Appealing once more to L3 we deduce that $\ell(e) = 1$.

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TORONTO SPACES, MINIMALITY, AND A THEOREM OF SIERPIŃSKI.

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In this note we gather together some theorems in the literature to resolve a problem suggested by P. J. Matthews and T. B. M. McMaster in a recent article, [1]. We also make an observation which allows one to deduce within ordinary set theory that neither the real line nor the Sorgenfrey line contains a Toronto space of cardinality the continuum (improving one of their results), and we establish some relative consistency results. To conclude the paper, we explain how a similar question arising from a theorem of Sierpiński (can every subset of the unit interval I of cardinality the continuum be mapped continuously onto I?) is independent of ordinary set theory.

1. Toronto spaces and minimality

Matthews and McMaster ask whether there are any reasonable set-theoretic assumptions which will enable one to prove or disprove the assertion $\mathrm{Qmin}(\kappa)$ where κ is an uncountable limit cardinal. Recall that the assertion $\mathrm{Qmin}(\kappa)$ says:

(a) neither $T(\kappa)$ nor $T(\kappa) \cap T_2$ is supported by its weakly quasi-minimal members,

and

(b) any subfamily of $T(\kappa)$ or $T(\kappa) \cap T_2$ which does support the whole family has more than κ members.

Full definitions are provided in the paper [1]. We note for convenience that a topological space X of cardinality κ is weakly quasi-minimal if X is embeddable in each of its subspaces of power κ. The notations $T(\kappa)$ and T_2 refer to the families of spaces of power κ and the Hausdorff spaces respectively. A family F supports a family G if every member of G contains a homeomorphic image of a member of F.

First of all, we show the following:

Proposition 1.

(1) *Every infinite Hausdorff topological space X contains an infinite discrete subset.*

(2) *If κ is a singular strong limit cardinal or $\kappa = \aleph_0$, then the discrete topological space $D(\kappa)$ of cardinality κ is strongly quasi-minimal and supports the family $T(\kappa) \cap T_2$. In particular, $\mathrm{Qmin}(\kappa)$ is false.*

Proof: (1) Since X is infinite and Hausdorff, and the intersection of a finite number of open sets is open, it follows that one can choose a discrete sequence $\{x_n \in X : n \in \omega\}$ by induction. Alternatively, apply Zorn's lemma to the family $\mathcal{S} = \{Y : Y$ is a discrete subset of $X\}$ partially ordered by inclusion, to obtain a maximal element D which must be infinite.

(2) Trivially, the discrete space $D(\kappa)$ is strongly quasi-minimal, i.e. it is homeomorphic to each of its subspaces of cardinality κ, and hence it is weakly quasi-minimal too. A theorem of Hajnal and Juhász, [2, 4 or 5], says that if κ is a singular strong limit cardinal, then every Hausdorff space X of cardinality at least κ has a discrete subset Y of cardinality κ. If $\kappa = \aleph_0$, then part (1) applies. In either eventuality, $D(\kappa)$ supports the family $T(\kappa) \cap T_2$, since any bijection from $D(\kappa)$ onto Y is a homeomorphism. So $\mathrm{Qmin}(\kappa)$ is false. ∎

Proposition 1.2 covers a proper class of singular cardinals: for any cardinal λ, define

$$\kappa_\lambda = \sup\{\lambda, \exp(\lambda), \exp(\exp(\lambda)), \dots\},$$

where $\exp(\lambda) = 2^\lambda$. Proposition 1.2 implies that $\mathrm{Qmin}(\kappa_\lambda)$ is false for all λ. Note however that every κ_λ has countable cofinality.

We shall show further on that if κ is singular and has uncountable cofinality, then there is a model of ZFC (ordinary set theory) in which $\mathrm{Qmin}(\kappa)$ is true (although obviously in this model κ is not a strong limit).

We remind the reader that a weakly inaccessible cardinal is a regular limit cardinal, and a strongly inaccessible cardinal is an uncountable regular cardinal κ such that $(\forall \lambda < \kappa)(2^\lambda < \kappa)$. The Generalized Continuum Hypothesis (GCH: $(\forall \kappa)(2^\kappa = \kappa^+)$) implies that every weakly inaccessible cardinal is strongly inaccessible, and that every singular cardinal κ is a strong limit.

Corollary 2. *Assume GCH. Then $\mathrm{Qmin}(\kappa)$ is false for every singular cardinal κ.*

Corollary 3. *Assume GCH and there are no inaccessible cardinals. Then $(\forall \kappa)(\mathrm{Qmin}(\kappa)$ is true if and only if κ is an uncountable regular cardinal).*

Proof: If κ is an uncountable regular cardinal, then $\kappa = \lambda^+ = 2^\lambda$, and now $\mathrm{Qmin}(\kappa)$ holds by the theorem of Matthews and McMaster, [1]. If κ is countable or singular, then Proposition 1.1 and Corollary 2 show that $\mathrm{Qmin}(\kappa)$ is false. ∎

Corollary 3 answers Problem 2 of Matthews and McMaster, [1]. It also establishes that if ZFC (ordinary set theory) is consistent, then so is ZFC + $((\forall \kappa)(\mathrm{Qmin}(\kappa)$ is true if and only if κ is an uncountable successor cardinal)). In particular, it is impossible to prove from ZFC the existence of an uncountable regular limit cardinal for which $\mathrm{Qmin}(\kappa)$ fails.

The Hajnal-Juhász theorem to which we appealed in proving Proposition 1.2 relies on a positive partition relation. To illustrate the ideas and arguments involved, we prove a simple theorem, explaining first some convenient notation. Suppose that κ, λ, and μ are cardinals. The family of μ element subsets of a set A is denoted by $[A]^\mu$. The notation $\kappa \to (\kappa)^\mu_\lambda$ means: for every function $f : [\kappa]^\mu \to \lambda$, there exists a set $H \in [\kappa]^\kappa$ such that $f|[H]^\mu$ is constant. In pictorial terms, if one colours the μ element subsets of κ using λ colours, then there will always be a κ element subset H all of whose μ element subsets get the same colour. In this notation, Ramsey's theorem reads: $\aleph_0 \to (\aleph_0)^m_k$

for all natural numbers m and k, where $\aleph_0$ is the cardinality of the natural numbers. A relation of the form $\kappa \to (\kappa)_\lambda^\mu$ is called a positive partition relation. Weakly compact cardinals are often defined as those cardinals κ for which $\kappa \to (\kappa)_2^2$ holds. We note that $\kappa \to (\kappa)_2^2$ implies $\kappa \to (\kappa)_\lambda^n$ for all $n < \omega$ and $\lambda < \kappa$. In general, if an uncountable cardinal κ satisfies a non-trivial positive partition relation, then κ is a large cardinal, and its existence cannot be proven in ZFC (ordinary set theory). The reader can easily check that for example $c \to (3)_\omega^2$ does not hold, where c is the cardinality of the real numbers (enumerate the set of rationals $\mathbf{Q} = \{q_n : n \in \omega\}$, and for $x < y \in \mathbf{R}$, put

$$g(\{x, y\}) = \min\{n : x < q_n < y\}).$$

The classic monograph of Erdös, Hajnal, Maté and Rado, [5], provides detailed information on the partition calculus.

Proposition 4. *Suppose that* $\kappa \to (\kappa)_\omega^2$. *If* X *is a first countable Hausdorff space of cardinality* κ, *then* X *has a discrete subset* D *of cardinality* κ.

Proof: For each $x \in X$, let $\{V(x, n) : n \in \omega\}$ be a shrinking neighbourhood basis at x. Define a colouring f of the pairs of elements of X as follows:

$$f(\{x, y\}) = \min\{n : V(x, n) \cap V(y, n) = \emptyset\}.$$

Apply the partition relation to obtain an n and a subset D of X of power κ such that $(\forall x \neq y \in D)(f(\{x, y\}) = n)$, i.e. D is a discrete subspace, since

$$(\forall x \in D)(D \cap V(x, n) = \{x\}). \ \blacksquare$$

Next we turn to the Toronto space problem, [3]. A minor improvement of a lemma from Matthews and McMaster allows one to prove (as a theorem in ordinary set theory) that $\mathrm{Qmin}(c)$ holds, where c is the cardinality of the real numbers.

Lemma A*. [1, Lemma A] *Suppose that* κ *is an infinite cardinal,* X *is a set of power* κ *and*

$$(\forall \alpha < \kappa)(S_\alpha \text{ is a subset of } X \text{ of power } \kappa).$$

Then there exists a subset Z *of* X *of power* κ *which does not contain any* S_α.

Proof: Without loss of generality, we identify X with κ and assume that κ is uncountable. Choose distinct elements x_0 and y_0 in S_0. Given x_α and z_α in S_α for $\alpha < \beta < \kappa$, note that $S_\beta \setminus \{x_\alpha, z_\alpha : \alpha < \beta\}$ has power κ, since $\beta < \kappa$, and so one can find distinct elements x_β, z_β in $S_\beta \setminus \{x_\alpha, z_\alpha : \alpha < \beta\}$. Put $Z = \{z_\alpha : \alpha < \kappa\}$. Then Z has power κ, and for all $\alpha < \kappa$, Z does not contain S_α since $x_\alpha \in S_\alpha \setminus Z$. $\blacksquare$

The essential results of Matthews and McMaster now go through without the assumption of regularity.

Lemma C* [1, Lemma C]. *Suppose that* X *is a Hausdorff space of cardinality* κ *all of whose subspaces have dense subsets of power at most* λ, *and* $\kappa^\lambda = \kappa$. *Suppose that*

$$(\forall \alpha < \kappa)(S_\alpha \text{ is a subset of } X \text{ of power } \kappa).$$

If Y *is a subspace of* X *of power* κ, *then* Y *has a subspace* Z *which contains no homeomorphic copy of any* S_α.

The Toronto problem, [3], asks whether it is possible to have a Toronto space, i.e. an uncountable non-discrete Hausdorff space which is homeomorphic to each of its uncountable subspaces. It is unknown whether the existence of a Toronto space is consistent with ZFC. A counting argument shows that if X has hereditary density λ and $|X|^\lambda < 2^{|X|}$, then X is not a Toronto space: there are $2^{|X|}$ subspaces of power $|X|$, but only $|X|^\lambda$ auto-homeomorphic images of X.

Corollary 5. *There are no Toronto spaces of singular strong limit cardinality. In particular, GCH implies that there are no Toronto spaces of singular cardinality.*

Proof: If κ is a singular strong limit cardinal, then every Hausdorff space X of cardinality κ has a discrete subset of cardinality κ, and so X is not a Toronto space. $\blacksquare$

Corollary 6.

(1) $\mathrm{Qmin}(c)$ *is true.*

(2) *The real line contains no Toronto space of power* c.

(3) *The Sorgenfrey line contains no Toronto space of power* c.

Matthews and McMaster, [1], proved the results 6.1 and 6.2 with the additional assumption that **c** is a regular cardinal. Similar results can also be demonstrated for the natural analogues of the real line in higher cardinalities.

Corollary 7. *Suppose that $\kappa \to (\kappa)^2_\omega$. If X is a first countable Hausdorff space of cardinality κ, then X does not contain a Toronto space of cardinality κ.*

Proof: By Proposition 4, every subspace of X of power κ contains a discrete subset of cardinality κ. ∎

Corollary 6 enables one to show that if κ is any cardinal of uncountable cofinality, then there is a model of ZFC in which $Qmin(\kappa)$ is true: for example, add κ Cohen reals to **L**, the universe of constructible sets (or more generally, to any model of ZFC + GCH). So Corollary 2 and Corollary 6 show that $Qmin(\kappa)$ is independent of ZFC for any singular cardinal κ with $\kappa > cf(\kappa) > \omega$.

There is a general phenomenon at work behind Corollaries 2 and 6: suppose that $P(\lambda)$ is a property of cardinals for which one can prove in ordinary set theory that $P(\mathbf{c})$ is true but $P(\kappa)$ is false for every singular strong limit cardinal κ; then $P(\kappa)$ is independent of ZFC for every singular cardinal κ of uncountable cofinality.

Returning to the question of $Qmin(\kappa)$, what happens if an uncountable cardinal κ has countable cofinality? First of all, κ is singular. If κ is a strong limit, then Proposition 1 says that $Qmin(\kappa)$ is false. We do not know what happens if κ is not a strong limit, for example if $\kappa = \aleph_\omega < \mathbf{c}$ (where part of the difficulty is that $\kappa^{cf(\kappa)} > \kappa$ (Koenig's theorem)). Some additional partial information can be gleaned from the papers of Hajnal and Juhász, [6], and Kunen and Roitman, [7].

Finally, let us consider what one can prove if one removes in the statement of Corollary 3 the assumption that there are no inaccessible cardinals. In particular, is there a model of ZFC in which $Qmin(\kappa)$ holds for some weakly inaccessible cardinal? The following example provides a positive answer.

Example 8. It is well-known that if there is a model of ZFC + $(\exists\kappa)(\kappa$ is weakly inaccessible), then there is also a model M

of ZFC + (**c** is weakly inaccessible) (for example, see [9]). By Corollary 6, $Qmin(\mathbf{c})$ holds in M, so that M is a model of ZFC in which $Qmin(\kappa)$ holds for a weakly inaccessible cardinal κ.

A defect of this example is that the weakly inaccessible cardinal κ which it exhibits is fairly small. To explain what happens for larger inaccessible cardinals, we require the notion of the spread of a topological space.

Definition. *The spread of a topological space X is*

$$\sup\{|D| : D \text{ is a discrete subset of } X\} + \omega.$$

We denote the spread of X by $s(X)$ and say that the spread is achieved if X has a discrete subset D of power $s(X)$.

Hodel, [4], remarks the spread is achieved at those regular limit cardinals κ which are weakly compact, and hence all Hausdorff spaces in these cardinalities contain discrete subsets of size κ. As in Proposition 1, it follows that $Qmin(\kappa)$ is false for weakly compact cardinals, and there are no Toronto spaces of weakly compact cardinality. This leads to a model of ZFC + GCH + $(\exists\kappa)(\kappa$ is a regular limit cardinal and $Qmin(\kappa)$ is false).

Example 9. Suppose that κ is a weakly compact cardinal.[2] Then κ is weakly compact in **L**, and since GCH holds in **L**, one obtains a model of ZFC + GCH + $(\exists\kappa)$ (κ is a weakly compact (regular limit) cardinal and $Qmin(\kappa)$ is false). So while ZFC + GCH suffices to determine that $Qmin(\kappa)$ is true for uncountable successor cardinals and false for singular cardinals, it is not powerful enough to settle whether $Qmin(\kappa)$ holds if κ is an inaccessible cardinal.

We summarize the import of these examples:

Corollary 10.

(1) *Suppose that κ is a singular cardinal of uncountable cofinality. Then $Qmin(\kappa)$ is independent of ZFC (ordinary set theory).*

(2) *If there is a weakly inaccessible cardinal, then there is a model of ZFC in which $Qmin(\kappa)$ is true for some weakly inaccessible cardinal κ.*

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[2] It suffices to suppose that κ is a regular cardinal with the tree property (i.e. there is no κ-Aronszajn tree).

(3) *If there is a weakly compact cardinal κ, then there is a model of (ZFC + GCH + Qmin(κ) is false). Note that κ is weakly inaccessible in this model.*

Jensen, [8], has shown that if the axiom of constructibility (**V** = **L**) holds, then for each regular limit cardinal λ which is not weakly compact, there is a Hausdorff linearly ordered space of power λ in which the spread is not achieved. We do not know whether **V** = **L** determines which truth value Qmin(λ) has in this case, nor what this truth value may be. And of course, it may still be a theorem of ZFC that Qmin(κ^+) is true for every infinite cardinal κ. (The reader curious about future progress on these problems can consult the *Topology Atlas*, located at http://www.unipissing.ca/topology)

2. A theorem of Sierpiński

Next, we turn to a theorem of Sierpiński, [11]: there exists an uncountable subset P of the unit interval **I** such that **I** is not a continuous image of P. In his classic work, [10], Kuratowski notes on page 428: "Without the continuum hypothesis, however, we are unable to prove the existence of a set P of power **c** such that the interval is not a continuous image of P." We explain in detail how to use Martin's Axiom (MA) to prove the existence of such a set P. In fact this result follows from a weaker hypothesis: **R** is not the union of less than **c** many nowhere dense sets. This hypothesis is true for example under Martin's Axiom for countable partial orders, [12], or for a slick proof, see [13, Theorem 16.1]. Arnold Miller constructed a model of ZFC in which **c** = $\aleph_2$ and every subset of **I** of cardinality **c** can be mapped continuously onto **I**. Thus whether every subset of **I** of power **c** can be mapped continuously onto **I** is independent of ordinary set theory.

To make the arguments fairly self-contained, we recall some definitions and standard results which can be found in the textbook [10]. A set A has the Baire property in the space X iff there is an open set G such that $A \setminus G$ and $G \setminus A$ are of first category (meagre, or, a countable union of nowhere dense sets). An equivalent characterization is that $A = (G \setminus N) \cup M$ where G is open and N and M are of first category. So open sets and closed sets have

the Baire property (every closed set is the union of its interior and its boundary (which is always nowhere dense)).

Lemma 11. [10, section 24, I, Theorem 3, p.256]. *Every family of disjoint sets $\{X_i : i \in I\}$ with the Baire property, of which none is of first category, is countable.*

The next lemma is a special case of a more general result. The proof is copied from that of the analogous result for $\aleph_1$ in [10], introducing the necessary modifications to avoid assuming the regularity of the continuum **c**.

Lemma 12. *Assume that **R** is not the union of less than **c** many nowhere dense sets. Suppose that $\{E_{\alpha\beta} : \alpha, \beta < \mathbf{c}\}$ is a sequence of subsets of the unit interval **I** with the Baire property. If $\beta < \beta'$ implies that $E_{\alpha,\beta} \cap E_{\alpha,\beta'} = \emptyset$, then there exists a sequence of distinct ordinals $\{\gamma(\alpha) : \alpha < \mathbf{c}\}$ such that $|\mathbf{I} \setminus \bigcup_{\alpha < \mathbf{c}} E_{\alpha\gamma(\alpha)}| = \mathbf{c}$.*
Proof: We define by induction on $\alpha < \mathbf{c}$, an ordinal $\gamma(\alpha)$, and an element $p_\alpha \in \mathbf{I}$, as follows. Note first that by Lemma 11,

$$(\forall \alpha < \mathbf{c})(\exists \beta_\alpha)(\forall \beta > \beta_\alpha)(E_{\alpha\beta} \text{ is of the first category}).$$

Fix $\alpha < \mathbf{c}$. Suppose that we have defined $\{\gamma(\xi), p_\xi : \xi < \alpha\}$. Since the sets $E_{\alpha\beta}$ are disjoint for different β,

$$\{\delta > \beta_\alpha : (\forall \xi < \alpha)(p_\xi \in (\mathbf{I} \setminus E_{\alpha\delta}))\}$$

has power **c**, and so

$$(\exists \gamma(\alpha) > \beta_\alpha)(\forall \xi < \alpha)(\gamma(\alpha) \neq \gamma(\xi) \text{ and } p_\xi \in (\mathbf{I} \setminus E_{\alpha\gamma(\alpha)})).$$

Observe now that $\{E_{\xi\gamma(\xi)} : \xi \leq \alpha\}$ is a family of less than **c** many sets of first category. We have assumed that **R** (and hence **I**) is not the union of less than **c** many nowhere dense sets, therefore

$$\mathbf{I} \setminus (\bigcup_{\xi \leq \alpha} E_{\xi\gamma(\xi)} \cup \bigcup_{\xi < \alpha} \{p_\xi\}) \neq \emptyset,$$

since $\alpha < \mathbf{c}$; take

$$p_\alpha \in \mathbf{I} \setminus (\bigcup_{\xi \leq \alpha} E_{\xi\gamma(\xi)} \cup \bigcup_{\xi < \alpha} \{p_\xi\}),$$

and let $P = \{p_\alpha : \alpha < c\}$. If $\xi < \alpha$, then $p_\xi \neq p_\alpha$, and hence $|P| = c$. Also

$$(\forall \alpha < c)(P \cap E_{\alpha\gamma(\alpha)} = \emptyset),$$

so

$$P \subseteq (I \setminus \cup_{\alpha < c} E_{\alpha\gamma(\alpha)}),$$

and hence

$$|I \setminus \cup_{\alpha < c} E_{\alpha\gamma(\alpha)}| = c. \quad \blacksquare$$

We remind the reader that if f is a real-valued continuous function defined on a subset A of $\mathbf{R}$, then there exists a continuous extension of f to a G_δ-set (a countable intersection of open sets). This follows from Theorem 1 in section 35, I, in [10]. And we remark that if Φ is the family of real-valued continuous functions defined on G_δ-subsets of I, then Φ has cardinality c.

Theorem 13. *Assume that $\mathbf{R}$ is not the union of less than c many nowhere dense sets. Let $\mathbf{F}$ be a family of at most c many uncountable subsets of the unit interval I. Then there exists a subset P of I of cardinality c such that no element of $\mathbf{F}$ is a continuous image of P.*

Proof: By the previous remark, for each $Y \in \mathbf{F}$, there are at most c many continuous real-valued functions f defined on G_δ-subsets of I with $Y \subseteq \mathrm{range}(f)$. Since $\mathbf{F}$ has cardinality c, and $c^2 = c$, we can list all the pairs (Y, f) such that f is a real-valued continuous function defined on a G_δ-subset of I with $Y \subseteq \mathrm{range}(f)$, in a list of length c: $\mathbf{A} = \{(Y_\alpha, f_\alpha) : \alpha < c\}$. We can also enumerate (possibly with repetitions) each set $Y_\alpha = \{y_{\alpha\beta} : \beta < c\}$. Let $E_{\alpha\beta} = f_\alpha^{-1}(y_{\alpha\beta})$. Since f_α is continuous, it follows that $E_{\alpha\beta}$ is closed and hence has the Baire property. Furthermore, for $\beta < \beta'$, $E_{\alpha\beta} \cap E_{\alpha\beta'} = \emptyset$. Now we can apply Lemma 12 to the family $\{E_{\alpha\beta} : \alpha, \beta < c\}$, to obtain a sequence $\{\gamma(\alpha) : \alpha < c\}$ and a set P of cardinality c disjoint from every $E_{\alpha\gamma(\alpha)}$.

It remains to show that no $Y \in \mathbf{F}$ is a continuous image of P. Suppose (towards a contradiction) that g is a continuous function and $Y \subseteq g[P]$. As we noted after Lemma 12, there is a continuous extension g^* of the partial function $g|P$ to a G_δ-subset of I. So $Y \subseteq g[P] \subseteq g*[P]$. Hence the pair $(Y, g*)$ must appear in the list

$\mathbf{A}$ as some pair (Y_α, f_α) for some $\alpha < c$. Since $P \cap E_{\alpha\gamma(\alpha)} = \emptyset$, we have

$$P \cap f_\alpha^{-1}(y_{\alpha\gamma(\alpha)}) = \emptyset,$$

so $y_{\alpha\gamma(\alpha)}$ does not belong to $f_\alpha[P]$, and so

$$y_{\alpha\gamma(\alpha)} \in Y_\alpha \setminus f_\alpha[P] \subseteq Y \setminus g[P],$$

which contradicts $Y \subseteq g[P]$. This completes the proof. $\blacksquare$

Corollary 14. *Assume that $\mathbf{R}$ is not the union of less than c many nowhere dense sets. Then there exists a subset $P \subseteq I$ of cardinality c such that I is not a continuous image of P.*

Proof: Let $\mathbf{F} = \{I\}$ in Theorem 13. $\blacksquare$

Corollary 15. *Martin's Axiom (or Martin's Axiom for countable partial orders) implies that there exists a subset $P \subseteq I$ of cardinality c such that I is not a continuous image of P.*

In his paper [14], Miller constructed a model of ZFC using forcing in which every subset of power c of I can be mapped continuously onto I. In his model, $c = \aleph_2$. As of writing, it is an open question[3] whether there is a model of ZFC in which $c > \aleph_2$ and every subset of power c of I can be mapped continuously onto I.

Our last theorem concerns totally imperfect subsets of real numbers. A totally imperfect subset of $\mathbf{R}$ is one which contains no non-empty perfect set. A set $E \subseteq \mathbf{R}$ is perfect iff E is closed and contains no isolated points. In section 40 of [10], Kuratowski proved that there exists an uncountable (totally imperfect) set $P \subseteq I$ each of whose continuous images (situated in I) is a totally imperfect set.

Theorem 16. *Suppose that $\mathbf{R}$ is not the union of less than c many nowhere dense sets. Then there exists a (totally imperfect) set $P \subseteq I$ of cardinality c each of whose continuous images is a totally imperfect set.*

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[3] Information kindly supplied by Professor Arnie Miller. The interested reader can find some of his papers and a list of problems on his website at: http://math.wisc.edu/miller

Proof: Let F be the family of non-empty perfect subsets of I. Recall that F and every uncountable perfect set have cardinality c. Now apply Theorem 13 to F. ▪

Corollary 17. *Martin's Axiom (or Martin's Axiom for countable partial orders) implies that there exists a set $P \subseteq I$ of cardinality c each of whose continuous images is a totally imperfect set.*

Of course, in Miller's model of ZFC mentioned above, these conclusions are false. So the question whether there exists a set $P \subseteq I$ of cardinality c each of whose continuous images is a totally imperfect set is again independent of ordinary set theory.

Further generalizations of these results to second-countable complete metric spaces are also possible.

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