Comments on Two Papers by R. Gow in the December 1995 IMS Bulletin

Concerning the article on Bourbaki's problem in the December 1995 issue of the IMS Bulletin, I discovered that a theorem of L. K. Hua (*On the automorphisms of a field*, Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 386-389) answers the problem that I raised. I am grateful to my colleagues Fergus Gaines and David Lewis for information on the problem, and also to Professor Larry Harris for related correspondence. If I had remembered an earlier paper by Fergus Gaines (*How to compose a problem for the International Mathematical Olympiad?*, Irish Math. Soc. Bulletin **28** (1992), 20-29), I would not have written the article.

Concerning another article written by us in the same issue of the Bulletin (*Some Galway professors of mathematics and of natural philosophy*), Professor Alastair Wood has kindly informed me that Morgan Crofton's father was not the successor of G. G. Stokes's father as Rector of Skreen, Co. Sligo, as stated in the article. His successor was the Rev. George Trulock, who was Rector from 1834 until 1847. The Rev. W. Crofton was in fact the successor of Trulock, and died in Skreen in 1851.

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A LINEAR SYSTEM OF IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract A linear system of impulsive differential equations that models an example from pharmacokinetics is investigated. The example is where a drug is administered periodically at certain fixed times resulting in a jump (called an *impulse*) in the concentration level of the drug. The case where the same dosage is applied at each of these fixed times is considered. Both necessary and sufficient conditions are sought to guarantee that an effective concentration level of the drug is maintained in the body.

Introduction

Impulsive differential equations are used to describe physical processes that undergo instantaneous perturbations. As in the study of ordinary differential equations, the study of impulsive differential equations is motivated by many practical examples from the physical sciences, [1, 2]. In this paper, we look at a linear system of impulsive differential equations, a special case of which may model the concentration of a drug in the bloodstream and an organ, like the heart or liver. Our results here generalize those obtained by Stewart, [3, pp. 758-759], who considered the scalar case.

The paper begins by describing a first order linear system of impulsive differential equations in terms of a constant real matrix K. We call the solution vector ϕ of the system an *admissible* solution when it lies in the region between two specified concentric spheres in $\mathbf{R}^d$. The initial dosage vector c and the time T (length

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of time between consecutive impulses) form a pair of parameters which are critical in determining the admissibility of ϕ. The paper's main result requires that K be both positive definite and symmetric, combined with a method of choosing first T and then c. The way we choose T and c depends on the eigenvalues of K and the radii of the two spheres.

1. Preliminaries

Consider the following linear system of impulsive differential equations:

$$\begin{aligned} &\dot{x}(t) = -Kx(t), \qquad t \neq T, 2T, \ldots, \\ &\Delta x(t) = x(t^+) - x(t^-) = c, \; t = T, 2T, \ldots, \\ &x(0^+) = c, \end{aligned} \tag{1}$$

where K is a constant real $d \times d$ matrix, c a vector in $\mathbf{R}^d$ and $T > 0$.

Definition 1 A solution to (1) is a piecewise continuous function $\phi : [0, \infty) \to \mathbf{R}^d$ such that
(i) $\dot{\phi}(t) = K\phi(t)$, for $t \in \mathbf{R} \setminus \{T, 2T, \ldots\}$;
(ii) at times nT, where $n \in \mathbf{N}$, it is the case that $\phi((nT)^+) = \phi((nT)^-) + c$;
(iii) $\phi(0^+) = c$;
(iv) $\lim_{t \to (nT)^+} \phi(t) = \phi(nT)$ for $n \in \mathbf{N}$.

Note that the existence and uniqueness of a solution to the linear system above is well known (see [2] for details). We now make some more definitions that will be needed later.

Definition 2 Let B be a real $d \times d$ matrix. Then B is said to be *positive definite* if for any $a \in \mathbf{R}^d$ we have $a^t Ba \geq 0$ with equality occurring if and only if $a = 0$.[1]

Here the transpose of the vector a is denoted by a^t. As usual, B is said to be *symmetric* if $B = B^t$. A consequence of B being

[1] We adhere to this definition throughout the paper, but readers should be warned that unlike other authors we do not require that a positive definite matrix be symmetric.

positive definite and symmetric is that its eigenvalues are both real and positive.

Definition 3 Given two parameters L and H where $0 \leq L < H < \infty$ we say that ϕ is an *admissible* solution of (1) if, for some choice of parameters T and c, ϕ solves the linear impulsive system (1) and, for all $t > 0$, $\phi(t)$ lies in the set $\{a \in \mathbf{R}^d : L \leq ||a|| \leq H\}$. For convenience we call this set $A(0; L, H)$.

Note that $||a||$ is the length of the vector a:

$$||a||^2 = a^t a = \sum_{i=1}^{d} |a_i|^2.$$

To describe the solution of (1) we define sequences of vectors A_n, B_n in $\mathbf{R}^d$ as follows: Let

$$\begin{aligned} &A_{-1} = 0, \\ &A_n = c + e^{-TK}c + e^{-2TK}c + \cdots + e^{-nTK}c, \quad n = 0, 1, 2 \ldots, \end{aligned}$$

and

$$B_n = e^{-TK} A_{n-1}, \quad n = 0, 1, 2, \ldots.$$

Note that

$$\begin{aligned} A_n &= (I + e^{-TK} + \cdots + e^{-nTK})c \\ &= (I - e^{-(n+1)TK})(I - e^{-TK})^{-1}c, \end{aligned}$$

where I is the unit matrix.

The solution of the differential equation is then given by

$$x(t) = e^{-(t-(n-1)T)K} A_{n-1}, \quad \text{if } t \in [(n-1)T, nT), \quad n = 1, 2, \ldots.$$

Our objective is to determine (practical) necessary and sufficient conditions on the various parameters defining the system that ensure the solution $x(t)$ is admissible.

2. Necessary and sufficient conditions for admissibility

Theorem 1 *Suppose that K is positive definite. Then $x(t) \in A(0; L, H)$ for all $t \in (0, \infty)$ if and only if*

$$L \leq \inf \|B_n\| \text{ and } \sup \|A_n\| \leq H.$$

Proof: From the differential equation we see that

$$\frac{d}{dt}\|x(t)\|^2 = 2x^t\dot{x} = -2x^tKx \leq 0,$$

for all $t \in ((n-1)T, nT)$, $n = 1, 2, \ldots$, and so $\|x(t)\|$ is decreasing on the intervals $((n-1)T, nT)$, $n = 1, 2, \ldots$. It follows that

$$\|B_n\| \leq \|x(t)\| \leq \|A_{n-1}\|,$$

for all $t \in ((n-1)T, nT)$. Hence, if

$$L \leq \inf \|B_n\| \text{ and } \sup \|A_n\| \leq H,$$

then

$$L \leq \|x(t)\| \leq H, \text{ for all } t > 0.$$

Conversely, if this holds, then

$$\|B_n\| = \|x((nT)^-)\| \geq L$$

and

$$\|A_n\| = \|x((nT)^+)\| \leq H$$

for $n = 1, 2, \ldots$. Hence the given condition is also necessary. •

Different aspects of the following example have been studied by various authors, [1, 2, 4]. The example below consists of a two-compartment model for the distribution of a drug in the body, say the gastro-intestinal tract and the bloodstream. The components x_1 and x_2 stand for the quantity of the drug in the gastro-intestinal tract and the bloodstream, respectively, and k_1

and k_2 are the respective rate constants. The differential equations and associated boundary conditions governing this model are given by the following:

$$\begin{aligned} \dot{x}_1(t) &= -k_1x(t),\ 0 < t \neq T, 2T, \ldots, \\ \dot{x}_2(t) &= k_1x_1(t) - k_2x_2(t),\ 0 < t \neq T, 2T, \ldots, \\ x_i(t^+) - x_i(t^-) &= c_i,\ i = 1, 2,\ t = T, 2T, \ldots \end{aligned}$$

and

$$x_i(0) = x_i(0^+) = c_i,\ i = 1, 2.$$

Theorem 1 applies to this system once we know for what values of k_1 and k_2 the matrix of the system is positive definite. The answer is provided by the following.

Example 1 *Suppose that $4k_2 > k_1 > 0$. Then the 2×2 matrix*

$$K = \begin{pmatrix} k_1 & 0 \\ -k_1 & k_2 \end{pmatrix}$$

is positive definite, but not symmetric.

Proof: For, if $a \in \mathbb{R}^2$, then

$$\begin{aligned} a^tKa &= k_1a_1^2 - k_1a_1a_2 + k_2a_2^2 \\ &= k_1(a_1^2 - a_1a_2) + k_2a_2^2 \\ &= k_1(a_1 - a_2/2)^2 + (k_2 - k_1/4)a_2^2 \\ &\geq 0 \end{aligned}$$

and equality holds if and only if $a_1 = a_2 = 0$. Thus K is positive definite. Clearly, $K \neq K^t$. •

It is also clear that this matrix is not positive definite if $0 < 4k_2 \leq k_1$.

Returning to the general situation, the fact that K is positive definite implies that

$$\|B_n\| \leq \|A_{n-1}\|,\ n = 1, 2, \ldots,$$

as we have just seen. We wish to investigate the monotonicity of these sequences under some additional assumptions about K. In what follows, we suppose that K is symmetric. This, coupled with the fact it is positive definite, implies that K is unitarily equivalent to a diagonal matrix. Thus $K = U^tDU$, where U is unitary, $U^tU = I$, and $D = \text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_d]$, where $\lambda_i > 0$, $i = 1, 2, \ldots, d$, and so

$$e^{-TK} = U^t e^{-TD} U = U^t \text{diag}[e^{-\lambda_1 T}, e^{-\lambda_2 T}, \ldots, e^{-\lambda_d T}]U.$$

Next,

$$A_{n-1} = (I - e^{-nTK})y, \text{ where } y = (I - e^{-TK})^{-1}c,$$

and so

$$\begin{aligned} \|A_{n-1}\|^2 &= A_{n-1}^t A_{n-1} \\ &= y^t(I - e^{-nTK})^t(I - e^{-nTK})y \\ &= y^tU^t(I - e^{-nTD})^2Uy \\ &= z^t(I - e^{-nTD})^2 z, \text{ where } z = Uy \\ &= \sum_{i=1}^{d}(1 - e^{-n\lambda_i T})^2|z_i|^2. \end{aligned}$$

But the sequence $(1-e^{-n\lambda T})$ is increasing if $\lambda > 0$. Hence $\|A_{n-1}\|$ is increasing. But it is also clear that the positive definiteness of K implies that

$$\lim A_n = y,$$

so that

$$\sup \|A_n\| = \|y\| = \|(I - e^{-TK})^{-1}c\|.$$

In the same way it can be seen that $\|B_n\|$ is increasing, and so

$$\inf \|B_n\| = \|B_1\| = \|e^{-TK}c\|.$$

These considerations enable us to express the conclusion of Theorem 1 in the following form.

Theorem 2 *Suppose that K is positive definite and symmetric. Then $x(t) \in A(0; L, H)$, for all $t \in (0, \infty)$ if and only if*

$$L \le \|e^{-TK}c\| \text{ and } \|(I - e^{-TK})^{-1}c\| \le H.$$

Still supposing that K is unitarily equivalent to a diagonal matrix, none of whose eigenvalues is zero, we have

$$\begin{aligned} \|e^{-TK}c\|^2 &= c^t e^{-2TK} c \\ &= v^t e^{-2TD} v, \text{ where } v = Uc \\ &= v^t \,\text{diag}[e^{-2T\lambda_1}, \ldots, e^{-2T\lambda_d}]\, v \\ &= \sum_{i=1}^{d} e^{-2T\lambda_i}|v_i|^2 \end{aligned}$$

and

$$\begin{aligned} \|(I - e^{-TK})^{-1}c\|^2 &= c^t(I - e^{-TK})^{-2}c \\ &= v^t(I - e^{-TD})^{-2}v \\ &= v^t \,\text{diag}[(1 - e^{-T\lambda_1})^{-2}, \ldots, (1 - e^{-T\lambda_d})^{-2}]\, v \\ &= \sum_{i=1}^{d} \frac{|v_i|^2}{(1 - e^{-T\lambda_i})^2}. \end{aligned}$$

Hence we can phrase the necessary and sufficient condition that $x(t) \in A(0; L, H)$ as follows.

Theorem 3 *Suppose that $K = U^t \,\text{diag}[\lambda_1, \ldots, \lambda_d]\, U$, where U is unitary and the λ_i's are positive. Let $v = Uc$. Then $x(t) \in A(0; L, H)$ for all $t \in (0, \infty)$ if and only if*

$$L^2 \le \sum_{i=1}^{d} e^{-2T\lambda_i}|v_i|^2 \text{ and } \sum_{i=1}^{d} \frac{|v_i|^2}{(1 - e^{-T\lambda_i})^2} \le H^2.$$

3. Necessary conditions for admissibility

We deduce some coupled restrictions on the parameters K, L, H, c, T that must hold if the system has an admissible solution.

Theorem 4 *Suppose that K is symmetric and positive definite. If $x(t) \in A(0; L, H)$ for all $t > 0$, then*
(i) $4L \leq H$, with equality if and only if $e^{T\lambda_i} = 2$, $i = 1, \ldots, d$;
(ii) $Le^{T\lambda} \leq \|c\| \leq (1 - e^{-T\Lambda})H$, where

$$\Lambda = \max\{\lambda_i : i = 1, 2, \ldots\} \text{ and } \lambda = \min\{\lambda_i : i = 1, 2, \ldots\}.$$

Proof: Under our assumptions,

$$\begin{aligned} L^2 \leq \sum_{i=1}^{d} e^{-2T\lambda_i}|v_i|^2 &= \sum_{i=1}^{d}\{e^{-T\lambda_i}(1 - e^{-T\lambda_i})\}^2 \frac{|v_i|^2}{(1 - e^{-T\lambda_i})^2} \\ &\leq \frac{1}{16}\sum_{i=1}^{d} \frac{|v_i|^2}{(1 - e^{-T\lambda_i})^2} \\ &\leq \frac{H^2}{16}, \end{aligned}$$

since $4y(1 - y) \leq 1$ for all $y \in \mathbb{R}$, with equality if and only if $y = 1/2$, and so $16e^{-2x}(1 - e^{-x})^2 \leq 1$ for all $x \in [0, \infty)$. If $H = 4L$, then equality holds throughout, which clearly happens if and only if $e^{-T\lambda_i} = \frac{1}{2}$, $i = 1, 2, \ldots, d$. This gives the first part. As for the second part, we have

$$\begin{aligned} \|c\|^2 &= \|v\|^2 \\ &= \sum_{i=1}^{d} e^{-2T\lambda_i}|v_i|^2 e^{2T\lambda_i} \\ &\geq e^{2T\lambda}\sum_{i=1}^{d} e^{-2T\lambda_i}|v_i|^2 \\ &\geq e^{2T\lambda}L^2 \end{aligned}$$

and so $e^{T\lambda}L \leq \|c\|$.

Similarly,

$$\begin{aligned} \|c\|^2 &= \|v\|^2 \\ &= \sum_{i=1}^{d}(1 - e^{-T\lambda_i})^2 \frac{|v_i|^2}{(1 - e^{-T\lambda_i})^2} \\ &\leq (1 - e^{-T\Lambda})^2 \sum_{i=1}^{d} \frac{|v_i|^2}{(1 - e^{-T\lambda_i})^2} \\ &\leq (1 - e^{-T\Lambda})^2 H^2 \end{aligned}$$

and so $\|c\| \leq (1 - e^{-T\Lambda})H$. •

4. Sufficient conditions for admissibility

For practical reasons it is essential to have easily verifiable conditions that will guarantee that the system has an admissible solution. We proceed to give one such condition that guarantees that $x(t) \in A(0; L, H)$ for all $t > 0$. This will then enable us to give a simple sufficient condition for admissibility; this is recorded in Theorem 7.

Theorem 5 *Suppose that K is symmetric and positive definite. Let*

$$Le^{T\Lambda} \leq \|c\| \leq (1 - e^{-T\lambda})H.$$

Then $x(t) \in A(0; L, H)$, for all $t > 0$.

Proof: It is enough to note that

$$\begin{aligned} L^2 &\leq e^{-2T\Lambda}\|c\|^2 \\ &= e^{-2T\Lambda}\|v\|^2 \\ &= \sum_{i=1}^{d} e^{-2T\Lambda}|v_i|^2 \\ &\leq \sum_{i=1}^{d} e^{-2T\lambda_i}|v_i|^2. \end{aligned}$$

Similarly,

$$\sum_{i=1}^{d} \frac{|v_i|^2}{(1-e^{-T\lambda_i})^2} \le (1-e^{-T\lambda})^{-2}\|v\|^2$$
$$= (1-e^{-T\lambda})^{-2}\|c\|^2$$
$$\le H^2.$$

The stated result now follows from Theorem 3. •

Before leaving this result, we analyse the sufficient condition more closely in an attempt to uncouple the relationship between T and c. Our findings are summarized in the next theorem.

Theorem 6 *Let $r = H/L$. Under the assumptions of the previous theorem,*
(i) $r \ge 4$;
(ii)

$$\frac{\log \frac{r-\sqrt{r^2-4r}}{2}}{\lambda} \le T \le \frac{\log \frac{r+\sqrt{r^2-4r}}{2}}{\Lambda};$$

(iii)

$$\frac{[r-\sqrt{r^2-4r}]L}{2} \le \|c\| \le \frac{[r+\sqrt{r^2-4r}]L}{2}.$$

Proof: It follows from our assumptions that

$$Le^{T\lambda} \le \|c\| \le (1-e^{-T\lambda})H$$

and so

$$e^{2T\lambda} \le (e^{T\lambda}-1)r,$$

whence, by elementary methods, part (i) and the left-hand inequality of part (ii) follow. But also,

$$Le^{T\Lambda} \le \|c\| \le (1-e^{-T\Lambda})H$$

and so

$$e^{2T\Lambda} \le (e^{T\Lambda}-1)r,$$

whence the right-hand inequality of part (ii) also follows. Equivalently,

$$\frac{r-\sqrt{r^2-4r}}{2} \le e^{T\lambda} \le e^{T\Lambda} \le \frac{r+\sqrt{r^2-4r}}{2}.$$

From this the left side of part (iii) follows. Using the right-hand inequality in the previous display, we see that

$$1-e^{-T\lambda} \le 1-\frac{2}{r+\sqrt{r^2-4r}}$$
$$= 1-\frac{2[r-\sqrt{r^2-4r}]}{4r}$$
$$= \frac{r+\sqrt{r^2-4r}}{2r},$$

which gives the right-hand side of the inequality in part (iii). •

Theorem 6 means that if any one of the conditions (i), (ii) and (iii) fails to hold, then we do not have an admissible solution. But, at the same time, the satisfaction of all three is no guarantee that an admissible solution exists. The following example is intended to illustrate this.

Example 2 *Suppose that $K = \text{diag}\,[1,2]$. Let $H/L = r = 9/2$. Choose $T = \ln\sqrt{3}$ and let c be any vector whose first component is zero and $\sqrt{3}L \le \|c\| < 3L$. Then the inequalities stated in Theorem 6 are satisfied, but the system has no admissible solutions.*

It is clear that the inequalities in Theorem 6 hold. But, if the system has an admissible solution for the given values of T and c, we see from Theorem 3 that

$$L^2 \le e^{-2T}|c_1|^2 + e^{-4T}|c_2|^2 = e^{-4T}|c_2|^2 = \frac{\|c\|^2}{9},$$

which is false. •

The same example shows that the converse of Theorem 4 is false.

We now state a simple criterion for admissibility.

Theorem 7 *Given a symmetric positive definite matrix K, whose smallest and largest eigenvalues are λ, Λ, respectively, and positive constants H, L, with $r = H/L$ such that*

$$\min\{\frac{e^{x\Lambda}}{1-e^{-x\lambda}} : 0 < x\} \leq r.$$

Choose T so that $e^{T\Lambda} \leq (1-e^{-T\lambda})r$. Having selected T, now choose the vector c so that

$$Le^{T\Lambda} \leq ||c|| \leq (1-e^{-T\lambda})H.$$

Then the system (1) has an admissible solution.
Proof: This is a consequence of Theorem 5. •

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WHY PHONON LINES DON'T CROSS

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Abstract We report on recent developments in quantum stochastic approximations of physical systems, the relative merits of Gauss and Wigner distributions and the physical reasons one should arise rather than the other in a model of an electron interacting with a phonon field.

1. Introduction

Eugene Wigner, [1], introduced ensembles of $N \times N$ real matrices to model the spectra of complex nuclei and noticed that the $N \to \infty$ limit corresponds to a non-commutative central limit theorem involving a non-gaussian distribution as the limit distribution. The new distribution, called the Wigner or semi-circle law, frequently appears in place of the gaussian when one departs from ordinary probability to quantum probability, that is, when random variables are represented as non-commutative operators and probability as a positive normalized functional.

In the 1980's Hudson and Parthasarathy, [2], attempted to construct quantum (i.e. non-commutative) stochastic analogues to the brownian motion, and indeed Poisson, processes and the calculi by using the inherent gaussianity of bose fields. Later Voiculescu, [3], and Kümmerer and Speicher, [4], used free fields to construct free noise processes which are related to the Wigner law.

Here we report about the emergence of a new type of noise from physical models which is closer to the Wigner class than the Gauss class. It was first discovered by Lu by examining moments and later proven in general in [5]. This report centres on the physical mechanism behind this, [6].