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S. K. Houston,
University of Ulster.
http://www.infj.ulst.ac.uk/staff/sk.houston

AN APPROACH TO THE NATURAL LOGARITHM FUNCTION

Finbarr Holland*

1. Introduction

In many, if not all, modern calculus texts, the logarithm function is usually defined, and its properties developed, following a discussion of the Riemann integral. It seems to me, however, that, for many students of the physical, engineering and biological sciences, this is much too late, and a *careful* treatment of the elementary functions should be given much earlier in any course aimed at such students. The emphasis here is on the word 'careful': I mean that every effort should be made to keep the technicalities to a minimum, without sacrificing rigour, even if this means that some results may have to be stated without proof. Instead, the utility and importance of these should be pointed out at every opportunity.

This note, then, is a contribution to the ongoing debate on what material should be taught in a modern calculus course, how it should be treated and at what stage it should be presented. Its main purpose is to outline an approach to the natural logarithm function that can be adopted in any a course that treats sequences and series early on in a serious manner, starting with a discussion of the completeness axiom for the real numbers. Its main novelty is that it deals with sequences which are indexed on the dyadic

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integers. This simplifies many of the technical details that would otherwise arise.

The paper is organized as follows: after first outlining in the next section our philosophy of how limits might be treated in such a course, we show how to define the dyadic root of a positive number, employing an iterative procedure that the Babylonians are credited with using to extract square roots, [Ev, Ne, Wa]. Next, we modify slightly the approach adopted by Euler, [Ed], to define the natural logarithm function as a limit of a sequence of functions, by passing to a *dyadic* subsequence, a suggestion I owe to Pat McCarthy. Although this idea is not new (see [La, pp 39–48], for instance), it doesn't appear to have been exploited in any of the recent popular textbooks. Finally, we relate the logarithm to the area of a hyperbolic segment by utilizing the method of exhaustion by triangles that Archimedes, [Ed, He, KL, To, Wa], used in his quadrature of the parabola, something which seems to have gone unnoticed before now.

In an Appendix, we show how to treat the number e in a similar fashion and relate it to the logarithm.

2. Limits of Sequences

It seems to me that sequences and series should be introduced early on in a calculus course; and that, in many courses, the treatment should encompass sequences of *complex* numbers as well. A course on limits of real sequences and series should begin by developing the students' intuitive notion of a limit of a sequence and they should be encouraged to use a calculator to study and predict the eventual behaviour of some standard sequences. At an appropriate time, they should be told the definition of a limit, and taught how to apply the definition in a few simple cases. And, at the very least, it should be demonstrated for them that a convergent sequence has a unique limit and that every convergent sequence is bounded. The following basic results should be stated for all students, with rigorous proofs supplied only to able students.

L1: The sum rule. If a_n and b_n are convergent, then $a_n + b_n$ is

convergent and

$$\lim(a_n + b_n) = \lim a_n + \lim b_n.$$

L2: The product rule. If a_n and b_n are convergent, then $a_n\, b_n$ is convergent and

$$\lim(a_n\, b_n) = \lim a_n \lim b_n.$$

L3: The positivity rule. If $a_n \geq 0$, $n = 1, 2, \ldots$, and a_n is convergent, then $\lim a_n \geq 0$.

L4: The shift rule. If a_n is convergent, then so is a_{n+1} and

$$\lim a_{n+1} = \lim a_n.$$

L5: The quotient rule. If $a_n \neq 0$, $n = 1, 2, \ldots$, and a_n converges to a non-zero limit, then

$$\lim \frac{1}{a_n} = \frac{1}{\lim a_n}.$$

In other words, they should be told, in some form or other, that the collection of convergent sequences is an algebra that is invariant under the shift operator that maps a_n to a_{n+1}, and that the limit function is a positive, linear and multiplicative functional on this algebra.

Examples illustrating the usefulness of these rules should be provided, stressing that they enable us to *evaluate* limits of sequences in terms of limits of more elementary ones, once these are recognized. Exercises should be given to ensure that students become comfortable when dealing with rational expressions of convergent sequences. Examples should also be given that alert them to the possibility that there are convergent sequences whose limits are not explicit quantities and motivate the following question: are there criteria that can be used to test a sequence for convergence? This and other questions should lead the classroom discussion to bounded monotonic sequences and the completeness of the reals, which we are content to state as the following axiom.

Axiom 1 *Every bounded monotonic sequence of real numbers is convergent.*

We refine this a little bit by establishing the following result.
Theorem 1 *Suppose that a_n is increasing and bounded above, with $a = \lim a_n$. Then*

$$a_n \leq a, n = 1, 2, \ldots.$$

Moreover, the inequality is strict, if a_n is strictly increasing.
Proof: Suppose that this is not the case. Then there is an integer $n' \geq 1$ such that $a_{n'} > a$. Let $\epsilon = a_{n'} - a$. Then $\epsilon > 0$. Since a_n is convergent, there is a positive integer n_0 such that

$$|a_n - a| < \epsilon, \forall n > n_0.$$

Since we're dealing with real sequences, this can be restated as

$$a - \epsilon < a_n < a + \epsilon, \forall n > n_0.$$

In particular, $a_n < a + \epsilon = a_{n'}, \forall n > n_0$, which conflicts with the fact that $a_n \geq a_{n'}$ if $n > n'$. This contradiction ends the proof of the main part. We leave it to the reader to supply the gloss.

3. Dyadic Roots

The Babylonians of old compiled tables of squares and extracted square roots of positive numbers, apparently using essentially the iterative scheme below, [Ev, Ne, Wa]. It's clear that by repeated application of their methods they could have obtained good approximations to fourth roots, eighth roots etc., of any positive number.

In what follows, and throughout the rest of the article, N will stand for a dyadic integer of the form $N = 2^n$, $n = 1, 2, \ldots$

Theorem 2 *N be a dyadic integer. Let $a \geq 0$. Then there is a unique real number $x \geq 0$ such that*

$$x^N = a.$$

Proof: Let $N = 2^m$, where m is a positive integer. We prove the statement by induction on m. The key step is to show that the equation $x^2 = a$ has a unique nonnegative solution. This is clear if $a = 0$. So, suppose that $a > 0$ and consider the (Babylonian) sequence a_n defined by

$$a_{n+1} = \frac{1}{2}(a_n + \frac{a}{a_n}), n = 1, 2, \ldots,$$

where a_1 is any positive number whose square is $\geq a$.[1] It is clear that the sequence consists of positive terms. Also, a simple computation shows that $a_{n+1}^2 \geq a$, $n = 1, 2, \ldots$, independently of the choice of a_1. But now this implies that

$$a_{n+1} - a_n = \frac{(a - a_n^2)}{2a_n^2} \leq 0, n = 1, 2, \ldots$$

In other words, a_n is a decreasing sequence of positive numbers and so, by Axiom 1, is convergent to x, say. By L3, $x \geq 0$. By L2, $x^2 = \lim a_n^2$. Hence, by an easy consequence of L3, $x^2 \geq a > 0$. So, $x > 0$. Next, applying L1, L4 and L5, we see that

$$x = \lim a_{n+1} = \frac{1}{2}(\lim a_n + \frac{a}{\lim a_n}) = \frac{1}{2}(x + \frac{a}{x}),$$

and so, $x^2 = a$. It's easy to see that this x is the only positive solution of this equation.

It is also clear how to build an inductive argument on this and establish the theorem. This ends the proof of Theorem 2.

The uniqueness part of this theorem enables us to define the Nth root of any $a \geq 0$. We use the notations $\sqrt[N]{a}$, $a^{1/N}$ interchangeably to denote the unique Nth root of $x^N = a$, where

[1] In other words, if a_0 is an approximation to the desired square root, a better one is obtained by taking the arithmetic mean of a_0 and a/a_0, the square of one of which is bigger, and of the other, smaller than a.

$N = 1, 2, 4, 8, \ldots$. Uniqueness also guarantees that $\sqrt[N]{ab} = \sqrt[N]{a}\ \sqrt[N]{b}$, $\forall a, b \geq 0$, a fact that will be needed below.

Experimentation with a hand-calculator that has a square-root key should lead students to the truth of the following.

Lemma 1 *Suppose that $a > 0$. Then*

$$\lim_{n\to\infty} \sqrt[N]{a} = 1.$$

Proof: First, suppose that $a \geq 1$ and put $a_n = \sqrt[N]{a}$, $n = 1, 2, \ldots$, so that the terms of the sequence are $\sqrt{a}$, $\sqrt[4]{a}$, $\sqrt[8]{a}$, ... We wish to show that the sequence is decreasing. But it is clear that $a_n \geq 1$, and $a_{n+1}^2 = a_n$. Hence $a_{n+1}^2 = a_n \leq a_n^2$, whence it follows that $a_{n+1} \leq a_n$. Thus $\lim a_n$ exists. Denote the limit by b. Then, by L2 and L4, $b = \lim a_n = (\lim a_{n+1})^2 = b^2$. But, by L3, $b \geq 1$. Hence $b = 1$. This proves the result when $a \geq 1$. Using this and L5 we see that

$$\lim \sqrt[N]{a} = \frac{1}{\lim \sqrt[N]{1/a}} = 1$$

if $0 < a < 1$. This completes the proof.

4. The logarithm function

Euler, [Ed], established that the sequence $n(\sqrt[n]{x}-1)$, $n = 1, 2, \ldots$, has a limit for every $x > 0$, and that the limit function satisfies the law of the logarithm. We consider the subsequence of this based on the dyadic integers, which we've just seen makes sense. Again, students should be encouraged to use a calculator to examine the behaviour of this for different values of x before being shown the following.

Theorem 3 *Let $x > 0$. Then the limit*

$$\lim_{n\to\infty} N(\sqrt[N]{x} - 1)$$

exists. Denoting this limit by $\ell(x)$ we have that
(a) $\ell(1) = 0$;
(b) $\ell(xy) = \ell(x) + \ell(y)$, $\forall x$ and $y > 0$;
(c) $(x-1)/x \leq \ell(x) \leq x - 1, \forall x > 0$.

Proof: We will show that the sequence

$$\ell_n(x) = N(\sqrt[N]{x} - 1),\ N = 2^n,\ n = 1, 2, \ldots,$$

is decreasing. But this is an immediate consequence of the simple inequality

$$2(\sqrt{x} - 1) \leq (x - 1),$$

which holds for all $x \geq 0$, with equality when and only when $x = 1$. Thus we have

$$\ell_{n+1}(x) \leq \ell_n(x) \leq x - 1,\ n = 1, 2, \ldots$$

To continue, suppose that $x \geq 1$. Then the terms of $\ell_n(x)$ are also nonnegative. Hence the sequence is decreasing and bounded below, and, so, it is convergent. We can remove the restriction on x by noting that

$$N(\sqrt[N]{x} - 1) = -N(\sqrt[N]{1/x} - 1)\ \sqrt[N]{x},$$

and using Lemma 1 and L2. Thus, in all cases, the limit exists. It emerges, too, that $\ell(x) = -\ell(1/x)$ and $\ell(x) \leq x - 1$. Hence (a) and (c) follow. Finally, (b) follows from the identity

$$N(\sqrt[N]{xy} - 1) = N(\sqrt[N]{x} - 1)\ \sqrt[N]{y} + N(\sqrt[N]{y} - 1),$$

on applying L1, L2 and Lemma 1.

5. The area of a hyperbolic segment

In 250 BC or thereabouts, Archimedes, [He, Kl, To, Wa], devised two rigorous methods–the method of *compression* and the method of *exhaustion*–to measure the area of a parabolic segment. He proved that the area of such a region is four thirds the area of the largest triangle that can be inscribed in it. Some 1800 years later, Cavalieri, [Ed, To], built on the method of compression to find the area under the curves $y = x^k$, $k = 3, 4, \ldots, 9$, and paved the way for Riemann's development of the integral. In between, in 1647, the Belgian Jesuit Fr. Gregorius a Santo Vincentio, [Ed, To], used similar ideas to make the following connection between the logarithm function and the area under an arc of the rectangular hyperbola $y = 1/x$, $x > 0$. Denote by $A(a,b)$ the area of the region $\{(x,y) : a \leq x \leq b, 0 \leq y \leq 1/x\}$. Let $t > 0$. Then

$$A(ta, tb) = A(a,b).$$

Had he used the second method of Archimedes, which we're now going to apply, Gregorius might have discovered the following result.

Theorem 4 *Let* $0 < a < b$. *Then the area,* $H(a,b)$, *of the hyperbolic segment*

$$S(a,b) = \{(x,y) : a \leq x \leq b, 1/x \leq y\}$$

is given by

$$H(a,b) = \frac{b-a}{2}\{\frac{1}{a} + \frac{1}{b}\} - \frac{1}{2}\{\lim \ell_n(\frac{b}{a}) - \lim \ell_n(\frac{a}{b})\}.$$

In particular, $H(ta,tb) = H(a,b)$ *if* $t > 0$.

Proof: The set $S(a,b)$ is clearly convex, and for any $c \in [a,b]$ the triangle, $T(a,b)(c)$, with vertices $A(a,1/a)$, $C(c,1/c)$, $B(b,1/b)$, is contained in $S(a,b)$. The area of $T(a,b)(c)$ is easily seen to be given by

$$\frac{b-a}{2}\{\frac{1}{a} + \frac{1}{b}\} - \frac{b-a}{2}\{\frac{c}{ab} + \frac{1}{c}\} \leq \frac{b-a}{2}\{\frac{1}{a} + \frac{1}{b}\} - \frac{b-a}{2}\{\frac{2}{\sqrt{ab}}\},$$

with equality if and only if $c = \sqrt{ab}$, the *geometric mean* of a and b.[2] Thus, the area of the largest triangle that can be inscribed in the hyperbolic segment is given by

$$\begin{aligned}\Delta(a,b) &= \frac{b-a}{2}\{\frac{1}{a} + \frac{1}{b}\} - \frac{b-a}{2}\{\frac{2}{\sqrt{ab}}\} \\ &= \frac{b-a}{2}\{\frac{1}{a} + \frac{1}{b}\} - \{\sqrt{\frac{b}{a}} - \sqrt{\frac{a}{b}}\}.\end{aligned}$$

[2] This result should be contrasted with the corresponding statement for the parabola $y = x^2$, when c turns out to be the *arithmetic mean* of a, b, as Archimedes discovered using purely geometric reasoning.

(Geometrically, just as for the parabola, the largest triangle occurs when C is the point on the arc joining A and B where the tangent is parallel to the chord AB.)

Note the homogeneity property:

$$\Delta(ta, tb) = \Delta(a,b), \forall t > 0.$$

We've thus obtained a decomposition of $S(a,b)$ into three disjoint regions–which is optimal in a certain sense: a triangle, which we label $T(0,0)$, with area $\Delta(a,b)$, and two segments $S(a,\sqrt{ab})$ and $S(\sqrt{ab},b)$.

Next, we consider the segments $S(a,\sqrt{ab})$ and $S(\sqrt{ab},b)$. The triangles of largest area that can be inscribed in these segments have areas $\Delta(a,\sqrt{ab})$ and $\Delta(\sqrt{ab},b)$, respectively. We have

$$\Delta(a,\sqrt{ab}) = \Delta(\sqrt{a},\sqrt{b}) = \Delta(\sqrt{ab},b),$$

by homogeneity.[3]

Up to this point, we have obtained a decomposition of $S(a,b)$ into seven disjoint regions consisting of three triangles, $T(0,0), T(1,0), T(1,1)$, say, with corresponding areas $\Delta(a,b)$, $\Delta(\sqrt{a},\sqrt{b})$, $\Delta(\sqrt{a},\sqrt{b})$, and four segments. Next we partition each of these residual segments in the same way into a triangle and two segments, noting that the triangles have the same area equal to $\Delta(\sqrt[4]{a},\sqrt[4]{b})$. Continuing in this way, we partition the segment $S(a,b)$ into a countable union of triangles $T(n,k)$, $k = 0, 1, \ldots, N-1$, $n = 0, 1, \ldots$ with corresponding areas $\Delta(n,k)$, $k = 0, 1, \ldots, N-1$, $n = 0, 1, \ldots$, where

$$\Delta(n,k) = \Delta(\sqrt[N]{a}, \sqrt[N]{b}), k = 0, 1, \ldots, N-1, n = 0, 1, \ldots.$$

We conclude that the area of $S(a,b)$ is given by the sum of the infinite series of nonnegative terms

$$H(a,b) = \sum_{n=0}^{\infty}\sum_{k=0}^{N-1} \Delta(n,k) = \sum_{n=0}^{\infty} N\Delta(\sqrt[N]{a}, \sqrt[N]{b}).$$

[3] In light of his success with the parabola, it seems inconceivable that Archimedes didn't know these facts about the hyperbola, but I haven't encountered any mention of them in the literature.

To establish the convergence of the series and determine its sum, consider its sequence of partial sums s_n. Since the terms are nonnegative, this sequence is increasing: $s_{n+1} \geq s_n$, $n = 0, 1, \ldots$. We have

$$s_0 = \Delta(0,0) = \Delta(a,b) = \frac{b-a}{2}\{\frac{1}{a} + \frac{1}{b}\} - \{\sqrt{\frac{b}{a}} - \sqrt{\frac{a}{b}}\},$$

and

$$\begin{aligned} s_1 &= \Delta(0,0) + \Delta(1,0) + \Delta(1,1) \\ &= \Delta(a,b) + 2\Delta(\sqrt{a}, \sqrt{b}) \\ &= \frac{b-a}{2}\{\frac{1}{a} + \frac{1}{b}\} - 2\{\sqrt[4]{\frac{b}{a}} - \sqrt[4]{\frac{a}{b}}\}. \end{aligned}$$

Inductively, we see that

$$\begin{aligned} s_n &= \frac{b-a}{2}\{\frac{1}{a} + \frac{1}{b}\} - 2^n\{\sqrt[2^{n+1}]{\frac{b}{a}} - \sqrt[2^{n+1}]{\frac{a}{b}}\} \\ &= \frac{b-a}{2}\{\frac{1}{a} + \frac{1}{b}\} - \frac{1}{2}\{\ell_{n+1}(b/a) - \ell_{n+1}(a/b)\}, \end{aligned}$$

$n = 0, 1, \ldots$. Already, this tells us that the increasing sequence s_n is bounded above by

$$\frac{b-a}{2}\{\frac{1}{a} + \frac{1}{b}\}$$

and so, by Axiom 1, $\lim s_n$ exists. (From this, of course, we can deduce also that the sequence

$$N\{\sqrt[2N]{\frac{b}{a}} - \sqrt[2N]{\frac{a}{b}}\},\ n = 0, 1, \ldots,$$

is convergent, if we didn't already know that fact.) In any event, the claimed result follows. This ends the proof of Theorem 4.

Of course, we have that

$$\begin{aligned} H(a,b) = \lim s_n &= \frac{b-a}{2}\{\frac{1}{a} + \frac{1}{b}\} - \lim N\{\sqrt[2N]{\frac{b}{a}} - \sqrt[2N]{\frac{a}{b}}\} \\ &= \frac{b-a}{2}\{\frac{1}{a} + \frac{1}{b}\} - \frac{1}{2}\{\ell(b/a) - \ell(a/b)\} \\ &= \frac{b-a}{2}\{\frac{1}{a} + \frac{1}{b}\} - \ell(b/a). \end{aligned}$$

Readers will recognize that the first term represents the area of the trapezium with vertices $(a,0)$, $(b,0)$, $(b,1/b)$, $(a,1/a)$. Since $H(a,b) \geq s_0 = \Delta(a,b)$, Kepler's inequality :

$$\frac{\ell(b) - \ell(a)}{b-a} \leq \frac{1}{\sqrt{ab}},$$

follows, [To].

5.1 Exercises

1. Show that $S(a,b)$ is a subset of the parallelogram formed by the chord AB, the tangent parallel to this and the ordinates $x = a$, $x = b$.
2. Deduce from the previous exercise that $H(a,b) < 2\Delta(a,b)$.
3. Now use this to obtain the inequality:

$$\frac{2}{\sqrt{ab}} - \frac{1}{2}(\frac{1}{a} + \frac{1}{b}) < \frac{\ell(b) - \ell(a)}{b-a},$$

a companion for Kepler's. (Readers will observe, *inter alia*, that an independent proof of this and Kepler's inequality implies the arithmetic-geometric mean inequality, which, of course, we've used in our derivation of the expression for $H(a,b)$.)

4. Let $a < c < b$. Show directly from the definition that

$$H(a,b) = H(a,c) + H(c,b) + \frac{(b-a)(c-a)(b-c)}{2abc}.$$

5. Suppose that $f(x,y)$ is defined and continuous on $(0,\infty)\times(0,\infty)$ and satisfies the homogeneity condition: $f(tx,ty)=f(x,y)$, $\forall x,y,t>0$, and the functional equation:

$$f(x,z)=f(x,y)+f(y,z)+\frac{(z-x)(y-x)(z-y)}{2xyz},\ \forall x,y,z>0.$$

Determine f.

6. Appendix

For those interested in relating the number e to the treatment of the logarithm given here, we describe how to introduce e using similar ideas, and to show that $\ell(e)=1$.

Since

$$(1+\frac{x}{2})^2\geq 1+x$$

for all real x it is easy to see that the sequences

$$(1+\frac{1}{N})^N,\quad (1-\frac{1}{N})^N,\quad (1-\frac{1}{N^2})^N,$$

where $N=2^n$, $n=1,2,\ldots$, are increasing. The second and third are clearly bounded, since

$$1/4\leq(1-\frac{1}{N})^N\leq 1,$$

and

$$9/16\leq(1-\frac{1}{N^2})^N\leq 1,\ n=1,2,\ldots,$$

and hence are convergent to non-zero limits. (We can deduce at this stage, if we want to, that the first is also convergent because

$$(1+\frac{1}{N})^N=\frac{(1-\frac{1}{N^2})^N}{(1-\frac{1}{N})^N}\leq\frac{1}{1/4}=4,\ n=1,2,\ldots.)$$

We show that

$$c=\lim(1-\frac{1}{N^2})^N=1.$$

To see this, note that

$$c^2=\lim(1-\frac{1}{N^2})^{2N}=\lim(1-\frac{1}{4N^2})^{4N},$$

by L2 and L4. But

$$(1-\frac{x}{4})^4=1-x+x^2\frac{(96-16x+x^2)}{256}>(1-x),\ \forall x\in(-\infty,\infty).$$

Hence

$$c^2\geq\lim(1-\frac{1}{N^2})^N=c.$$

But $0<c\leq 1$. Thus $c=1$. It now follows that

$$\begin{aligned}\lim_{n\to\infty}(1+\frac{1}{N})^N&=\lim\frac{(1-\frac{1}{N^2})^N}{(1-\frac{1}{N})^N}\\&=\frac{\lim(1-\frac{1}{N^2})^N}{\lim(1-\frac{1}{N})^N}\\&=\frac{1}{\lim(1-\frac{1}{N})^N}.\end{aligned}$$

So, following Euler, and setting

$$e=\lim_{n\to\infty}(1+\frac{1}{N})^N,$$

we see that

$$\frac{1}{e}=\lim(1-\frac{1}{N})^N.$$

But, since the sequences are increasing, Theorem 1 tells us that

$$(1+\frac{1}{N})^N\leq e\leq\frac{1}{(1-\frac{1}{N})^N}=(\frac{N}{N-1})^N,\ n=1,2,\ldots,$$

whence it results that

$$1\leq\ell_n(e)\leq\frac{N}{N-1},\ n=1,2,\ldots.$$

Appealing once more to L3 we deduce that $\ell(e) = 1$.

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Finbarr Holland
Department of Mathematics,
University College,
Cork,
Ireland.

TORONTO SPACES, MINIMALITY, AND A THEOREM OF SIERPIŃSKI.

Eoin Coleman[1]

In this note we gather together some theorems in the literature to resolve a problem suggested by P. J. Matthews and T. B. M. McMaster in a recent article, [1]. We also make an observation which allows one to deduce within ordinary set theory that neither the real line nor the Sorgenfrey line contains a Toronto space of cardinality the continuum (improving one of their results), and we establish some relative consistency results. To conclude the paper, we explain how a similar question arising from a theorem of Sierpiński (can every subset of the unit interval $\mathbf{I}$ of cardinality the continuum be mapped continuously onto $\mathbf{I}$?) is independent of ordinary set theory.

1. Toronto spaces and minimality

Matthews and McMaster ask whether there are any reasonable set-theoretic assumptions which will enable one to prove or disprove the assertion Qmin(κ) where κ is an uncountable limit cardinal. Recall that the assertion Qmin(κ) says:
(a) neither $T(\kappa)$ nor $T(\kappa) \cap T_2$ is supported by its weakly quasi-minimal members,
and
(b) any subfamily of $T(\kappa)$ or $T(\kappa) \cap T_2$ which does support the whole family has more than κ members.

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