \longmapsto (iz_1, 2z_2)$, M is locally given by $z_2 = f(z_1)$, where

$$\begin{aligned} f(z_1) = & \quad \tfrac{1}{2}|z_1|^2 - \tfrac{1}{4}(z_1^2 + \bar{z}_1^2) + o(|z_1|^2) \\ = & \quad y_1^2 + o(|z_1|^2) \end{aligned} \tag{3.6}$$

and $z_1 = x_1 + iy_1$. The condition of having an isolated complex point at the origin means that $f_{\bar{z}_1}(z_1)$ has an isolated zero at the origin.

The index of an isolated parabolic point depends on the higher order terms. The following results are from [18].

Theorem 3.16. *Let M be a C^2 surface in $\mathbb{C}^2$ having an isolated parabolic point at the origin. The parabolic point may have index +1, 0 or -1. Isolated parabolic points of index +1 are locally non-polynomially convex, while if the index is -1 then the surface is locally polynomially convex.*

We note that the case of index $+1$ was considered by Wiegerinck [32, Corollary 3.2, p. 907] too, who showed that the local polynomially convex hull is formed by a Levi-flat hypersurface foliated by analytic discs with boundary in M.

In the particular case that the function $f(z)$ in (3.6) is real-valued a complete picture is contained in the next theorem [18, Lemma 3, p. 803]. First, we introduce the following notation. Given a compact set K contained in the C^2 boundary of a bounded strictly pseudoconvex domain Ω in $\mathbb{C}^2$, the *non-trivial* or *essential part* of the hull is defined by $\hat{K}_{\text{ess}} = \hat{K} \setminus K$, and its *trace* on the compact set K is defined by $K_{\text{tr}} = K \cap \operatorname{clos} \hat{K}_{\text{ess}}$.

Theorem 3.17. *If the index of the origin is 0 or -1 then the topological disc M defined by the real-valued function f is locally polynomially convex near the origin. If the index is +1 then the essential local hull is the union of analytic discs with boundary in M surrounding the origin.*

The general situation in the index 0 case is different as is shown by the following example.

Example 3.18. Consider the C^∞ function $g(x) = x\mathrm{e}^{-1/|x|}$ for $x \neq 0$ and $g(0) = 0$. Let U be a small neighbourhood of zero in $\mathbb{C}^2$. The C^∞ hypersurface $H = \{(z, w) \in U : \Re e\ w = y_1^2 + g(x_1)\}$ contains a C^2 disc for which zero is an isolated parabolic point of index zero with non-trivial local polynomial hull.

The description of the local polynomial hull of isolated parabolic points of index 0 is done in term of "onions".

Definition 3.19. Let M be an open C^2 disc in a strictly pseudoconvex boundary in $\mathbb{C}^2$ and $p \in M$. An *onion* $\mathcal{C}$ at the point p is a compact subset of M with p in its boundary such that

i) M has no complex tangents in int $\mathcal{C}$,
ii) bdy $\mathcal{C}$ is a rectifiable closed Jordan curve,
iii) $\mathcal{C}$ is the union of rectifiable closed Jordan curves Γ_α containing p, with $\Gamma_\alpha \setminus \{p\}$ pairwise disjoint and such that each Γ_α bounds a continuous analytic disc D_α, and
iv) the compact sets clos w_α on M bounded by the Γ_α are increasing, their union is $\mathcal{C}$, and their intersection is p.

Theorem 3.20. *Suppose M is a C^2 disc in $\mathbb{C}^2$ with the origin as an isolated complex point of parabolic type of index zero. Then either M is locally polynomially convex near zero or for any small closed disc K on M around* 0 *the trace of its essential hull on K, K_{tr} is a single onion.*

Remark 3.21. We note that generically isolated parabolic points are locally polynomially convex. A description of the local polynomial hull of parabolic points of index $+1$ is given as well in terms of onions. Several different structures can occur. We refer to [18] for further details.

The tools employed in the proof of the latter results are related to the bifurcation theory of a planar dynamical system associated with the complex point. It is not known whether non-isolated parabolic points of surfaces are locally polynomially convex or not.

Acknowledgements. I would like to thank Professor Anthony O'Farrell for his helpful advice, as well as the referee for multiple suggestions and corrections.

References

[1] H. Alexander and J. Wermer, *Several complex variables and Banach algebras*, 3rd ed., Springer-Verlag, New York, 1998.

[2] E. Bishop, *Differentiable manifolds in complex Euclidean space,* Duke Math. J. **32** (1965), 1–21.

[3] J. Duval, *Un exemple de disque polynômialement convexe*, Math. Ann. **281** (1988), 583–588.

[4] J. Duval, *Convexité rationnelle des surfaces lagrangiennes*, Invent. Math. **104** (1991), 581–599.

[5] J. Duval, *Surfaces convexes dans un bord pseudoconvexe*, Astérisque **217** (1993), 6, 103–118.

[6] J. Duval and N. Sibony, *Polynomial convexity, rational convexity, and currents*, Duke Math. J. **79** (1995), 487–513.

[7] F. Forstnerič, *Analytic disks with boundaries in a maximal real submanifold of* $\mathbb{C}^2$, Ann. Inst. Fourier (Grenoble) **37** (1987), 1–44.

[8] F. Forstnerič, *A smooth holomorphically convex disc in* $\mathbb{C}^2$ *that is not locally polynomially convex*, Proc. Amer. Math. Soc. **116** (1992), 411–415.

[9] F. Forstnerič and E.L. Stout, *A new class of polynomially convex sets*, Ark. Mat. **29** (1991), 51–62.

[10] M. Freeman, *The polynomial hull of a thin two-manifold*, Pacific J. Math. **38** (1971), 377–389.

[11] T. Gamelin, *Uniform Algebras*, Prentice-Hall, Englewood Cliffs, N. J., 1969.

[12] X. Gong, *Fixed points of elliptic reversible transformations with integrals*, Ergodic Theory Dynam. Systems **16** (1996), 683–702.

[13] F.R. Harvey and R.O. Wells, *Holomorphic approximation and hyperfunction theory on a* C^1 *totally real submanifold of a complex manifold*, Math. Ann. **197** (1972), 287–318.

[14] L. Hörmander and J. Wermer, *Uniform approximation on compact sets in* $\mathbb{C}^n$, Math. Scand. **23** (1968), 5–21.

[15] L. Hörmander, *An introduction to complex analysis in several variables*, 2nd ed., North-Holland Publishing Co., Amsterdam, 1973.

[16] X. Huang and S.G. Krantz, *On a problem of Moser,* Duke Math. J. **78** (1995), 213–228.

[17] X. Huang, *On an* n*-manifold in* $\mathbb{C}^n$ *near an elliptic complex tangent,* Jour. Amer. Math. Soc. **11** (1998) 669–692.

[18] B. Jöricke, *Local Polynomial Hulls of Discs near Isolated Parabolic Points,* Indiana Univ. Math. J. **46** (1997), 789–826.

[19] C.E. Kenig and S.M. Webster, *The Local Hull of Holomorphy of a Surface in the Space of Two Complex Variables,* Invent. Math. **67** (1982), 1–21.

[20] S.G. Krantz, *Function theory of several complex variables*, Wiley-Interscience Publication, John Wiley & Sons, New York, 1982.

[21] S.N. Mergelylan, *Uniform approximations to functions of a complex variable*, Uspehi Matem. Nauk **7** (48) (1952), 31–122: English transl., Amer. Math. Soc. Translation 1954, (1954) no. 101

[22] J.K. Moser and S.M. Webster, *Normal forms for real surfaces in* $\mathbb{C}^2$ *near complex tangents and hyperbolic surface transformations,* Acta. Math. **150** (1983), 255–296.

[23] J.K. Moser, *Analytic surfaces in* $\mathbb{C}^2$ *and their local hull of holomorphy,* Ann. Acad. Sci. Fenn. A. I. Math. **10** (1985), 397–410.
[24] A.G. O'Farrell and K.J. Preskenis, *Approximation by polynomials in two diffeomorphisms*, Bull. Amer. Math. Soc. **10** (1984), 105–107.
[25] A.G. O'Farrell and K.J. Preskenis, *Uniform approximation by polynomials in two functions*, Math. Ann. **284** (1989), 529–535.
[26] P.J. de Paepe, *Eva Kallin's lemma on polynomial convexity*, Bull. London Math. Soc., to appear.
[27] K.J. Preskenis, *Approximation on disks*, Trans. Amer. Math. Soc. **171** (1972), 445–467.
[28] R.M. Range, *Holomorphic functions and integral representations in several complex variables*, Springer-Verlag, New York-Berlin, 1986.
[29] G. Stolzenberg, *Polynomially and rationally convex sets*, Acta Math. **109** (1963), 259–289.
[30] E.L. Stout, *Polynomially convex neighborhoods of hyperbolic points*, Abstracts AMS **7** (1986), 174.
[31] J. Wermer, *Approximation on a disk*, Math. Ann. **155** (1964), 331–333.
[32] J. Wiegerinck, *Local polynomially convex hulls at degenerated CR singularities of surfaces in* $\mathbb{C}^2$, Indiana Univ. Math. J. **44** (1995), 897–915.

Alejandro Sanabria-García,
Department of Mathematics,
National University of Ireland Maynooth,
Maynooth, Co. Kildare,
Ireland,
manuel@maths.may.ie

Received on 14 June 2000 and in revised form on 20 November 2000.