We now state a simple criterion for admissibility.

Theorem 7 *Given a symmetric positive definite matrix K, whose smallest and largest eigenvalues are λ, Λ, respectively, and positive constants H, L, with $r = H/L$ such that*

$$\min\{\frac{e^{x\Lambda}}{1-e^{-x\lambda}} : 0 < x\} \leq r.$$

Choose T so that $e^{T\Lambda} \leq (1-e^{-T\lambda})r$. Having selected T, now choose the vector c so that

$$Le^{T\Lambda} \leq \|c\| \leq (1-e^{-T\lambda})H.$$

Then the system (1) has an admissible solution.
Proof: This is a consequence of Theorem 5. •

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WHY PHONON LINES DON'T CROSS

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Abstract We report on recent developments in quantum stochastic approximations of physical systems, the relative merits of Gauss and Wigner distributions and the physical reasons one should arise rather than the other in a model of an electron interacting with a phonon field.

1. Introduction

Eugene Wigner, [1], introduced ensembles of $N \times N$ real matrices to model the spectra of complex nuclei and noticed that the $N \to \infty$ limit corresponds to a non-commutative central limit theorem involving a non-gaussian distribution as the limit distribution. The new distribution, called the Wigner or semi-circle law, frequently appears in place of the gaussian when one departs from ordinary probability to quantum probability, that is, when random variables are represented as non-commutative operators and probability as a positive normalized functional.

In the 1980's Hudson and Parthasarathy, [2], attempted to construct quantum (i.e. non-commutative) stochastic analogues to the brownian motion, and indeed Poisson, processes and the calculi by using the inherent gaussianity of bose fields. Later Voiculescu, [3], and Kümmerer and Speicher, [4], used free fields to construct free noise processes which are related to the Wigner law.

Here we report about the emergence of a new type of noise from physical models which is closer to the Wigner class than the Gauss class. It was first discovered by Lu by examining moments and later proven in general in [5]. This report centres on the physical mechanism behind this, [6].

2. Wigner versus Gauss

Let h be a separable Hilbert space. The Fock space over h is defined to be $\mathcal{H} = \oplus_{n=0}^{\infty}\{\otimes^n h\}$, where $\otimes^0 h \equiv \mathbf{C}$. An operator $a^\dagger(f)$, with argument $f \in h$, is defined by linearity from the mapping : $\otimes^n h \mapsto \otimes^{n+1} h$ with

$$a^\dagger(f)\phi_1 \otimes ... \otimes \phi_n \mapsto f \otimes \phi_1 \otimes ... \otimes \phi_n \tag{2.1}$$

The operator $a^\dagger(f)$ is called a creator; its adjoint $a(f)$, called an annihilator, can then be defined as the mapping : $\otimes^n h \mapsto \otimes^{n-1} h$ with

$$a(f)\phi_1 \otimes ... \otimes \phi_n =< f, \phi_1 > \phi_2 \otimes ... \otimes \phi_n. \tag{2.2}$$

They satisfy the relation

$$a(f)a^\dagger(g) =< f, g > . \tag{2.3}$$

This relation is often called the free relation. The vacuum vector is defined to be the vector $\Psi = 1 \oplus 0 \oplus 0 \oplus ....$ The vacuum expectation of an observable X is then taken as

$$E[X] =< \Psi, X\Psi > .$$

In particular, let $X(f) = a(f) + a^\dagger(f)$: the distribution ρ of $X(f)$ is obtained via the characteristic formula

$$E[e^{isX(f)}] = \int_{-\infty}^{\infty} e^{isx}\rho(x)dx \tag{2.4}$$

In order to calculate an expression like

$$< \Psi, a^{\epsilon_n}(g_n) \dots a^{\epsilon_1}(g_1)\Psi >,$$

where $a^{\epsilon_i}(g_i)$ denotes either $a(g_i)$ or $a^\dagger(g_i)$, we note that n must be even and that the number of creators equals the number of annihilators in order that the expression is non-zero. Furthermore there must exist a non-crossing pair partition for the sequence ϵ_n, ..., ϵ_1 as outlined below.

Following the well-known notation we write

$$\overset{\frown}{A.a(f).B.a^\dagger}(g).C =< f, g > A.B.C \tag{2.5}$$

for arbitrary operators A, B and C. This is a contraction and we reserve the notation for the case of an annihilator with a creator only, with the annihilator to the left of the creator.

The relation (2.3) and the fact that $a(g)\Psi = 0$, $\forall g \in h$, is enough to calculate the vacuum expectation of any product of creators and annihilators. However we may give the rule of thumb that any such expression equals the complete contraction decomposition where none of the contraction lines are allowed to cross (if no such non-crossing set of contractions exists then the expression vanishes).

Thus

$$E[a(g_8)a^\dagger(g_7)a(g_6)a(g_5)a(g_4)a^\dagger(g_3)a^\dagger(g_2)a^\dagger(g_1)] =$$

$$\overset{\frown}{a(g_8)}a^\dagger(g_7)\overset{\frown}{a(g_6)}\overset{\frown}{a(g_5)}\overset{\frown}{a(g_4)}a^\dagger(g_3)a^\dagger(g_2)a^\dagger(g_1) =$$

$$< g_8, g_7 >< g_4, g_3 >< g_5, g_2 >< g_6, g_1 > . \tag{2.6}$$

We note that, if a product of creators and annihilators admits a non-crossing contraction decomposition, then it is unique. So

$$E[X(f)^{2n}] = c_n ||f||^{2n},$$

where c_n is the number of non-crossing n contractions, that is

$$c_n = \frac{1}{n+1}\binom{2n}{n},$$

which is also known as the Catalan number.

One may then show that

$$\rho(x) = \frac{1}{\|f\|} w(\frac{x}{\|f\|}),$$

where

$$w(x) = \frac{1}{2\pi}\sqrt{4-x^2}\chi_{(-2,2)}(x). \tag{2.7}$$

This is the so-called Wigner semi-circle[1] law.

If we symmetrize the n-space by means of the projector P defined by

$$P(\phi_1 \otimes ... \otimes \phi_n) = \frac{1}{n!}\sum_{\sigma \in S_n} \phi_{\sigma_1} \otimes ... \otimes \phi_{\sigma_n},$$

where S_n denotes the symmetric group of degree n, consisting of all permutations on $\{1, ..., n\}$, then we may define the Bose Fock space over h to be $\oplus_{n=0}^{\infty}\{P \otimes^n h\}$. The Bose Fock space over $\mathbf{C}$, for instance, is Hardy space. Bose creators and annihilators are defined by

$$b^{\epsilon}(g) = Pa^{\epsilon}(g)P \tag{2.8}$$

and one has the commutation relations

$$[b(f), b^{\dagger}(g)] = b(f)b^{\dagger}(g) - b^{\dagger}(g)b(f) = < f, g > . \tag{2.9}$$

The self-adjoint operator $Y(f) := b(f) + b^{\dagger}(f)$, as is well-known, is gaussian distributed with mean zero and variance $\|f\|^2$ in the vacuum state, i.e.

$$E[e^{isY(f)}] = e^{-\frac{1}{2}s^2\|f\|^2}. \tag{2.10}$$

In fact, expressions of the type $E[b^{\epsilon_n}(f_n)...b^{\epsilon_1}(g_1)]$ can be calculated by summing over all sets of decompositions into pair contractions. So

$$E[Y(f)^{2n}] = d_n\|f\|^2,$$

where

[1] The mathematically correct terminology of semi-elliptic distribution is often used.

$$d_n = \frac{1}{2^n}\frac{(2n)!}{n!}.$$

The number of crossing diagrams with n contractions is then equal to $d_n - c_n$. Note that

$$\frac{c_n}{d_n} = \frac{2^n}{(n+1)!},$$

so the crossing contraction diagrams quickly out-proliferate the non-crossing ones, see figure 1, p. 28.

3. Quantum Damping

Consider a quantum mechanical system S consisting of a single electron in a metal. If the electron is close to the edge of a conduction band we may treat it as a free particle: if its momentum is p, then its energy is

$$e(p) = \frac{p^2}{2m},$$

where m is the effective mass. As a model of electric resistance we couple the electron to a reservoir R which damps the motion: R will be a field of quantum particles called phonons. We describe the dynamical evolution using the Hamiltonian H_λ acting on the combined state space $\mathcal{H}_S \otimes \mathcal{H}_R$ (where $\mathcal{H}_S$ is the Hilbert state space of the system and $\mathcal{H}_R$ is the Hilbert state space of the reservoir):

$$H_\lambda = \{H_S \otimes 1_R + 1_S \otimes H_R\} + \lambda H_I, \tag{3.1}$$

where λ is a coupling constant. The system Hamiltonian is $H_S = e(p)$, where p denotes canonical momentum, while

$$H_R = \int d^3k \quad \omega(k)b^{\dagger}(k)b(k)$$

$$H_I = i\int d^3k\{\theta(k) \otimes b^{\dagger}(k) - \theta^{\dagger}(k) \otimes b(k)\}$$

and the operators $b^{\epsilon}(k)$ satisfy the relations

$$[b(k), b^{\dagger}(k')] = \delta^3(k-k'), \quad [b(k), b(k')] = 0. \tag{3.2}$$

Here, $b^\dagger(k)$ is the operator describing the creation of a phonon of momentum k.

The operators $\theta(k)$ act on $\mathcal{H}_S$ and should take the form

$$\theta(k) = e^{-ik.q} \tag{3.3}$$

where q is canonical position. This is the responsive part of the interaction H_I. It is responsible for recording the recoil of the electron when it emits or absorbs a phonon. For instance, in figure 2, p. 28, we have an emission and absorption vertex. The presence of the response ensures momentum conservation: so for both diagrams $p' = p - k$. Note that associated with the emission vertex is the the energy non-conservation by an amount $\Delta(p,k) = e(p-k) + \omega(k) - e(p)$ and there is an equal and opposite amount for the absorption vertex.

In some situations it is possible to make an approximation of the type

$$\theta(k) \cong Dg(k) \tag{3.4}$$

where D is independent of k and g is a scalar function (say Schwartz on $\mathbf{R}^3$): this is the responseless approximation[2]. In this case the interaction simplifies to

$$H_I = i\{D \otimes B^\dagger(g) - D^\dagger \otimes B(g)\}$$

where

$$B^\dagger(g) = \int d^3k \quad g(k)b^\dagger(k).$$

The operators $B^\epsilon(g)$ are in fact creators/annihilators on the Bose Fock space over $L^2(\mathbf{R}^3)$. Under the responseless approximation one has $p \equiv p'$ for D diagonal in p in figure 2, p. 28.

It is known that under a van Hove scaling limit (where time is rescaled by $\frac{1}{\lambda^2}$ and one takes the limit $\lambda \to 0$), the responseless interaction leads to a quantum brownian motion, [2], as limit noise. This is due to the underlying gaussianity of the bose fields

[2] In quantum electro-dynamics, this is known as the dipole approximation; however, here the damping vanishes in general.

and the simplicity of the responseless interaction. We now wish to give an indication of why the same scaling limit for the proper responsive interaction leads to a limit noise which is closer in spirit to Wigner type than to gaussian type.

The key to understanding the limit noise in general is the fact that the van Hove limit extracts the behaviour predicted by the Golden Rule approximation of quantum physics. Consider the diagram in figure 3, p. 28, which shows a crossing of phonon lines. The phonons are virtual particles of momentum k and k' respectively: it is implicit that in order to calculate the coefficient associated with this diagram we integrate over all k and k'. However we must include the terms $\delta(\Delta_1 + \Delta_3) \times \delta(\Delta_2 + \Delta_4)$, where Δ_j is the energy non-conservation at the j^{th} vertex:

$$\Delta_1 + \Delta_3 = \{e(p-k) + \omega(k) - e(p)\} + \{e(p-k') - e(p-k-k') - \omega(k)\}$$

$$= -\frac{1}{m}k.k' \tag{3.5}$$

so there is a restriction of the k, k' integration to a set of measure zero in $\mathbf{R}^6$. As a rule, all diagrams which are crossing vanish identically, while all non-crossing diagrams give a non-trivial coefficient.

Thus the combination of the Golden Rule applied to each pair of contracted vertices and the constraint of momentum conservation leads to the non-triviality of only the non-crossing diagrams.

The effect is of a universal nature. It even holds if the reservoir quanta are fermionic: in fact we may change the relations (3.2) to anti-commutation relations $b(k)b^\dagger(k) + b^\dagger(k)b(k)$ without changing the final numerical result as the non-crossing diagrams do not have an associated sign change.

Figure 1:

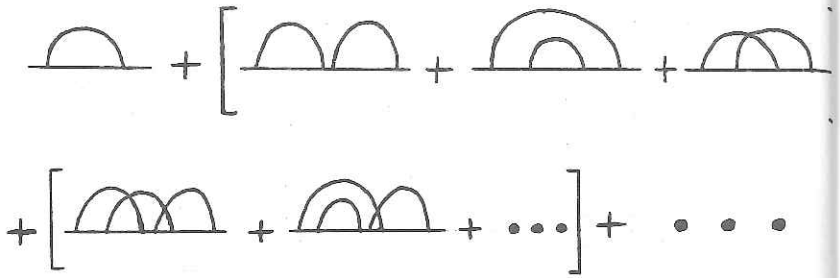


Figure 2:

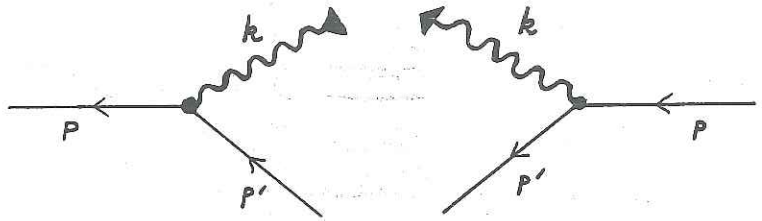
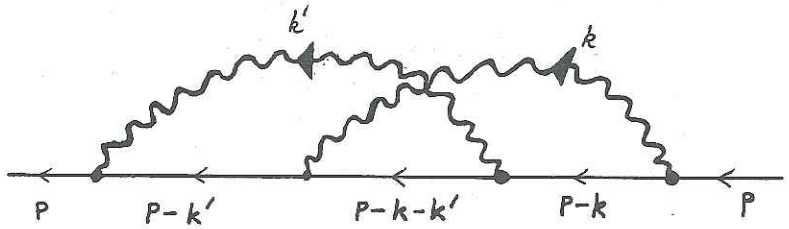


Figure 3:



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