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The aim of the Bulletin is to inform Society members about the activities of the Society and about items of general mathematical interest. It appears twice each year, at Easter and at Christmas. The Bulletin is supplied free of charge to members; it is sent abroad by surface mail. Libraries may subscribe to the Bulletin for IR£20.00 per annum.

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THE IRISH MATHEMATICAL SOCIETY

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NOTES ON APPLYING FOR I.M.S. MEMBERSHIP

1. The Irish Mathematical Society has reciprocity agreements with the American Mathematical Society and the Irish Mathematics Teachers Association.

2. The current subscription fees are given below.

Institutional member	IR£50.00
Ordinary member	IR£15.00
Student member	IR£6.00
I.M.T.A. reciprocity member	IR£5.00

   The subscription fees listed above should be paid in Irish pounds (puint) by means of a cheque drawn on a bank in the Irish Republic, a Eurocheque, or an international money-order.

3. The subscription fee for ordinary membership can also be paid in a currency other than Irish pounds using a cheque drawn on a foreign bank according to the following schedule:

   If paid in United States currency then the subscription fee is US$25.00.
   If paid in sterling then the subscription fee is £15.00 stg.
   If paid in any other currency then the subscription fee is the amount in that currency equivalent to US$25.00.

   The amounts given in the table above have been set for the current year to allow for bank charges and possible changes in exchange rates.

4. Any member with a bank account in the Irish Republic may pay his or her subscription by a bank standing order using the form supplied by the Society.

5. The subscription fee for reciprocity membership by members of the American Mathematical Society is US$10.00.

6. Any ordinary member who has reached the age of 65 years and has been a fully paid up member for the previous five years may pay at the student membership rate of subscription.

7. Subscriptions normally fall due on 1 February each year.

8. Cheques should be made payable to the Irish Mathematical Society. If a Eurocheque is used then the card number should be written on the back of the cheque.

9. Any application for membership must be presented to the Committee of the I.M.S. before it can be accepted. This Committee meets twice each year.

10. Please send the completed application form with one year's subscription fee to

The Treasurer, I.M.S.
Department of Math. Physics
University College, Dublin
Ireland

Minutes of the Meeting of the Irish Mathematical Society

Ordinary Meeting
20th December 1996

The Irish Mathematical Society held an Ordinary Meeting at 12.15pm on Friday 20th December 1996 in the Dublin Institute for Advanced Studies, 10 Burlington Road. There were 16 members present. The Vice-President, C. Nash was in the chair. Apologies were received from D. Hurley (President) and P. Mellon (Secretary).

1. Minutes

The minutes of the meeting of April 1996 were approved and signed.

2. Matters arising

It was noted that D. Hurley has a contact who can sculpt the proposed I.M.S. Mathematics Olympiad trophy. This will be discussed at the next meeting.
Concern was expressed by members from U.C.D. about the narrowly won decision of the U.C.D. Academic Council to dispense with the bonus points for Honours Leaving Certificate Mathematics.
The joint I.M.S.–Irish Mathematics Teachers' Association venture to organize a lecture for transition (4th) year secondary school students took place successfully at U.C.D. The speakers were Martin Newell and Phil Boland.

3. Bulletin

It was noted that the Christmas 1996 edition of the Bulletin is now available and is being distributed by G. Lessells.
R. Gow (editor) appealed for articles for consideration for the next issue.
It was agreed that the Bulletin should include a reference to the IMS web-page URL http://www.maths.tcd.ie/pub/ims

The Society recorded its gratitude to R. Gow (editor) for his efforts in producing the Bulletin.

4. Treasurer's Business

The Treasurer presented his interim report for 1996. The surplus for the year stands at £2,812 but this includes many U.S. members who have paid $50 in advance for five years' membership.
The Treasurer reported that he has invested £2,000 in a 5-year Saving Certificate.
The Society is £300 in arrears with its E.M.S. subscription. In view of the improved financial situation of the Society, it was agreed that this should now be paid in full. However, some concern was expressed about the fact that there are only seven I.M.S. members who are also in the E.M.S. and that the benefit of membership includes just a newsletter.
It was agreed to allocate £800 towards the costs of organizing the 1997 September Meeting in D.I.T.
It was agreed that £600 be set aside in each year 1997 and 1998 to cover the projected costs of the proposed joint meeting with the L.M.S. in May 1998.

5. September Meetings

It was noted that the September 1996 meeting at the Queen's University of Belfast was very successful and, thanks to the sponsorship of QUB, was self-financing. The Society recorded its gratitude to the organizers, and especially to Prof. D. Armitage and Prof. A. Wickstead.
The 1997 September meeting will take place at D.I.T. on Thursday and Friday, 4th and 5th September 1997. J. M. Golden is the main organizer.
It was agreed that the 1998 September meeting will take place in the University of Ulster, Jordanstown or Coleraine, with K. Houston as the main organizer.

6. Elections

It was noted that D. Hurley, C. Nash, M. Clancy, B. Goldsmith, K. Hutchinson, G. Lessells and M. Tuite have all reached the end of their two-year terms of office on the committee.

An election took place to fill the six vacant committee positions. The nominations were as follows:

Position	Nominee	Proposer	Seconder
President	C.Nash††	J.Pulé	D.Ó Mathúna
Vice-Pres	D.Armitage	C.Nash	N.Buttimore
Ordinary Members	G.Lessells††	D.Ó Mathúna	R.Timoney
	D.Hurley††	D.Ó Mathúna	R.Dark
	K.Hutchinson†	D.Ó Mathúna	R.Dark
	M.Clancy†	D.Ó Mathúna	R. Dark

Since there was only one candidate for each position all six nominees were elected. Each † denotes a previous term of office. It was noted that P. Mellon (Secretary), J. Pulé (Treasurer), E. Gath, R. Gow and A. Wickstead each has one more year remaining in their current term of office. It was proposed that the committee consider coopting J. M. Golden and M. Tuite as members.

7. Any other business

7.1 Proposal for joint IMS/LMS meeting

R. Timoney recalled that in 1986 the I.M.S. and the London Mathematical Society hosted a joint meeting in Dublin. He proposed that a joint meeting take place in London in May 1998, with the title *Complex Analysis and Dynamical Systems*. Six speakers would be invited, four from outside the U.K. and Ireland, and two from within. The I.M.S. would nominate two of these speakers. The estimated cost is £3,500. R. Timoney proposed that the I.M.S. contribute £1,200 of this (*vide* Treasurer's Report). The L.M.S. has nominated a subcommittee consisting of Alan Beardon and Shaun Bullett to organize this event. R. Timoney and D. Hurley are the I.M.S. local organizers. The proposal was approved.

It was suggested that sponsorship in the form of travel grants *etc* should be sought from the British Council and the Department of Foreign Affairs.

7.2 White Paper on Science, Technology and Innovation

N. Buttimore drew to the attention of the meeting, the recent government *White Paper on Science, Technology and Innovation.* The I.M.S. made a submission in advance of this white paper two years ago. It was agreed that the Society should write to the Minister for Commerce, Science and Technology, in advance of the 1997 budget, outlining the need for government support for fundamental research in Mathematics and Science. It was suggested that the letter be short (about one page) and that it avoid using international comparisons.

The meeting closed at 1.20pm.

Eugene Gath
University of Limerick

THE IMS SEPTEMBER MEETING 1996

D. H. Armitage and A. W. Wickstead

The ninth September meeting of the Irish Mathematical Society took place on 2nd and 3rd September 1996 at the Queen's University of Belfast. The lectures were held in the David Bates Building, adjacent to the Botanic Gardens. About 50 people attended, including participants from all parts of Ireland, from Britain, and from further afield. Visitors were accommodated in Queen's Elms Halls of Residence, about one kilometre from the main campus.

Queen's Senior Pro-Vice-Chancellor, Professor R. G. Shanks, delivered a welcoming address, and the Vice-President of the I.M.S., Dr. C. Nash, opened the proceedings. The guest speakers were Dr T. A. Gillespie (University of Edinburgh), Professor S. K. Houston (UU), Professor S. J. Tobin (UCG), Professor T. T. West (TCD) and Professor A. D. Wood (DCU). The hosts received an almost overwhelming response to their call for volunteered lectures. A wide range of topics was covered: algebra, analysis, geometry, topology, mathematics in computer science, history of mathematics, and mathematics education. The full programme is reproduced below.

A very enjoyable conference banquet was held in the University's Great Hall. Professor Shanks, Professor A. E. Kingston (Provost of the College of Science and Agriculture, QUB) and Professor D. G. Walmsley (Director of the School of Mathematics and Physics, QUB) were guests at the banquet.

We are grateful to the School of Mathematics and Physics at QUB for financial support which covered most expenses. A small remaining deficit was met by our Department of Pure Mathematics. We thank colleagues who helped with the organization of the meeting, especially Miss S. O'Brien whose secretarial work was invaluable. We also extend our thanks to all the speakers and everyone who contributed to the success of the meeting.

The full program was:

Monday 2nd September, 1996

Professor R. G. Shanks, Senior Pro-Vice Chancellor, QUB *Welcoming address*
Dr C. Nash (IMS Vice-President) *Opening Remarks*
Professor T. T. West (Trinity College Dublin) *Perron-Frobenius theorems for positive matrices*
Professor I. Düntsch (University of Ulster) *Introduction to the rough set model for data analysis*
M. Marjoram (University College Dublin) *Irreducible characters of Sylow p-subgroups of classical groups*
Professsor S. J. Tobin (University College Galway) *The Burnside saga*
A. Hughes (Trinity College Dublin) *A connection between Constructive Mathematics and Pure Mathematics*
Professor A. G. O'Farrell (St Patrick's College, Maynooth) *Algebras of smooth functions*
Professor A. D. Wood (Dublin City University) *G. G. Stokes: the man and his phenomenon today*

Tuesday 3rd September, 1996

Dr C. Nash (St Patrick's College, Maynooth) *Modular invariance of topological quantum field theories*
Dr M. Mac an Airchinnigh (Trinity College Dublin) *Some applications of algebraic topology in computer science*
Professor P. D. Barry (University College Cork) *Cross-currents in the development of geometry*
Dr T. A. Gillespie (University of Edinburgh) *Making the most of quality assessment*
Professor S. K. Houston (University of Ulster) *The role of mathematical modelling in undergraduate courses*
Dr S. T. Swift (Southampton) *Spaces of symplectic submanifolds*
K. Abodayeh *Semigroups associated to dynamical systems*
Dr S. Pabst (Dublin Institute of Technology) *On almost free abelian groups with trivial dual*
Professor M. Nishihara (Fukuoka University) *On holomorphic mappings of weak type*

D. H. Armitage and A. W. Wickstead,
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Queen's University,
Belfast BT7 1NN.

Comments on Two Papers by R. Gow in the December 1995 IMS Bulletin

Concerning the article on Bourbaki's problem in the December 1995 issue of the IMS Bulletin, I discovered that a theorem of L. K. Hua (*On the automorphisms of a field*, Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 386-389) answers the problem that I raised. I am grateful to my colleagues Fergus Gaines and David Lewis for information on the problem, and also to Professor Larry Harris for related correspondence. If I had remembered an earlier paper by Fergus Gaines (*How to compose a problem for the International Mathematical Olympiad?*, Irish Math. Soc. Bulletin **28** (1992), 20-29), I would not have written the article.

Concerning another article written by us in the same issue of the Bulletin (*Some Galway professors of mathematics and of natural philosophy*), Professor Alastair Wood has kindly informed me that Morgan Crofton's father was not the successor of G. G. Stokes's father as Rector of Skreen, Co. Sligo, as stated in the article. His successor was the Rev. George Trulock, who was Rector from 1834 until 1847. The Rev. W. Crofton was in fact the successor of Trulock, and died in Skreen in 1851.

Rod Gow,
Department of Mathematics,
University College Dublin.

A LINEAR SYSTEM OF IMPULSIVE DIFFERENTIAL EQUATIONS

Michael Brennan and Finbarr Holland

Abstract A linear system of impulsive differential equations that models an example from pharmacokinetics is investigated. The example is where a drug is administered periodically at certain fixed times resulting in a jump (called an *impulse*) in the concentration level of the drug. The case where the same dosage is applied at each of these fixed times is considered. Both necessary and sufficient conditions are sought to guarantee that an effective concentration level of the drug is maintained in the body.

Introduction

Impulsive differential equations are used to describe physical processes that undergo instantaneous perturbations. As in the study of ordinary differential equations, the study of impulsive differential equations is motivated by many practical examples from the physical sciences, [1, 2]. In this paper, we look at a linear system of impulsive differential equations, a special case of which may model the concentration of a drug in the bloodstream and an organ, like the heart or liver. Our results here generalize those obtained by Stewart, [3, pp. 758-759], who considered the scalar case.

The paper begins by describing a first order linear system of impulsive differential equations in terms of a constant real matrix K. We call the solution vector ϕ of the system an *admissible* solution when it lies in the region between two specified concentric spheres in $\mathbf{R}^d$. The initial dosage vector c and the time T (length

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of time between consecutive impulses) form a pair of parameters which are critical in determining the admissibility of ϕ. The paper's main result requires that K be both positive definite and symmetric, combined with a method of choosing first T and then c. The way we choose T and c depends on the eigenvalues of K and the radii of the two spheres.

1. Preliminaries

Consider the following linear system of impulsive differential equations:

$$\begin{aligned} &\dot{x}(t) = -Kx(t), \qquad\qquad t \neq T, 2T, \ldots, \\ &\Delta x(t) = x(t^+) - x(t^-) = c,\ t = T, 2T, \ldots, \\ &x(0^+) = c, \end{aligned} \tag{1}$$

where K is a constant real $d \times d$ matrix, c a vector in $\mathbf{R}^d$ and $T > 0$.

Definition 1 A solution to (1) is a piecewise continuous function $\phi : [0, \infty) \to \mathbf{R}^d$ such that
(i) $\dot{\phi}(t) = K\phi(t)$, for $t \in \mathbf{R} \setminus \{T, 2T, \ldots\}$;
(ii) at times nT, where $n \in \mathbf{N}$, it is the case that $\phi((nT)^+) = \phi((nT)^-) + c$;
(iii) $\phi(0^+) = c$;
(iv) $\lim_{t \to (nT)^+} \phi(t) = \phi(nT)$ for $n \in \mathbf{N}$.

Note that the existence and uniqueness of a solution to the linear system above is well known (see [2] for details). We now make some more definitions that will be needed later.

Definition 2 Let B be a real $d \times d$ matrix. Then B is said to be *positive definite* if for any $a \in \mathbf{R}^d$ we have $a^t Ba \geq 0$ with equality occurring if and only if $a = 0$.[1]

Here the transpose of the vector a is denoted by a^t. As usual, B is said to be *symmetric* if $B = B^t$. A consequence of B being

[1] We adhere to this definition throughout the paper, but readers should be warned that unlike other authors we do not require that a positive definite matrix be symmetric.

positive definite and symmetric is that its eigenvalues are both real and positive.

Definition 3 Given two parameters L and H where $0 \leq L < H < \infty$ we say that ϕ is an *admissible* solution of (1) if, for some choice of parameters T and c, ϕ solves the linear impulsive system (1) and, for all $t > 0$, $\phi(t)$ lies in the set $\{a \in \mathbf{R}^d : L \leq \|a\| \leq H\}$. For convenience we call this set $A(0; L, H)$.

Note that $\|a\|$ is the length of the vector a:

$$\|a\|^2 = a^t a = \sum_{i=1}^{d} |a_i|^2.$$

To describe the solution of (1) we define sequences of vectors A_n, B_n in $\mathbf{R}^d$ as follows: Let

$$\begin{aligned} &A_{-1} = 0, \\ &A_n = c + e^{-TK}c + e^{-2TK}c + \cdots + e^{-nTK}c, \quad n = 0, 1, 2 \ldots, \end{aligned}$$

and

$$B_n = e^{-TK} A_{n-1}, \quad n = 0, 1, 2, \ldots.$$

Note that

$$\begin{aligned} A_n &= (I + e^{-TK} + \cdots + e^{-nTK})c \\ &= (I - e^{-(n+1)TK})(I - e^{-TK})^{-1}c, \end{aligned}$$

where I is the unit matrix.

The solution of the differential equation is then given by

$$x(t) = e^{-(t-(n-1)T)K} A_{n-1}, \quad \text{if } t \in [(n-1)T, nT), \quad n = 1, 2, \ldots.$$

Our objective is to determine (practical) necessary and sufficient conditions on the various parameters defining the system that ensure the solution $x(t)$ is admissible.

2. Necessary and sufficient conditions for admissibility

Theorem 1 *Suppose that K is positive definite. Then $x(t) \in A(0; L, H)$ for all $t \in (0, \infty)$ if and only if*

$$L \leq \inf \|B_n\| \text{ and } \sup \|A_n\| \leq H.$$

Proof: From the differential equation we see that

$$\frac{d}{dt}\|x(t)\|^2 = 2x^t\dot{x} = -2x^tKx \leq 0,$$

for all $t \in ((n-1)T, nT)$, $n = 1, 2, \ldots$, and so $\|x(t)\|$ is decreasing on the intervals $((n-1)T, nT)$, $n = 1, 2, \ldots$. It follows that

$$\|B_n\| \leq \|x(t)\| \leq \|A_{n-1}\|,$$

for all $t \in ((n-1)T, nT)$. Hence, if

$$L \leq \inf \|B_n\| \text{ and } \sup \|A_n\| \leq H,$$

then

$$L \leq \|x(t)\| \leq H, \text{ for all } t > 0.$$

Conversely, if this holds, then

$$\|B_n\| = \|x((nT)^-)\| \geq L$$

and

$$\|A_n\| = \|x((nT)^+)\| \leq H$$

for $n = 1, 2, \ldots$. Hence the given condition is also necessary. •

Different aspects of the following example have been studied by various authors, [1, 2, 4]. The example below consists of a two-compartment model for the distribution of a drug in the body, say the gastro-intestinal tract and the bloodstream. The components x_1 and x_2 stand for the quantity of the drug in the gastro-intestinal tract and the bloodstream, respectively, and k_1 and k_2 are the respective rate constants. The differential equations and associated boundary conditions governing this model are given by the following:

$$\begin{aligned} \dot{x}_1(t) &= -k_1x(t),\ 0 < t \neq T, 2T, \ldots, \\ \dot{x}_2(t) &= k_1x_1(t) - k_2x_2(t),\ 0 < t \neq T, 2T, \ldots, \\ x_i(t^+) - x_i(t^-) &= c_i,\ i = 1, 2,\ t = T, 2T, \ldots \end{aligned}$$

and

$$x_i(0) = x_i(0^+) = c_i,\ i = 1, 2.$$

Theorem 1 applies to this system once we know for what values of k_1 and k_2 the matrix of the system is positive definite. The answer is provided by the following.

Example 1 *Suppose that $4k_2 > k_1 > 0$. Then the 2×2 matrix*

$$K = \begin{pmatrix} k_1 & 0 \\ -k_1 & k_2 \end{pmatrix}$$

is positive definite, but not symmetric.

Proof: For, if $a \in \mathbb{R}^2$, then

$$\begin{aligned} a^tKa &= k_1a_1^2 - k_1a_1a_2 + k_2a_2^2 \\ &= k_1(a_1^2 - a_1a_2) + k_2a_2^2 \\ &= k_1(a_1 - a_2/2)^2 + (k_2 - k_1/4)a_2^2 \\ &\geq 0 \end{aligned}$$

and equality holds if and only if $a_1 = a_2 = 0$. Thus K is positive definite. Clearly, $K \neq K^t$. •

It is also clear that this matrix is not positive definite if $0 < 4k_2 \leq k_1$.

Returning to the general situation, the fact that K is positive definite implies that

$$\|B_n\| \leq \|A_{n-1}\|,\ n = 1, 2, \ldots,$$

as we have just seen. We wish to investigate the monotonicity of these sequences under some additional assumptions about K. In what follows, we suppose that K is symmetric. This, coupled with the fact it is positive definite, implies that K is unitarily equivalent to a diagonal matrix. Thus $K = U^tDU$, where U is unitary, $U^tU = I$, and $D = \text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_d]$, where $\lambda_i > 0$, $i = 1, 2, \ldots, d$, and so

$$e^{-TK} = U^te^{-TD}U = U^t\text{diag}[e^{-\lambda_1 T}, e^{-\lambda_2 T}, \ldots, e^{-\lambda_d T}]U.$$

Next,

$$A_{n-1} = (I - e^{-nTK})y, \text{ where } y = (I - e^{-TK})^{-1}c,$$

and so

$$\begin{aligned} \|A_{n-1}\|^2 &= A_{n-1}^t A_{n-1} \\ &= y^t(I - e^{-nTK})^t(I - e^{-nTK})y \\ &= y^tU^t(I - e^{-nTD})^2Uy \\ &= z^t(I - e^{-nTD})^2z, \text{ where } z = Uy \\ &= \sum_{i=1}^{d}(1 - e^{-n\lambda_i T})^2|z_i|^2. \end{aligned}$$

But the sequence $(1-e^{-n\lambda T})$ is increasing if $\lambda > 0$. Hence $\|A_{n-1}\|$ is increasing. But it is also clear that the positive definiteness of K implies that

$$\lim A_n = y,$$

so that

$$\sup \|A_n\| = \|y\| = \|(I - e^{-TK})^{-1}c\|.$$

In the same way it can be seen that $\|B_n\|$ is increasing, and so

$$\inf \|B_n\| = \|B_1\| = \|e^{-TK}c\|.$$

These considerations enable us to express the conclusion of Theorem 1 in the following form.

Theorem 2 *Suppose that K is positive definite and symmetric. Then $x(t) \in A(0; L, H)$, for all $t \in (0, \infty)$ if and only if*

$$L \leq \|e^{-TK}c\| \text{ and } \|(I - e^{-TK})^{-1}c\| \leq H.$$

Still supposing that K is unitarily equivalent to a diagonal matrix, none of whose eigenvalues is zero, we have

$$\begin{aligned} \|e^{-TK}c\|^2 &= c^te^{-2TK}c \\ &= v^te^{-2TD}v, \text{ where } v = Uc \\ &= v^t\,\text{diag}[e^{-2T\lambda_1}, \ldots, e^{-2T\lambda_d}]\,v \\ &= \sum_{i=1}^{d} e^{-2T\lambda_i}|v_i|^2 \end{aligned}$$

and

$$\begin{aligned} \|(I - e^{-TK})^{-1}c\|^2 &= c^t(I - e^{-TK})^{-2}c \\ &= v^t(I - e^{-TD})^{-2}v \\ &= v^t\,\text{diag}[(1 - e^{-T\lambda_1})^{-2}, \ldots, (1 - e^{-T\lambda_d})^{-2}]\,v \\ &= \sum_{i=1}^{d} \frac{|v_i|^2}{(1 - e^{-T\lambda_i})^2}. \end{aligned}$$

Hence we can phrase the necessary and sufficient condition that $x(t) \in A(0; L, H)$ as follows.

Theorem 3 *Suppose that $K = U^t\,\text{diag}[\lambda_1, \ldots, \lambda_d]\,U$, where U is unitary and the λ_i's are positive. Let $v = Uc$. Then $x(t) \in A(0; L, H)$ for all $t \in (0, \infty)$ if and only if*

$$L^2 \leq \sum_{i=1}^{d} e^{-2T\lambda_i}|v_i|^2 \text{ and } \sum_{i=1}^{d} \frac{|v_i|^2}{(1 - e^{-T\lambda_i})^2} \leq H^2.$$

3. Necessary conditions for admissibility

We deduce some coupled restrictions on the parameters K, L, H, c, T that must hold if the system has an admissible solution.

Theorem 4 *Suppose that K is symmetric and positive definite. If $x(t) \in A(0; L, H)$ for all $t > 0$, then*
(i) $4L \leq H$, with equality if and only if $e^{T\lambda_i} = 2$, $i = 1, \ldots, d$;
(ii) $Le^{T\lambda} \leq \|c\| \leq (1 - e^{-T\Lambda})H$, where

$$\Lambda = \max\{\lambda_i : i = 1, 2, \ldots\} \text{ and } \lambda = \min\{\lambda_i : i = 1, 2, \ldots\}.$$

Proof: Under our assumptions,

$$\begin{aligned} L^2 \leq \sum_{i=1}^{d} e^{-2T\lambda_i} |v_i|^2 &= \sum_{i=1}^{d} \{e^{-T\lambda_i}(1 - e^{-T\lambda_i})\}^2 \frac{|v_i|^2}{(1 - e^{-T\lambda_i})^2} \\ &\leq \frac{1}{16} \sum_{i=1}^{d} \frac{|v_i|^2}{(1 - e^{-T\lambda_i})^2} \\ &\leq \frac{H^2}{16}, \end{aligned}$$

since $4y(1 - y) \leq 1$ for all $y \in \mathbb{R}$, with equality if and only if $y = 1/2$, and so $16e^{-2x}(1 - e^{-x})^2 \leq 1$ for all $x \in [0, \infty)$. If $H = 4L$, then equality holds throughout, which clearly happens if and only if $e^{-T\lambda_i} = \frac{1}{2}$, $i = 1, 2, \ldots, d$. This gives the first part. As for the second part, we have

$$\begin{aligned} \|c\|^2 &= \|v\|^2 \\ &= \sum_{i=1}^{d} e^{-2T\lambda_i} |v_i|^2 e^{2T\lambda_i} \\ &\geq e^{2T\lambda} \sum_{i=1}^{d} e^{-2T\lambda_i} |v_i|^2 \\ &\geq e^{2T\lambda} L^2 \end{aligned}$$

and so $e^{T\lambda} L \leq \|c\|$.

Similarly,

$$\begin{aligned} \|c\|^2 &= \|v\|^2 \\ &= \sum_{i=1}^{d} (1 - e^{-T\lambda_i})^2 \frac{|v_i|^2}{(1 - e^{-T\lambda_i})^2} \\ &\leq (1 - e^{-T\Lambda})^2 \sum_{i=1}^{d} \frac{|v_i|^2}{(1 - e^{-T\lambda_i})^2} \\ &\leq (1 - e^{-T\Lambda})^2 H^2 \end{aligned}$$

and so $\|c\| \leq (1 - e^{-T\Lambda})H$. •

4. Sufficient conditions for admissibility

For practical reasons it is essential to have easily verifiable conditions that will guarantee that the system has an admissible solution. We proceed to give one such condition that guarantees that $x(t) \in A(0; L, H)$ for all $t > 0$. This will then enable us to give a simple sufficient condition for admissibility; this is recorded in Theorem 7.

Theorem 5 *Suppose that K is symmetric and positive definite. Let*

$$Le^{T\Lambda} \leq \|c\| \leq (1 - e^{-T\lambda})H.$$

Then $x(t) \in A(0; L, H)$, for all $t > 0$.

Proof: It is enough to note that

$$\begin{aligned} L^2 &\leq e^{-2T\Lambda} \|c\|^2 \\ &= e^{-2T\Lambda} \|v\|^2 \\ &= \sum_{i=1}^{d} e^{-2T\Lambda} |v_i|^2 \\ &\leq \sum_{i=1}^{d} e^{-2T\lambda_i} |v_i|^2. \end{aligned}$$

Similarly,

$$\sum_{i=1}^{d} \frac{|v_i|^2}{(1-e^{-T\lambda_i})^2} \le (1-e^{-T\lambda})^{-2}\|v\|^2$$
$$= (1-e^{-T\lambda})^{-2}\|c\|^2$$
$$\le H^2.$$

The stated result now follows from Theorem 3. •

Before leaving this result, we analyse the sufficient condition more closely in an attempt to uncouple the relationship between T and c. Our findings are summarized in the next theorem.

Theorem 6 *Let $r = H/L$. Under the assumptions of the previous theorem,*

(i) $r \ge 4$;

(ii)

$$\frac{\log \frac{r-\sqrt{r^2-4r}}{2}}{\lambda} \le T \le \frac{\log \frac{r+\sqrt{r^2-4r}}{2}}{\Lambda};$$

(iii)

$$\frac{[r-\sqrt{r^2-4r}]L}{2} \le \|c\| \le \frac{[r+\sqrt{r^2-4r}]L}{2}.$$

Proof: It follows from our assumptions that

$$Le^{T\lambda} \le \|c\| \le (1-e^{-T\lambda})H$$

and so

$$e^{2T\lambda} \le (e^{T\lambda}-1)r,$$

whence, by elementary methods, part (i) and the left-hand inequality of part (ii) follow. But also,

$$Le^{T\Lambda} \le \|c\| \le (1-e^{-T\Lambda})H$$

and so

$$e^{2T\Lambda} \le (e^{T\Lambda}-1)r,$$

whence the right-hand inequality of part (ii) also follows. Equivalently,

$$\frac{r-\sqrt{r^2-4r}}{2} \le e^{T\lambda} \le e^{T\Lambda} \le \frac{r+\sqrt{r^2-4r}}{2}.$$

From this the left side of part (iii) follows. Using the right-hand inequality in the previous display, we see that

$$1-e^{-T\lambda} \le 1-\frac{2}{r+\sqrt{r^2-4r}}$$
$$= 1-\frac{2[r-\sqrt{r^2-4r}]}{4r}$$
$$= \frac{r+\sqrt{r^2-4r}}{2r},$$

which gives the right-hand side of the inequality in part (iii). •

Theorem 6 means that if any one of the conditions (i), (ii) and (iii) fails to hold, then we do not have an admissible solution. But, at the same time, the satisfaction of all three is no guarantee that an admissible solution exists. The following example is intended to illustrate this.

Example 2 *Suppose that $K = \text{diag}[1,2]$. Let $H/L = r = 9/2$. Choose $T = \ln\sqrt{3}$ and let c be any vector whose first component is zero and $\sqrt{3}L \le \|c\| < 3L$. Then the inequalities stated in Theorem 6 are satisfied, but the system has no admissible solutions.*

It is clear that the inequalities in Theorem 6 hold. But, if the system has an admissible solution for the given values of T and c, we see from Theorem 3 that

$$L^2 \le e^{-2T}|c_1|^2 + e^{-4T}|c_2|^2 = e^{-4T}|c_2|^2 = \frac{\|c\|^2}{9},$$

which is false. •

The same example shows that the converse of Theorem 4 is false.

We now state a simple criterion for admissibility.

Theorem 7 *Given a symmetric positive definite matrix K, whose smallest and largest eigenvalues are λ, Λ, respectively, and positive constants H, L, with $r = H/L$ such that*

$$\min\{\frac{e^{x\Lambda}}{1-e^{-x\lambda}} : 0 < x\} \leq r.$$

Choose T so that $e^{T\Lambda} \leq (1-e^{-T\lambda})r$. Having selected T, now choose the vector c so that

$$Le^{T\Lambda} \leq ||c|| \leq (1-e^{-T\lambda})H.$$

Then the system (1) has an admissible solution.
Proof: This is a consequence of Theorem 5. •

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WHY PHONON LINES DON'T CROSS

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Abstract We report on recent developments in quantum stochastic approximations of physical systems, the relative merits of Gauss and Wigner distributions and the physical reasons one should arise rather than the other in a model of an electron interacting with a phonon field.

1. Introduction

Eugene Wigner, [1], introduced ensembles of $N \times N$ real matrices to model the spectra of complex nuclei and noticed that the $N \to \infty$ limit corresponds to a non-commutative central limit theorem involving a non-gaussian distribution as the limit distribution. The new distribution, called the Wigner or semi-circle law, frequently appears in place of the gaussian when one departs from ordinary probability to quantum probability, that is, when random variables are represented as non-commutative operators and probability as a positive normalized functional.

In the 1980's Hudson and Parthasarathy, [2], attempted to construct quantum (i.e. non-commutative) stochastic analogues to the brownian motion, and indeed Poisson, processes and the calculi by using the inherent gaussianity of bose fields. Later Voiculescu, [3], and Kümmerer and Speicher, [4], used free fields to construct free noise processes which are related to the Wigner law.

Here we report about the emergence of a new type of noise from physical models which is closer to the Wigner class than the Gauss class. It was first discovered by Lu by examining moments and later proven in general in [5]. This report centres on the physical mechanism behind this, [6].

2. Wigner versus Gauss

Let h be a separable Hilbert space. The Fock space over h is defined to be $\mathcal{H} = \oplus_{n=0}^{\infty}\{\otimes^n h\}$, where $\otimes^0 h \equiv \mathbf{C}$. An operator $a^\dagger(f)$, with argument $f \in h$, is defined by linearity from the mapping : $\otimes^n h \mapsto \otimes^{n+1} h$ with

$$a^\dagger(f)\phi_1 \otimes ... \otimes \phi_n \mapsto f \otimes \phi_1 \otimes ... \otimes \phi_n \tag{2.1}$$

The operator $a^\dagger(f)$ is called a creator; its adjoint $a(f)$, called an annihilator, can then be defined as the mapping : $\otimes^n h \mapsto \otimes^{n-1} h$ with

$$a(f)\phi_1 \otimes ... \otimes \phi_n = < f, \phi_1 > \phi_2 \otimes ... \otimes \phi_n . \tag{2.2}$$

They satisfy the relation

$$a(f)a^\dagger(g) = < f, g > . \tag{2.3}$$

This relation is often called the free relation. The vacuum vector is defined to be the vector $\Psi = 1 \oplus 0 \oplus 0 \oplus ....$ The vacuum expectation of an observable X is then taken as

$$E[X] = < \Psi, X\Psi > .$$

In particular, let $X(f) = a(f) + a^\dagger(f)$: the distribution ρ of $X(f)$ is obtained via the characteristic formula

$$E[e^{isX(f)}] = \int_{-\infty}^{\infty} e^{isx}\rho(x)dx \tag{2.4}$$

In order to calculate an expression like

$$< \Psi, a^{\epsilon_n}(g_n) \dots a^{\epsilon_1}(g_1)\Psi >,$$

where $a^{\epsilon_i}(g_i)$ denotes either $a(g_i)$ or $a^\dagger(g_i)$, we note that n must be even and that the number of creators equals the number of annihilators in order that the expression is non-zero. Furthermore there must exist a non-crossing pair partition for the sequence ϵ_n, $\dots$, ϵ_1 as outlined below.

Following the well-known notation we write

$$A.a(f).B.a^\dagger(g).C = < f, g > A.B.C \tag{2.5}$$

for arbitrary operators A, B and C. This is a contraction and we reserve the notation for the case of an annihilator with a creator only, with the annihilator to the left of the creator.

The relation (2.3) and the fact that $a(g)\Psi = 0$, $\forall g \in h$, is enough to calculate the vacuum expectation of any product of creators and annihilators. However we may give the rule of thumb that any such expression equals the complete contraction decomposition where none of the contraction lines are allowed to cross (if no such non-crossing set of contractions exists then the expression vanishes).

Thus

$$E[a(g_8)a^\dagger(g_7)a(g_6)a(g_5)a(g_4)a^\dagger(g_3)a^\dagger(g_2)a^\dagger(g_1)] =$$

$$a(g_8)a^\dagger(g_7)a(g_6)a(g_5)a(g_4)a^\dagger(g_3)a^\dagger(g_2)a^\dagger(g_1) =$$

$$< g_8, g_7 >< g_4, g_3 >< g_5, g_2 >< g_6, g_1 > . \tag{2.6}$$

We note that, if a product of creators and annihilators admits a non-crossing contraction decomposition, then it is unique. So

$$E[X(f)^{2n}] = c_n ||f||^{2n},$$

where c_n is the number of non-crossing n contractions, that is

$$c_n = \frac{1}{n+1}\binom{2n}{n},$$

which is also known as the Catalan number.

One may then show that

$$\rho(x) = \frac{1}{||f||} w(\frac{x}{||f||}),$$

where

$$w(x) = \frac{1}{2\pi}\sqrt{4-x^2}\chi_{(-2,2)}(x). \tag{2.7}$$

This is the so-called Wigner semi-circle[1] law.

If we symmetrize the n-space by means of the projector P defined by

$$P(\phi_1 \otimes ... \otimes \phi_n) = \frac{1}{n!}\sum_{\sigma \in S_n} \phi_{\sigma_1} \otimes ... \otimes \phi_{\sigma_n},$$

where S_n denotes the symmetric group of degree n, consisting of all permutations on $\{1, ..., n\}$, then we may define the Bose Fock space over h to be $\oplus_{n=0}^{\infty}\{P \otimes^n h\}$. The Bose Fock space over $\mathbf{C}$, for instance, is Hardy space. Bose creators and annihilators are defined by

$$b^\epsilon(g) = Pa^\epsilon(g)P \tag{2.8}$$

and one has the commutation relations

$$[b(f), b^\dagger(g)] = b(f)b^\dagger(g) - b^\dagger(g)b(f) = < f, g > . \tag{2.9}$$

The self-adjoint operator $Y(f) := b(f) + b^\dagger(f)$, as is well-known, is gaussian distributed with mean zero and variance $||f||^2$ in the vacuum state, i.e.

$$E[e^{isY(f)}] = e^{-\frac{1}{2}s^2||f||^2}. \tag{2.10}$$

In fact, expressions of the type $E[b^{\epsilon_n}(f_n)...b^{\epsilon_1}(g_1)]$ can be calculated by summing over all sets of decompositions into pair contractions. So

$$E[Y(f)^{2n}] = d_n||f||^2,$$

where

[1] The mathematically correct terminology of semi-elliptic distribution is often used.

$$d_n = \frac{1}{2^n}\frac{(2n)!}{n!}.$$

The number of crossing diagrams with n contractions is then equal to $d_n - c_n$. Note that

$$\frac{c_n}{d_n} = \frac{2^n}{(n+1)!},$$

so the crossing contraction diagrams quickly out-proliferate the non-crossing ones, see figure 1, p. 28.

3. Quantum Damping

Consider a quantum mechanical system S consisting of a single electron in a metal. If the electron is close to the edge of a conduction band we may treat it as a free particle: if its momentum is p, then its energy is

$$e(p) = \frac{p^2}{2m},$$

where m is the effective mass. As a model of electric resistance we couple the electron to a reservoir R which damps the motion: R will be a field of quantum particles called phonons. We describe the dynamical evolution using the Hamiltonian H_λ acting on the combined state space $\mathcal{H}_S \otimes \mathcal{H}_R$ (where $\mathcal{H}_S$ is the Hilbert state space of the system and $\mathcal{H}_R$ is the Hilbert state space of the reservoir):

$$H_\lambda = \{H_S \otimes 1_R + 1_S \otimes H_R\} + \lambda H_I, \tag{3.1}$$

where λ is a coupling constant. The system Hamiltonian is $H_S = e(p)$, where p denotes canonical momentum, while

$$H_R = \int d^3k \quad \omega(k)b^\dagger(k)b(k)$$

$$H_I = i\int d^3k\{\theta(k) \otimes b^\dagger(k) - \theta^\dagger(k) \otimes b(k)\}$$

and the operators $b^\epsilon(k)$ satisfy the relations

$$[b(k), b^\dagger(k')] = \delta^3(k-k'), \quad [b(k), b(k')] = 0. \tag{3.2}$$

Here, $b^{\dagger}(k)$ is the operator describing the creation of a phonon of momentum k.

The operators $\theta(k)$ act on $\mathcal{H}_S$ and should take the form

$$\theta(k) = e^{-ik.q} \tag{3.3}$$

where q is canonical position. This is the responsive part of the interaction H_I. It is responsible for recording the recoil of the electron when it emits or absorbs a phonon. For instance, in figure 2, p. 28, we have an emission and absorption vertex. The presence of the response ensures momentum conservation: so for both diagrams $p' = p - k$. Note that associated with the emission vertex is the the energy non-conservation by an amount $\Delta(p,k) = e(p-k) + \omega(k) - e(p)$ and there is an equal and opposite amount for the absorption vertex.

In some situations it is possible to make an approximation of the type

$$\theta(k) \cong Dg(k) \tag{3.4}$$

where D is independent of k and g is a scalar function (say Schwartz on $\mathbf{R}^3$): this is the responseless approximation[2]. In this case the interaction simplifies to

$$H_I = i\{D \otimes B^{\dagger}(g) - D^{\dagger} \otimes B(g)\}$$

where

$$B^{\dagger}(g) = \int d^3k \quad g(k)b^{\dagger}(k).$$

The operators $B^{\epsilon}(g)$ are in fact creators/annihilators on the Bose Fock space over $L^2(\mathbf{R}^3)$. Under the responseless approximation one has $p \equiv p'$ for D diagonal in p in figure 2, p. 28.

It is known that under a van Hove scaling limit (where time is rescaled by $\frac{1}{\lambda^2}$ and one takes the limit $\lambda \to 0$), the responseless interaction leads to a quantum brownian motion, [2], as limit noise. This is due to the underlying gaussianity of the bose fields

[2] In quantum electro-dynamics, this is known as the dipole approximation; however, here the damping vanishes in general.

and the simplicity of the responseless interaction. We now wish to give an indication of why the same scaling limit for the proper responsive interaction leads to a limit noise which is closer in spirit to Wigner type than to gaussian type.

The key to understanding the limit noise in general is the fact that the van Hove limit extracts the behaviour predicted by the Golden Rule approximation of quantum physics. Consider the diagram in figure 3, p. 28, which shows a crossing of phonon lines. The phonons are virtual particles of momentum k and k' respectively: it is implicit that in order to calculate the coefficient associated with this diagram we integrate over all k and k'. However we must include the terms $\delta(\Delta_1 + \Delta_3) \times \delta(\Delta_2 + \Delta_4)$, where Δ_j is the energy non-conservation at the j^{th} vertex:

$$\Delta_1 + \Delta_3 = \{e(p-k) + \omega(k) - e(p)\} + \{e(p-k') - e(p-k-k') - \omega(k)\}$$

$$= -\frac{1}{m}k.k' \tag{3.5}$$

so there is a restriction of the k, k' integration to a set of measure zero in $\mathbf{R}^6$. As a rule, all diagrams which are crossing vanish identically, while all non-crossing diagrams give a non-trivial coefficient.

Thus the combination of the Golden Rule applied to each pair of contracted vertices and the constraint of momentum conservation leads to the non-triviality of only the non-crossing diagrams.

The effect is of a universal nature. It even holds if the reservoir quanta are fermionic: in fact we may change the relations (3.2) to anti-commutation relations $b(k)b^{\dagger}(k) + b^{\dagger}(k)b(k)$ without changing the final numerical result as the non-crossing diagrams do not have an associated sign change.

Figure 1:

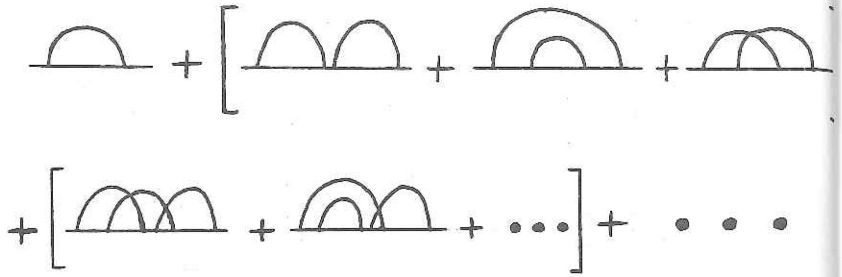


Figure 2:

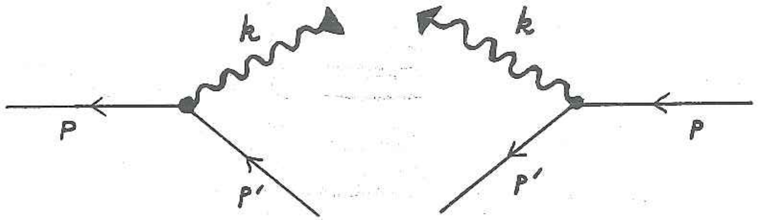
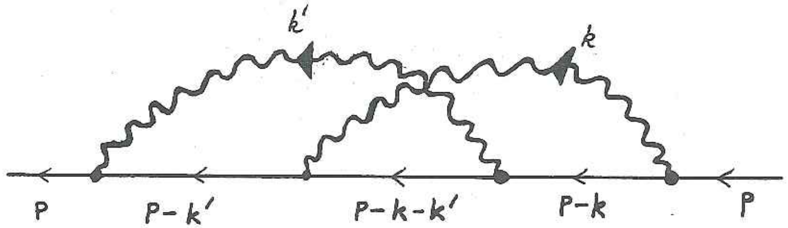


Figure 3:



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THE ROLE OF MATHEMATICAL MODELLING IN UNDERGRADUATE COURSES

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Abstract This paper examines the role of mathematical modelling in undergraduate courses. After setting modelling in the context of the three M's of mathematics-methods, models and modelling-and after describing the process of mathematical modelling, it suggests that mathematical modelling can help achieve the following aims in a course

- have a unifying effect
- create interest in novel applications
- develop interpersonal transferable skills
- develop deep learning
- develop self knowledge.

The implications for staff are that they will need to embrace radically new methods of teaching and assessing.

Introduction

It is my thesis that mathematical modelling is the way of life of a professional applied mathematician. More than that, mathematical modelling is a way of life, full stop; a way of life for everyone whether they realize it or not. So, it seems to me, every undergraduate student, of mathematics or of something else, and every pupil at school should meet the *concept* of a model, particularly a mathematical model; they should be aware of the *process* of modelling, and they should have some understanding of the *philosophy* of modelling. It seems to me that if there were a more widespread understanding and acceptance of the relationship between reality and models of reality and how they are created, then fundamentalists, both scientific and religious, would cause us all fewer problems.

While I would describe mathematical modelling as an art rather than a science because, at its best, it requires creative and imaginative thinking, that is not to say that the scientific method of investigating, conjecturing and proving is not valuable to the mathematical modeller. Of course it is, and, indeed, it is possible to take almost an algorithmic approach to modelling, but, as we shall see, even this includes the verb "create."

The three M's of mathematics

Mathematical Modelling is one of what I describe as the "three M's" of mathematics-Mathematical Methods, Mathematical Models and Mathematical Modelling. (The reader may be able to suggest some more.)

A mathematical model is a representation of some aspect of reality which, however complex, is necessarily a simplified representation. A model is used to describe and/or to predict the aspect of reality. A mathematical model consists of some mathematical entity such as an equation together with a statement of the simplifying assumptions that have been made in going from the reality to the model. Perhaps I should say, in going from our perception of the reality to the model. Thus Newton's second law of motion is a model of planetary motion. It is a differential equation which describes and predicts the motion of a planet and it embodies all sorts of assumptions such as the relationship between acceleration and force and the nature of the force of gravity as expressed in another model, Newton's universal law of gravity. It makes assumptions about planets being point masses in order to simplify the calculation. Increasingly mathematical models are being used in many walks of life and many academic disciplines.

Mathematical modelling is the activity of creating or modifying a model and using it. It is, I believe, the essence of applied mathematics. It requires a wide range of knowledge and skills. We must know something about mathematics before we can start, and similarly we need to know quite a lot about the aspect of reality we want to model. A knowledge and understanding of other models of this or other aspects of reality is also useful, as is an appreciation of how other modellers have worked in the past. It is particularly useful if we can obtain some insights into the way they went about it, the ideas they had, the questions they asked, even the blind alleys they went up. And we need to develop the same skills-reading, questioning, conjecturing and, indeed, proving. We need at least to attempt to validate or justify the model we have created.

Mathematical methods are the tools we use to create models and to answer questions about them. Arithmetic, algebra, calculus, geometry and so on-all are useful and some one, for some aspect of reality, is necessary. We need to learn which tools to use; we need to know of their existence and their scope; we may have to invent new ones.

Learning methods, studying models and engaging in modelling should all be part and parcel of an undergraduate course in mathematics. Also they should be part and parcel of high school mathematics. Enterprising primary school teachers might be able to find ways of getting their pupils in on the act as well. But there needs to be an awareness on the part of the student or pupil as to what is going on. When they are learning new tools they need to know in what context this might be useful; when studying models they need to look out for the assumptions made, the questions asked, the methods used and the extent to which the model is valid; when engaging in modelling they need to be aware of what they are doing at any particular time, what they have already done and what still remains to be done. In other words, they need a methodology for modelling.

The process of mathematical modelling

I expect that this is generally well known. Let me summarize it as an algorithm.

```
        Study the aspect of reality
                Identify main features
                Define variables
Label 1 Make simplifying assumptions
        Create model
                Establish relationships between variables
        Use model to answer questions
                Translate into mathematical problems
                Use methods to find solutions
        Interpret solutions
        Attempt to validate solutions
        If valid (or valid enough) finish cycle and report back
        If not valid, revise model by returning to Label 1 or earlier.
```


To repeat what I wrote earlier, I believe it is important for modellers to know when they are making simplifying assumptions and to articulate them always. This, if you like, qualifies the answer they may give to a problem. It is important for them to be aware of the range of validity of a solution and to be able to interpret their solution in terms of the original aspect of reality. It helps if they can say where they are in the modelling cycle when they are engaged in any particular task.

The role of mathematical modelling

Mathematical modelling can contribute in a number of ways to achieving the aim of a course.

Unifying effect

Modelling has a unifying effect on an undergraduate course in applied mathematics. Whether the student is studying mechanics or its derivatives, statistics or operational research, the ideas behind *methods*, *models* and *modelling* can be applied in each of those subject areas. Pure mathematics may be taught to give a rigorous foundation to methods, or it may be taught for its own sake, and in this case it may not be possible to link it to the theme of modelling. But wherever possible, it is desirable to see a topic as a study of a model, or a method to solve a problem deriving from a modelling activity. The idea of learning about the way of life of an applied mathematician can be used to bring those diverse topics together.

Creating interest

When used in a novel situation, the theme of modelling can help stimulate interest in an application area. Of course, some students will not want to get involved with reality, but will want to stay within the safe world of mathematics that they know and love.

The trouble is that not very many people earn their livings as pure mathematicians and at some stage (and the sooner the better in my view) they will have to engage with problems out there in physics, or economics, or society, or wherever. If they can bring a trusted philosophy and methodology to a new situation, then it will not be such a daunting task. Modelling provides opportunities for the lecturer to introduce students to many areas of human endeavour wherein mathematics is useful.

Developing personal skills

In 1987 in the UK, the Department of Employment introduced the Enterprise in Higher Education Initiative, [1]. The term *enterprise* was defined widely and the proposed objectives of the initiative included the ideas that students should

- be more ready to be enterprising
- have developed personal transferable skills
- be better prepared to contribute to and take responsibility in their professional and working lives.

Personal transferable skills are "the generic capabilities which allow people to succeed in a wide range of different tasks and jobs" and include the development of

- group work skills-leadership and followership
- verbal communication skills
- written communication skills
- problem solving skills
- numeracy skills
- computer literacy
- the ability to achieve results
- self assessment skills.

Mathematical modelling can help develop these skills and turn passive receptors into active learners.

In industry, much work is accomplished by teams of people. The mathematician may be only one person of several working on a problem. Employers tell us that besides mathematical skills and knowledge, they look particularly for good interpersonal skills in those whom they employ. The ability to work with others, perhaps

as a leader, perhaps as a follower, the ability to communicate with others, especially "lay" men, the ability to write cogent and persuasive English-these skills are highly valued by employers, [1].

Mathematical modelling provides opportunities for students to learn and practise their skills. When students are involved in creative modelling, it is best they do so in groups. They can be taught about group dynamics and group work and they can practise it. Yes, it takes up some time, but it is a more valuable use of their time than studying yet another topic on an overlong syllabus.

Students can report their group project work to the rest of the class via a student-led seminar and this gives them experience and practice at making presentations using an OHP or even, nowadays, PowerPoint. They can also write up their work in a report which is also assessed. Recently I have been experimenting with student poster sessions, [2], wherein they present their work in a poster instead of a seminar. This introduces students to yet another aspect of professional life and it presents to a student different challenges from a seminar.

Developing learning

Mathematical modelling helps convert students from being passive receptors into active learners. It is all too easy for a student to attend class, take notes and submit homeworks. They may use the library only as a place to sit without ever opening any of its books or consulting any hyper-media instructional packages. Their conversation with their peers may not extend beyond football, beer and sex. They may pass their exams through having a good memory and through spotting questions. I know that this caricature is mythical to an extent, but the point I want to make is that students learn more thoroughly and with a deeper understanding if they are actively engaged with their learning and are prepared to take responsibility for all aspects of it, [3]. Talking with their peers about mathematics helps them express themselves more clearly; teaching others (or explaining things to them) encourages a deeper understanding (or else exposes their misconceptions, and this is also an important aspect of learning).

Through the requirement to carry out research, through the use of comprehension tests, students are encouraged to read mathematics. More than that, they are encouraged to *study* mathematics *independently*. They begin to learn to take control of their own lives and their own learning and this is a terribly important step on the way to maturity.

Moreover, students learn from one another and this informal peer tutoring is a valuable learning resource for students to have. Recently I have been experimenting with ways of using peer tutoring to help students learn methods and models as well as modelling, [4].

Developing self knowledge

> *Had some power the gift to gie us*
> *To see ourselves as others see us* Burns.

The ability to know one's own capabilities and to assess one's own performance is an important one to develop, and modelling provides opportunities for engaging in self and peer assessment. Students will learn more and perform better if they know what the assessment criteria are. To start with, students are inexperienced assessors and so need to be taught how to construct assessment criteria and they need practice in applying this and in making judgements. As they progress, they will engage more and more in critical self-reflection (i.e. self assessment) before submitting work for summative assessment by their lecturers and so should perform better because they now have a better idea of what they are striving to achieve.

It is important to develop assessment criteria with students so that, as far as possible, they belong to the students in that they have made them their own, understand them and can apply them. This involves discussion and the use of exemplar material. Again, this is, in my view, time well spent. See [5] for examples of such assessment criteria.

Implications for staff

It is of course possible to introduce mathematical modelling to one or two modules of a course, but it is better if the whole course and

the whole course team embrace the philosophy and the unifying theme of modelling. This requires the head of department to embrace this idea and to exercise leadership. It is not enough for one or two enthusiasts at a junior level to get involved.

Radical changes are required

- from staff-centred to student-centred management of learning
- from precise (!) methods to more fuzzy methods of assessment.

Staff need to be prepared to "let go" to some extent and to function more as enablers of learning than as central performers. This is, for many, a journey into uncharted waters, although they can of course read the adventure tales of those who have explored these ways before them. It requires people to do things differently and to be prepared to embrace new ideas and methods.

Staff also have to get involved in assessing oral and written work. Colleagues in the humanities have been doing this for years, but it requires us to accept a greater measure of fuzziness in our marking than before. Until people have had experience of assessing oral and written work, they will find it hard to agree, as they can do now, that a particular piece of work is worth (exactly) a mark of 61, say, out of 100.

Accordingly, staff development is a necessary precursor to the introduction of modelling on a widespread basis. Staff must themselves engage in the same activities they are planning for their students. They must learn how to develop assessment criteria and to apply them consistently. Above all, they need an enthusiasm for the job.

Conclusion

In this paper I have described the concept of mathematical modelling and the role it can play in undergraduate courses. Modelling can have a unifying theme and can create interest in novel areas of application; it can develop interpersonal, transferable skills and it can encourage deep learning and self assessment.

The implications for staff who wish to introduce modelling are that they must be prepared to engage in continuing professional development of their teaching and to adopt a new, different role in relation to their students.

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AN APPROACH TO THE NATURAL LOGARITHM FUNCTION

Finbarr Holland*

1. Introduction

In many, if not all, modern calculus texts, the logarithm function is usually defined, and its properties developed, following a discussion of the Riemann integral. It seems to me, however, that, for many students of the physical, engineering and biological sciences, this is much too late, and a *careful* treatment of the elementary functions should be given much earlier in any course aimed at such students. The emphasis here is on the word 'careful': I mean that every effort should be made to keep the technicalities to a minimum, without sacrificing rigour, even if this means that some results may have to be stated without proof. Instead, the utility and importance of these should be pointed out at every opportunity.

This note, then, is a contribution to the ongoing debate on what material should be taught in a modern calculus course, how it should be treated and at what stage it should be presented. Its main purpose is to outline an approach to the natural logarithm function that can be adopted in any a course that treats sequences and series early on in a serious manner, starting with a discussion of the completeness axiom for the real numbers. Its main novelty is that it deals with sequences which are indexed on the dyadic

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integers. This simplifies many of the technical details that would otherwise arise.

The paper is organized as follows: after first outlining in the next section our philosophy of how limits might be treated in such a course, we show how to define the dyadic root of a positive number, employing an iterative procedure that the Babylonians are credited with using to extract square roots, [Ev, Ne, Wa]. Next, we modify slightly the approach adopted by Euler, [Ed], to define the natural logarithm function as a limit of a sequence of functions, by passing to a *dyadic* subsequence, a suggestion I owe to Pat McCarthy. Although this idea is not new (see [La, pp 39–48], for instance), it doesn't appear to have been exploited in any of the recent popular textbooks. Finally, we relate the logarithm to the area of a hyperbolic segment by utilizing the method of exhaustion by triangles that Archimedes, [Ed, He, KL, To, Wa], used in his quadrature of the parabola, something which seems to have gone unnoticed before now.

In an Appendix, we show how to treat the number e in a similar fashion and relate it to the logarithm.

2. Limits of Sequences

It seems to me that sequences and series should be introduced early on in a calculus course; and that, in many courses, the treatment should encompass sequences of *complex* numbers as well. A course on limits of real sequences and series should begin by developing the students' intuitive notion of a limit of a sequence and they should be encouraged to use a calculator to study and predict the eventual behaviour of some standard sequences. At an appropriate time, they should be told the definition of a limit, and taught how to apply the definition in a few simple cases. And, at the very least, it should be demonstrated for them that a convergent sequence has a unique limit and that every convergent sequence is bounded. The following basic results should be stated for all students, with rigorous proofs supplied only to able students.

L1: The sum rule. If a_n and b_n are convergent, then $a_n + b_n$ is

convergent and

$$\lim(a_n + b_n) = \lim a_n + \lim b_n.$$

L2: The product rule. If a_n and b_n are convergent, then $a_n b_n$ is convergent and

$$\lim(a_n b_n) = \lim a_n \lim b_n.$$

L3: The positivity rule. If $a_n \geq 0$, $n = 1, 2, \ldots$, and a_n is convergent, then $\lim a_n \geq 0$.

L4: The shift rule. If a_n is convergent, then so is a_{n+1} and

$$\lim a_{n+1} = \lim a_n.$$

L5: The quotient rule. If $a_n \neq 0$, $n = 1, 2, \ldots$, and a_n converges to a non-zero limit, then

$$\lim \frac{1}{a_n} = \frac{1}{\lim a_n}.$$

In other words, they should be told, in some form or other, that the collection of convergent sequences is an algebra that is invariant under the shift operator that maps a_n to a_{n+1}, and that the limit function is a positive, linear and multiplicative functional on this algebra.

Examples illustrating the usefulness of these rules should be provided, stressing that they enable us to *evaluate* limits of sequences in terms of limits of more elementary ones, once these are recognized. Exercises should be given to ensure that students become comfortable when dealing with rational expressions of convergent sequences. Examples should also be given that alert them to the possibility that there are convergent sequences whose limits are not explicit quantities and motivate the following question: are there criteria that can be used to test a sequence for convergence? This and other questions should lead the classroom discussion to bounded monotonic sequences and the completeness of the reals, which we are content to state as the following axiom.

Axiom 1 *Every bounded monotonic sequence of real numbers is convergent.*

We refine this a little bit by establishing the following result.
Theorem 1 *Suppose that a_n is increasing and bounded above, with $a = \lim a_n$. Then*

$$a_n \leq a, n = 1, 2, \ldots.$$

Moreover, the inequality is strict, if a_n is strictly increasing.
Proof: Suppose that this is not the case. Then there is an integer $n' \geq 1$ such that $a_{n'} > a$. Let $\epsilon = a_{n'} - a$. Then $\epsilon > 0$. Since a_n is convergent, there is a positive integer n_0 such that

$$|a_n - a| < \epsilon, \forall n > n_0.$$

Since we're dealing with real sequences, this can be restated as

$$a - \epsilon < a_n < a + \epsilon, \forall n > n_0.$$

In particular, $a_n < a + \epsilon = a_{n'}, \forall n > n_0$, which conflicts with the fact that $a_n \geq a_{n'}$ if $n > n'$. This contradiction ends the proof of the main part. We leave it to the reader to supply the gloss.

3. Dyadic Roots

The Babylonians of old compiled tables of squares and extracted square roots of positive numbers, apparently using essentially the iterative scheme below, [Ev, Ne, Wa]. It's clear that by repeated application of their methods they could have obtained good approximations to fourth roots, eighth roots etc., of any positive number.

In what follows, and throughout the rest of the article, N will stand for a dyadic integer of the form $N = 2^n$, $n = 1, 2, \ldots$

Theorem 2 *N be a dyadic integer. Let $a \geq 0$. Then there is a unique real number $x \geq 0$ such that*

$$x^N = a.$$

Proof: Let $N = 2^m$, where m is a positive integer. We prove the statement by induction on m. The key step is to show that the equation $x^2 = a$ has a unique nonnegative solution. This is clear if $a = 0$. So, suppose that $a > 0$ and consider the (Babylonian) sequence a_n defined by

$$a_{n+1} = \frac{1}{2}(a_n + \frac{a}{a_n}), \; n = 1, 2, \ldots,$$

where a_1 is any positive number whose square is $\geq a$.[1] It is clear that the sequence consists of positive terms. Also, a simple computation shows that $a_{n+1}^2 \geq a$, $n = 1, 2, \ldots$, independently of the choice of a_1. But now this implies that

$$a_{n+1} - a_n = \frac{(a - a_n^2)}{2a_n^2} \leq 0, n = 1, 2, \ldots$$

In other words, a_n is a decreasing sequence of positive numbers and so, by Axiom 1, is convergent to x, say. By L3, $x \geq 0$. By L2, $x^2 = \lim a_n^2$. Hence, by an easy consequence of L3, $x^2 \geq a > 0$. So, $x > 0$. Next, applying L1, L4 and L5, we see that

$$x = \lim a_{n+1} = \frac{1}{2}(\lim a_n + \frac{a}{\lim a_n}) = \frac{1}{2}(x + \frac{a}{x}),$$

and so, $x^2 = a$. It's easy to see that this x is the only positive solution of this equation.

It is also clear how to build an inductive argument on this and establish the theorem. This ends the proof of Theorem 2.

The uniqueness part of this theorem enables us to define the Nth root of any $a \geq 0$. We use the notations $\sqrt[N]{a}$, $a^{1/N}$ interchangeably to denote the unique Nth root of $x^N = a$, where

[1] In other words, if a_0 is an approximation to the desired square root, a better one is obtained by taking the arithmetic mean of a_0 and a/a_0, the square of one of which is bigger, and of the other, smaller than a.

$N = 1, 2, 4, 8, \ldots$. Uniqueness also guarantees that $\sqrt[N]{ab} = \sqrt[N]{a}\,\sqrt[N]{b}$, $\forall a, b \geq 0$, a fact that will be needed below.

Experimentation with a hand-calculator that has a square-root key should lead students to the truth of the following.

Lemma 1 *Suppose that $a > 0$. Then*

$$\lim_{n\to\infty} \sqrt[N]{a} = 1.$$

Proof: First, suppose that $a \geq 1$ and put $a_n = \sqrt[N]{a}$, $n = 1, 2, \ldots$, so that the terms of the sequence are $\sqrt{a}$, $\sqrt[4]{a}$, $\sqrt[8]{a}$, ... We wish to show that the sequence is decreasing. But it is clear that $a_n \geq 1$, and $a_{n+1}^2 = a_n$. Hence $a_{n+1}^2 = a_n \leq a_n^2$, whence it follows that $a_{n+1} \leq a_n$. Thus $\lim a_n$ exists. Denote the limit by b. Then, by L2 and L4, $b = \lim a_n = (\lim a_{n+1})^2 = b^2$. But, by L3, $b \geq 1$. Hence $b = 1$. This proves the result when $a \geq 1$. Using this and L5 we see that

$$\lim \sqrt[N]{a} = \frac{1}{\lim \sqrt[N]{1/a}} = 1$$

if $0 < a < 1$. This completes the proof.

4. The logarithm function

Euler, [Ed], established that the sequence $n(\sqrt[n]{x}-1)$, $n = 1, 2, \ldots$, has a limit for every $x > 0$, and that the limit function satisfies the law of the logarithm. We consider the subsequence of this based on the dyadic integers, which we've just seen makes sense. Again, students should be encouraged to use a calculator to examine the behaviour of this for different values of x before being shown the following.

Theorem 3 *Let $x > 0$. Then the limit*

$$\lim_{n\to\infty} N(\sqrt[N]{x} - 1)$$

exists. Denoting this limit by $\ell(x)$ we have that
(a) $\ell(1) = 0$;
(b) $\ell(xy) = \ell(x) + \ell(y)$, $\forall x$ and $y > 0$;
(c) $(x-1)/x \leq \ell(x) \leq x - 1, \forall x > 0$.

Proof: We will show that the sequence

$$\ell_n(x) = N(\sqrt[N]{x} - 1),\ N = 2^n,\ n = 1, 2, \ldots,$$

is decreasing. But this is an immediate consequence of the simple inequality

$$2(\sqrt{x} - 1) \leq (x - 1),$$

which holds for all $x \geq 0$, with equality when and only when $x = 1$. Thus we have

$$\ell_{n+1}(x) \leq \ell_n(x) \leq x - 1,\ n = 1, 2, \ldots$$

To continue, suppose that $x \geq 1$. Then the terms of $\ell_n(x)$ are also nonnegative. Hence the sequence is decreasing and bounded below, and, so, it is convergent. We can remove the restriction on x by noting that

$$N(\sqrt[N]{x} - 1) = -N(\sqrt[N]{1/x} - 1)\,\sqrt[N]{x},$$

and using Lemma 1 and L2. Thus, in all cases, the limit exists. It emerges, too, that $\ell(x) = -\ell(1/x)$ and $\ell(x) \leq x - 1$. Hence (a) and (c) follow. Finally, (b) follows from the identity

$$N(\sqrt[N]{xy} - 1) = N(\sqrt[N]{x} - 1)\,\sqrt[N]{y} + N(\sqrt[N]{y} - 1),$$

on applying L1, L2 and Lemma 1.

5. The area of a hyperbolic segment

In 250 BC or thereabouts, Archimedes, [He, Kl, To, Wa], devised two rigorous methods–the method of *compression* and the method of *exhaustion*–to measure the area of a parabolic segment. He proved that the area of such a region is four thirds the area of the largest triangle that can be inscribed in it. Some 1800 years later, Cavalieri, [Ed, To], built on the method of compression to find the area under the curves $y = x^k$, $k = 3, 4, \ldots, 9$, and paved the way for Riemann's development of the integral. In between, in 1647, the Belgian Jesuit Fr. Gregorius a Santo Vincentio, [Ed, To], used similar ideas to make the following connection between the logarithm function and the area under an arc of the rectangular hyperbola $y = 1/x$, $x > 0$. Denote by $A(a,b)$ the area of the region $\{(x,y) : a \leq x \leq b, 0 \leq y \leq 1/x\}$. Let $t > 0$. Then

$$A(ta, tb) = A(a, b).$$

Had he used the second method of Archimedes, which we're now going to apply, Gregorius might have discovered the following result.

Theorem 4 *Let* $0 < a < b$. *Then the area,* $H(a,b)$, *of the hyperbolic segment*

$$S(a,b) = \{(x,y) : a \leq x \leq b, 1/x \leq y\}$$

is given by

$$H(a,b) = \frac{b-a}{2}\{\frac{1}{a} + \frac{1}{b}\} - \frac{1}{2}\{\lim \ell_n(\frac{b}{a}) - \lim \ell_n(\frac{a}{b})\}.$$

In particular, $H(ta,tb) = H(a,b)$ *if* $t > 0$.

Proof: The set $S(a,b)$ is clearly convex, and for any $c \in [a,b]$ the triangle, $T(a,b)(c)$, with vertices $A(a,1/a)$, $C(c,1/c)$, $B(b,1/b)$, is contained in $S(a,b)$. The area of $T(a,b)(c)$ is easily seen to be given by

$$\frac{b-a}{2}\{\frac{1}{a} + \frac{1}{b}\} - \frac{b-a}{2}\{\frac{c}{ab} + \frac{1}{c}\} \leq \frac{b-a}{2}\{\frac{1}{a} + \frac{1}{b}\} - \frac{b-a}{2}\{\frac{2}{\sqrt{ab}}\},$$

with equality if and only if $c = \sqrt{ab}$, the *geometric mean* of a and b.[2] Thus, the area of the largest triangle that can be inscribed in the hyperbolic segment is given by

$$\Delta(a,b) = \frac{b-a}{2}\{\frac{1}{a} + \frac{1}{b}\} - \frac{b-a}{2}\{\frac{2}{\sqrt{ab}}\}$$
$$= \frac{b-a}{2}\{\frac{1}{a} + \frac{1}{b}\} - \{\sqrt{\frac{b}{a}} - \sqrt{\frac{a}{b}}\}.$$

[2] This result should be contrasted with the corresponding statement for the parabola $y = x^2$, when c turns out to be the *arithmetic mean* of a, b, as Archimedes discovered using purely geometric reasoning.

(Geometrically, just as for the parabola, the largest triangle occurs when C is the point on the arc joining A and B where the tangent is parallel to the chord AB.)

Note the homogeneity property:

$$\Delta(ta, tb) = \Delta(a,b), \forall t > 0.$$

We've thus obtained a decomposition of $S(a,b)$ into three disjoint regions–which is optimal in a certain sense: a triangle, which we label $T(0,0)$, with area $\Delta(a,b)$, and two segments $S(a,\sqrt{ab})$ and $S(\sqrt{ab},b)$.

Next, we consider the segments $S(a,\sqrt{ab})$ and $S(\sqrt{ab},b)$. The triangles of largest area that can be inscribed in these segments have areas $\Delta(a,\sqrt{ab})$ and $\Delta(\sqrt{ab},b)$, respectively. We have

$$\Delta(a,\sqrt{ab}) = \Delta(\sqrt{a},\sqrt{b}) = \Delta(\sqrt{ab},b),$$

by homogeneity.[3]

Up to this point, we have obtained a decomposition of $S(a,b)$ into seven disjoint regions consisting of three triangles, $T(0,0), T(1,0), T(1,1)$, say, with corresponding areas $\Delta(a,b)$, $\Delta(\sqrt{a},\sqrt{b})$, $\Delta(\sqrt{a},\sqrt{b})$, and four segments. Next we partition each of these residual segments in the same way into a triangle and two segments, noting that the triangles have the same area equal to $\Delta(\sqrt[4]{a},\sqrt[4]{b})$. Continuing in this way, we partition the segment $S(a,b)$ into a countable union of triangles $T(n,k)$, $k = 0, 1, \ldots, N-1$, $n = 0, 1, \ldots$ with corresponding areas $\Delta(n,k)$, $k = 0, 1, \ldots, N-1$, $n = 0, 1, \ldots$, where

$$\Delta(n,k) = \Delta(\sqrt[N]{a}, \sqrt[N]{b}),\ k = 0,1,\ldots,N-1,\ n = 0,1,\ldots.$$

We conclude that the area of $S(a,b)$ is given by the sum of the infinite series of nonnegative terms

$$H(a,b) = \sum_{n=0}^{\infty}\sum_{k=0}^{N-1}\Delta(n,k) = \sum_{n=0}^{\infty} N\Delta(\sqrt[N]{a}, \sqrt[N]{b}).$$

[3] In light of his success with the parabola, it seems inconceivable that Archimedes didn't know these facts about the hyperbola, but I haven't encountered any mention of them in the literature.

To establish the convergence of the series and determine its sum, consider its sequence of partial sums s_n. Since the terms are nonnegative, this sequence is increasing: $s_{n+1} \geq s_n$, $n = 0, 1, \ldots$. We have

$$s_0 = \Delta(0,0) = \Delta(a,b) = \frac{b-a}{2}\{\frac{1}{a}+\frac{1}{b}\} - \{\sqrt{\frac{b}{a}} - \sqrt{\frac{a}{b}}\},$$

and

$$\begin{aligned} s_1 &= \Delta(0,0) + \Delta(1,0) + \Delta(1,1) \\ &= \Delta(a,b) + 2\Delta(\sqrt{a}, \sqrt{b}) \\ &= \frac{b-a}{2}\{\frac{1}{a}+\frac{1}{b}\} - 2\{\sqrt[4]{\frac{b}{a}} - \sqrt[4]{\frac{a}{b}}\}. \end{aligned}$$

Inductively, we see that

$$\begin{aligned} s_n &= \frac{b-a}{2}\{\frac{1}{a}+\frac{1}{b}\} - 2^n\{\sqrt[2^{n+1}]{\frac{b}{a}} - \sqrt[2^{n+1}]{\frac{a}{b}}\} \\ &= \frac{b-a}{2}\{\frac{1}{a}+\frac{1}{b}\} - \frac{1}{2}\{\ell_{n+1}(b/a) - \ell_{n+1}(a/b)\}, \end{aligned}$$

$n = 0, 1, \ldots$. Already, this tells us that the increasing sequence s_n is bounded above by

$$\frac{b-a}{2}\{\frac{1}{a}+\frac{1}{b}\}$$

and so, by Axiom 1, $\lim s_n$ exists. (From this, of course, we can deduce also that the sequence

$$N\{\sqrt[2N]{\frac{b}{a}} - \sqrt[2N]{\frac{a}{b}}\},\ n = 0, 1, \ldots,$$

is convergent, if we didn't already know that fact.) In any event, the claimed result follows. This ends the proof of Theorem 4.

Of course, we have that

$$\begin{aligned} H(a,b) = \lim s_n &= \frac{b-a}{2}\{\frac{1}{a}+\frac{1}{b}\} - \lim N\{\sqrt[2N]{\frac{b}{a}} - \sqrt[2N]{\frac{a}{b}}\} \\ &= \frac{b-a}{2}\{\frac{1}{a}+\frac{1}{b}\} - \frac{1}{2}\{\ell(b/a) - \ell(a/b)\} \\ &= \frac{b-a}{2}\{\frac{1}{a}+\frac{1}{b}\} - \ell(b/a). \end{aligned}$$

Readers will recognize that the first term represents the area of the trapezium with vertices $(a,0)$, $(b,0)$, $(b,1/b)$, $(a,1/a)$. Since $H(a,b) \geq s_0 = \Delta(a,b)$, Kepler's inequality :

$$\frac{\ell(b)-\ell(a)}{b-a} \leq \frac{1}{\sqrt{ab}},$$

follows, [To].

5.1 Exercises

1. Show that $S(a,b)$ is a subset of the parallelogram formed by the chord AB, the tangent parallel to this and the ordinates $x = a$, $x = b$.
2. Deduce from the previous exercise that $H(a,b) < 2\Delta(a,b)$.
3. Now use this to obtain the inequality:

$$\frac{2}{\sqrt{ab}} - \frac{1}{2}(\frac{1}{a}+\frac{1}{b}) < \frac{\ell(b)-\ell(a)}{b-a},$$

   a companion for Kepler's. (Readers will observe, *inter alia*, that an independent proof of this and Kepler's inequality implies the arithmetic-geometric mean inequality, which, of course, we've used in our derivation of the expression for $H(a,b)$.)
4. Let $a < c < b$. Show directly from the definition that

$$H(a,b) = H(a,c) + H(c,b) + \frac{(b-a)(c-a)(b-c)}{2abc}.$$

5. Suppose that $f(x,y)$ is defined and continuous on $(0,\infty)\times(0,\infty)$ and satisfies the homogeneity condition: $f(tx,ty) = f(x,y)$, $\forall x,y,t>0$, and the functional equation:

$$f(x,z) = f(x,y) + f(y,z) + \frac{(z-x)(y-x)(z-y)}{2xyz}, \ \forall x,y,z>0.$$

Determine f.

6. Appendix

For those interested in relating the number e to the treatment of the logarithm given here, we describe how to introduce e using similar ideas, and to show that $\ell(e)=1$.

Since

$$(1+\frac{x}{2})^2 \geq 1+x$$

for all real x it is easy to see that the sequences

$$(1+\frac{1}{N})^N, \quad (1-\frac{1}{N})^N, \quad (1-\frac{1}{N^2})^N,$$

where $N=2^n$, $n=1,2,\ldots$, are increasing. The second and third are clearly bounded, since

$$1/4 \leq (1-\frac{1}{N})^N \leq 1,$$

and

$$9/16 \leq (1-\frac{1}{N^2})^N \leq 1, \ n=1,2,\ldots,$$

and hence are convergent to non-zero limits. (We can deduce at this stage, if we want to, that the first is also convergent because

$$(1+\frac{1}{N})^N = \frac{(1-\frac{1}{N^2})^N}{(1-\frac{1}{N})^N} \leq \frac{1}{1/4} = 4, \ n=1,2,\ldots.)$$

We show that

$$c = \lim(1-\frac{1}{N^2})^N = 1.$$

To see this, note that

$$c^2 = \lim(1-\frac{1}{N^2})^{2N} = \lim(1-\frac{1}{4N^2})^{4N},$$

by L2 and L4. But

$$(1-\frac{x}{4})^4 = 1-x+x^2\frac{(96-16x+x^2)}{256} > (1-x), \ \forall x\in(-\infty,\infty).$$

Hence

$$c^2 \geq \lim(1-\frac{1}{N^2})^N = c.$$

But $0<c\leq 1$. Thus $c=1$. It now follows that

$$\lim_{n\to\infty}(1+\frac{1}{N})^N = \lim\frac{(1-\frac{1}{N^2})^N}{(1-\frac{1}{N})^N} = \frac{\lim(1-\frac{1}{N^2})^N}{\lim(1-\frac{1}{N})^N} = \frac{1}{\lim(1-\frac{1}{N})^N}.$$

So, following Euler, and setting

$$e = \lim_{n\to\infty}(1+\frac{1}{N})^N,$$

we see that

$$\frac{1}{e} = \lim(1-\frac{1}{N})^N.$$

But, since the sequences are increasing, Theorem 1 tells us that

$$(1+\frac{1}{N})^N \leq e \leq \frac{1}{(1-\frac{1}{N})^N} = (\frac{N}{N-1})^N, \ n=1,2,\ldots,$$

whence it results that

$$1 \leq \ell_n(e) \leq \frac{N}{N-1}, \ n=1,2,\ldots.$$

Appealing once more to L3 we deduce that $\ell(e) = 1$.

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TORONTO SPACES, MINIMALITY, AND A THEOREM OF SIERPIŃSKI.

Eoin Coleman[1]

In this note we gather together some theorems in the literature to resolve a problem suggested by P. J. Matthews and T. B. M. McMaster in a recent article, [1]. We also make an observation which allows one to deduce within ordinary set theory that neither the real line nor the Sorgenfrey line contains a Toronto space of cardinality the continuum (improving one of their results), and we establish some relative consistency results. To conclude the paper, we explain how a similar question arising from a theorem of Sierpiński (can every subset of the unit interval **I** of cardinality the continuum be mapped continuously onto **I**?) is independent of ordinary set theory.

1. Toronto spaces and minimality

Matthews and McMaster ask whether there are any reasonable set-theoretic assumptions which will enable one to prove or disprove the assertion Qmin(κ) where κ is an uncountable limit cardinal. Recall that the assertion Qmin(κ) says:
(a) neither $T(\kappa)$ nor $T(\kappa) \cap T_2$ is supported by its weakly quasi-minimal members,
and
(b) any subfamily of $T(\kappa)$ or $T(\kappa) \cap T_2$ which does support the whole family has more than κ members.

[1] I am very grateful to Dr Peter Collins for an invitation to present this and related material to the seminar in Analytic Topology at the Mathematical Institute, Oxford, in November 1996.

Full definitions are provided in the paper [1]. We note for convenience that a topological space X of cardinality κ is weakly quasi-minimal if X is embeddable in each of its subspaces of power κ. The notations $T(\kappa)$ and T_2 refer to the families of spaces of power κ and the Hausdorff spaces respectively. A family F supports a family G if every member of G contains a homeomorphic image of a member of F.

First of all, we show the following:

Proposition 1.
(1) *Every infinite Hausdorff topological space X contains an infinite discrete subset.*
(2) *If κ is a singular strong limit cardinal or $\kappa = \aleph_0$, then the discrete topological space $D(\kappa)$ of cardinality κ is strongly quasi-minimal and supports the family $T(\kappa) \cap T_2$. In particular, Qmin(κ) is false.*

Proof: (1) Since X is infinite and Hausdorff, and the intersection of a finite number of open sets is open, it follows that one can choose a discrete sequence $\{x_n \in X : n \in \omega\}$ by induction. Alternatively, apply Zorn's lemma to the family $S = \{Y : Y$ is a discrete subset of $X\}$ partially ordered by inclusion, to obtain a maximal element D which must be infinite.
(2) Trivially, the discrete space $D(\kappa)$ is strongly quasi-minimal, i.e. it is homeomorphic to each of its subspaces of cardinality κ, and hence it is weakly quasi-minimal too. A theorem of Hajnal and Juhász, [2, 4 or 5], says that if κ is a singular strong limit cardinal, then every Hausdorff space X of cardinality at least κ has a discrete subset Y of cardinality κ. If $\kappa = \aleph_0$, then part (1) applies. In either eventuality, $D(\kappa)$ supports the family $T(\kappa) \cap T_2$, since any bijection from $D(\kappa)$ onto Y is a homeomorphism. So Qmin(κ) is false. ■

Proposition 1.2 covers a proper class of singular cardinals: for any cardinal λ, define

$$\kappa_\lambda = \sup\{\lambda, \exp(\lambda), \exp(\exp(\lambda)), \ldots\},$$

where $\exp(\lambda) = 2^\lambda$. Proposition 1.2 implies that Qmin(κ_λ) is false for all λ. Note however that every κ_λ has countable cofinality.

We shall show further on that if κ is singular and has uncountable cofinality, then there is a model of ZFC (ordinary set theory) in which Qmin(κ) is true (although obviously in this model κ is not a strong limit).

We remind the reader that a weakly inaccessible cardinal is a regular limit cardinal, and a strongly inaccessible cardinal is an uncountable regular cardinal κ such that $(\forall\lambda < \kappa)(2^\lambda < \kappa)$. The Generalized Continuum Hypothesis (GCH: $(\forall\kappa)(2^\kappa = \kappa^+)$) implies that every weakly inaccessible cardinal is strongly inaccessible, and that every singular cardinal κ is a strong limit.

Corollary 2. *Assume GCH. Then* Qmin(κ) *is false for every singular cardinal κ.*

Corollary 3. *Assume GCH and there are no inaccessible cardinals. Then $(\forall\kappa)($*Qmin(κ) *is true if and only if κ is an uncountable regular cardinal).*

Proof: If κ is an uncountable regular cardinal, then $\kappa = \lambda^+ = 2^\lambda$, and now Qmin$(\kappa)$ holds by the theorem of Matthews and McMaster, [1]. If κ is countable or singular, then Proposition 1.1 and Corollary 2 show that Qmin(κ) is false. ■

Corollary 3 answers Problem 2 of Matthews and McMaster, [1]. It also establishes that if ZFC (ordinary set theory) is consistent, then so is ZFC + $((\forall\kappa)($Qmin(κ) is true if and only if κ is an uncountable successor cardinal)). In particular, it is impossible to prove from ZFC the existence of an uncountable regular limit cardinal for which Qmin(κ) fails.

The Hajnal-Juhász theorem to which we appealed in proving Proposition 1.2 relies on a positive partition relation. To illustrate the ideas and arguments involved, we prove a simple theorem, explaining first some convenient notation. Suppose that κ, λ, and μ are cardinals. The family of μ element subsets of a set A is denoted by $[A]^\mu$. The notation $\kappa \to (\kappa)^\mu_\lambda$ means: for every function $f : [\kappa]^\mu \to \lambda$, there exists a set $H \in [\kappa]^\kappa$ such that $f|[H]^\mu$ is constant. In pictorial terms, if one colours the μ element subsets of κ using λ colours, then there will always be a κ element subset H all of whose μ element subsets get the same colour. In this notation, Ramsey's theorem reads: $\aleph_0 \to (\aleph_0)^m_k$

for all natural numbers m and k, where $\aleph_0$ is the cardinality of the natural numbers. A relation of the form $\kappa \to (\kappa)^\mu_\lambda$ is called a positive partition relation. Weakly compact cardinals are often defined as those cardinals κ for which $\kappa \to (\kappa)^2_2$ holds. We note that $\kappa \to (\kappa)^2_2$ implies $\kappa \to (\kappa)^n_\lambda$ for all $n < \omega$ and $\lambda < \kappa$. In general, if an uncountable cardinal κ satisfies a non-trivial positive partition relation, then κ is a large cardinal, and its existence cannot be proven in ZFC (ordinary set theory). The reader can easily check that for example $\mathfrak{c} \to (3)^2_\omega$ does not hold, where $\mathfrak{c}$ is the cardinality of the real numbers (enumerate the set of rationals $\mathbb{Q} = \{q_n : n \in \omega\}$, and for $x < y \in \mathbb{R}$, put

$$g(\{x,y\}) = \min\{n : x < q_n < y\}).$$

The classic monograph of Erdös, Hajnal, Maté and Rado, [5], provides detailed information on the partition calculus.

Proposition 4. *Suppose that $\kappa \to (\kappa)^2_\omega$. If X is a first countable Hausdorff space of cardinality κ, then X has a discrete subset D of cardinality κ.*

Proof: For each $x \in X$, let $\{V(x,n) : n \in \omega\}$ be a shrinking neighbourhood basis at x. Define a colouring f of the pairs of elements of X as follows:

$$f(\{x,y\}) = \min\{n : V(x,n) \cap V(y,n) = \emptyset\}.$$

Apply the partition relation to obtain an n and a subset D of X of power κ such that $(\forall x \neq y \in D)(f(\{x,y\}) = n)$, i.e. D is a discrete subspace, since

$$(\forall x \in D)(D \cap V(x,n) = \{x\}). \blacksquare$$

Next we turn to the Toronto space problem, [3]. A minor improvement of a lemma from Matthews and McMaster allows one to prove (as a theorem in ordinary set theory) that Qmin($\mathfrak{c}$) holds, where $\mathfrak{c}$ is the cardinality of the real numbers.

Lemma A*. [1, Lemma A] *Suppose that κ is an infinite cardinal, X is a set of power κ and*

$$(\forall\alpha < \kappa)(S_\alpha \text{ is a subset of } X \text{ of power } \kappa).$$

Then there exists a subset Z of X of power κ which does not contain any S_α.

Proof: Without loss of generality, we identify X with κ and assume that κ is uncountable. Choose distinct elements x_0 and y_0 in S_0. Given x_α and z_α in S_α for $\alpha < \beta < \kappa$, note that $S_\beta \setminus \{x_\alpha, z_\alpha : \alpha < \beta\}$ has power κ, since $\beta < \kappa$, and so one can find distinct elements x_β, z_β in $S_\beta \setminus \{x_\alpha, z_\alpha : \alpha < \beta\}$. Put $Z = \{z_\alpha : \alpha < \kappa\}$. Then Z has power κ, and for all $\alpha < \kappa$, Z does not contain S_α since $x_\alpha \in S_\alpha \setminus Z$. $\blacksquare$

The essential results of Matthews and McMaster now go through without the assumption of regularity.

Lemma C* [1, Lemma C]. *Suppose that X is a Hausdorff space of cardinality κ all of whose subspaces have dense subsets of power at most λ, and $\kappa^\lambda = \kappa$. Suppose that*

$$(\forall\alpha < \kappa)(S_\alpha \text{ is a subset of } X \text{ of power } \kappa).$$

If Y is a subspace of X of power κ, then Y has a subspace Z which contains no homeomorphic copy of any S_α.

The Toronto problem, [3], asks whether it is possible to have a Toronto space, i.e. an uncountable non-discrete Hausdorff space which is homeomorphic to each of its uncountable subspaces. It is unknown whether the existence of a Toronto space is consistent with ZFC. A counting argument shows that if X has hereditary density λ and $|X|^\lambda < 2^{|X|}$, then X is not a Toronto space: there are $2^{|X|}$ subspaces of power $|X|$, but only $|X|^\lambda$ auto-homeomorphic images of X.

Corollary 5. *There are no Toronto spaces of singular strong limit cardinality. In particular, GCH implies that there are no Toronto spaces of singular cardinality.*

Proof: If κ is a singular strong limit cardinal, then every Hausdorff space X of cardinality κ has a discrete subset of cardinality κ, and so X is not a Toronto space. $\blacksquare$

Corollary 6.

(1) Qmin($\mathfrak{c}$) *is true.*

(2) *The real line contains no Toronto space of power* $\mathfrak{c}$.

(3) *The Sorgenfrey line contains no Toronto space of power* $\mathfrak{c}$.

Matthews and McMaster, [1], proved the results 6.1 and 6.2 with the additional assumption that **c** is a regular cardinal. Similar results can also be demonstrated for the natural analogues of the real line in higher cardinalities.

Corollary 7. *Suppose that $\kappa \to (\kappa)^2_\omega$. If X is a first countable Hausdorff space of cardinality κ, then X does not contain a Toronto space of cardinality κ.*

Proof: By Proposition 4, every subspace of X of power κ contains a discrete subset of cardinality κ. ■

Corollary 6 enables one to show that if κ is any cardinal of uncountable cofinality, then there is a model of ZFC in which Qmin(κ) is true: for example, add κ Cohen reals to **L**, the universe of constructible sets (or more generally, to any model of ZFC + GCH). So Corollary 2 and Corollary 6 show that Qmin(κ) is independent of ZFC for any singular cardinal κ with $\kappa > \mathrm{cf}(\kappa) > \omega$.

There is a general phenomenon at work behind Corollaries 2 and 6: suppose that $P(\lambda)$ is a property of cardinals for which one can prove in ordinary set theory that $P(\mathbf{c})$ is true but $P(\kappa)$ is false for every singular strong limit cardinal κ; then $P(\kappa)$ is independent of ZFC for every singular cardinal κ of uncountable cofinality.

Returning to the question of Qmin(κ), what happens if an uncountable cardinal κ has countable cofinality? First of all, κ is singular. If κ is a strong limit, then Proposition 1 says that Qmin(κ) is false. We do not know what happens if κ is not a strong limit, for example if $\kappa = \aleph_\omega < \mathbf{c}$ (where part of the difficulty is that $\kappa^{cf(\kappa)} > \kappa$ (Koenig's theorem)). Some additional partial information can be gleaned from the papers of Hajnal and Juhász, [6], and Kunen and Roitman, [7].

Finally, let us consider what one can prove if one removes in the statement of Corollary 3 the assumption that there are no inaccessible cardinals. In particular, is there a model of ZFC in which Qmin(κ) holds for some weakly inaccessible cardinal? The following example provides a positive answer.

Example 8. It is well-known that if there is a model of ZFC + $(\exists\kappa)(\kappa$ is weakly inaccessible), then there is also a model M of ZFC + (**c** is weakly inaccessible) (for example, see [9]). By Corollary 6, Qmin(**c**) holds in M, so that M is a model of ZFC in which Qmin(κ) holds for a weakly inaccessible cardinal κ.

A defect of this example is that the weakly inaccessible cardinal κ which it exhibits is fairly small. To explain what happens for larger inaccessible cardinals, we require the notion of the spread of a topological space.

Definition. *The spread of a topological space X is*

$$\sup\{|D| : D \text{ is a discrete subset of } X\} + \omega.$$

We denote the spread of X by $s(X)$ and say that the spread is achieved if X has a discrete subset D of power $s(X)$.

Hodel, [4], remarks the spread is achieved at those regular limit cardinals κ which are weakly compact, and hence all Hausdorff spaces in these cardinalities contain discrete subsets of size κ. As in Proposition 1, it follows that Qmin(κ) is false for weakly compact cardinals, and there are no Toronto spaces of weakly compact cardinality. This leads to a model of ZFC + GCH + $(\exists\kappa)(\kappa$ is a regular limit cardinal and Qmin(κ) is false).

Example 9. Suppose that κ is a weakly compact cardinal.[2] Then κ is weakly compact in **L**, and since GCH holds in **L**, one obtains a model of ZFC + GCH + $(\exists\kappa)$ (κ is a weakly compact (regular limit) cardinal and Qmin(κ) is false). So while ZFC + GCH suffices to determine that Qmin(κ) is true for uncountable successor cardinals and false for singular cardinals, it is not powerful enough to settle whether Qmin(κ) holds if κ is an inaccessible cardinal.

We summarize the import of these examples:

Corollary 10.

(1) *Suppose that κ is a singular cardinal of uncountable cofinality. Then* Qmin(κ) *is independent of ZFC (ordinary set theory).*

(2) *If there is a weakly inaccessible cardinal, then there is a model of ZFC in which Qmin(κ) is true for some weakly inaccessible cardinal κ.*

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[2] It suffices to suppose that κ is a regular cardinal with the tree property (i.e. there is no κ-Aronszajn tree).

(3) *If there is a weakly compact cardinal κ, then there is a model of (ZFC + GCH + Qmin(κ) is false). Note that κ is weakly inaccessible in this model.*

Jensen, [8], has shown that if the axiom of constructibility ($\mathbf{V} = \mathbf{L}$) holds, then for each regular limit cardinal λ which is not weakly compact, there is a Hausdorff linearly ordered space of power λ in which the spread is not achieved. We do not know whether $\mathbf{V} = \mathbf{L}$ determines which truth value Qmin(λ) has in this case, nor what this truth value may be. And of course, it may still be a theorem of ZFC that Qmin(κ^+) is true for every infinite cardinal κ. (The reader curious about future progress on these problems can consult the *Topology Atlas*, located at http://www.unipissing.ca/topology)

2. A theorem of Sierpiński

Next, we turn to a theorem of Sierpiński, [11]: there exists an uncountable subset P of the unit interval $\mathbf{I}$ such that $\mathbf{I}$ is not a continuous image of P. In his classic work, [10], Kuratowski notes on page 428: "Without the continuum hypothesis, however, we are unable to prove the existence of a set P of power $\mathfrak{c}$ such that the interval is not a continuous image of P." We explain in detail how to use Martin's Axiom (MA) to prove the existence of such a set P. In fact this result follows from a weaker hypothesis: $\mathbf{R}$ is not the union of less than $\mathfrak{c}$ many nowhere dense sets. This hypothesis is true for example under Martin's Axiom for countable partial orders, [12], or for a slick proof, see [13, Theorem 16.1]. Arnold Miller constructed a model of ZFC in which $\mathfrak{c} = \aleph_2$ and every subset of $\mathbf{I}$ of cardinality $\mathfrak{c}$ can be mapped continuously onto $\mathbf{I}$. Thus whether every subset of $\mathbf{I}$ of power $\mathfrak{c}$ can be mapped continuously onto $\mathbf{I}$ is independent of ordinary set theory.

To make the arguments fairly self-contained, we recall some definitions and standard results which can be found in the textbook [10]. A set A has the Baire property in the space X iff there is an open set G such that $A \setminus G$ and $G \setminus A$ are of first category (meagre, or, a countable union of nowhere dense sets). An equivalent characterization is that $A = (G \setminus N) \cup M$ where G is open and N and M are of first category. So open sets and closed sets have

the Baire property (every closed set is the union of its interior and its boundary (which is always nowhere dense)).

Lemma 11. [10, section 24, I, Theorem 3, p.256]. *Every family of disjoint sets $\{X_i : i \in I\}$ with the Baire property, of which none is of first category, is countable.*

The next lemma is a special case of a more general result. The proof is copied from that of the analogous result for $\aleph_1$ in [10], introducing the necessary modifications to avoid assuming the regularity of the continuum $\mathfrak{c}$.

Lemma 12. *Assume that $\mathbf{R}$ is not the union of less than $\mathfrak{c}$ many nowhere dense sets. Suppose that $\{E_{\alpha\beta} : \alpha, \beta < \mathfrak{c}\}$ is a sequence of subsets of the unit interval $\mathbf{I}$ with the Baire property. If $\beta < \beta'$ implies that $E_{\alpha,\beta} \cap E_{\alpha,\beta'} = \emptyset$, then there exists a sequence of distinct ordinals $\{\gamma(\alpha) : \alpha < \mathfrak{c}\}$ such that $|\mathbf{I} \setminus \cup_{\alpha<\mathfrak{c}} E_{\alpha\gamma(\alpha)}| = \mathfrak{c}$.*

Proof: We define by induction on $\alpha < \mathfrak{c}$, an ordinal $\gamma(\alpha)$, and an element $p_\alpha \in \mathbf{I}$, as follows. Note first that by Lemma 11,

$$(\forall \alpha < \mathfrak{c})(\exists \beta_\alpha)(\forall \beta > \beta_\alpha)(E_{\alpha\beta} \text{ is of the first category}).$$

Fix $\alpha < \mathfrak{c}$. Suppose that we have defined $\{\gamma(\xi), p_\xi : \xi < \alpha\}$. Since the sets $E_{\alpha\beta}$ are disjoint for different β,

$$\{\delta > \beta_\alpha : (\forall \xi < \alpha)(p_\xi \in (\mathbf{I} \setminus E_{\alpha\delta}))\}$$

has power $\mathfrak{c}$, and so

$$(\exists \gamma(\alpha) > \beta_\alpha)(\forall \xi < \alpha)(\gamma(\alpha) \neq \gamma(\xi) \text{ and } p_\xi \in (\mathbf{I} \setminus E_{\alpha\gamma(\alpha)})).$$

Observe now that $\{E_{\xi\gamma(\xi)} : \xi \leq \alpha\}$ is a family of less than $\mathfrak{c}$ many sets of first category. We have assumed that $\mathbf{R}$ (and hence $\mathbf{I}$) is not the union of less than $\mathfrak{c}$ many nowhere dense sets, therefore

$$\mathbf{I} \setminus (\cup_{\xi\leq\alpha} E_{\xi\gamma(\xi)} \cup \cup_{\xi<\alpha}\{p_\xi\}) \neq \emptyset,$$

since $\alpha < \mathfrak{c}$; take

$$p_\alpha \in \mathbf{I} \setminus (\cup_{\xi\leq\alpha} E_{\xi\gamma(\xi)} \cup \cup_{\xi<\alpha}\{p_\xi\}),$$

and let $P = \{p_\alpha : \alpha < \mathbf{c}\}$. If $\xi < \alpha$, then $p_\xi \neq p_\alpha$, and hence $|P| = \mathbf{c}$. Also

$$(\forall \alpha < \mathbf{c})(P \cap E_{\alpha\gamma(\alpha)} = \emptyset),$$

so

$$P \subseteq (\mathbf{I} \setminus \cup_{\alpha<\mathbf{c}} E_{\alpha\gamma(\alpha)}),$$

and hence

$$|\mathbf{I} \setminus \cup_{\alpha<\mathbf{c}} E_{\alpha\gamma(\alpha)}| = \mathbf{c}. \blacksquare$$

We remind the reader that if f is a real-valued continuous function defined on a subset A of $\mathbf{R}$, then there exists a continuous extension of f to a $\mathbf{G}_\delta$-set (a countable intersection of open sets). This follows from Theorem 1 in section 35, I, in [10]. And we remark that if Φ is the family of real-valued continuous functions defined on $\mathbf{G}_\delta$-subsets of $\mathbf{I}$, then Φ has cardinality $\mathbf{c}$.

Theorem 13. *Assume that* $\mathbf{R}$ *is not the union of less than* $\mathbf{c}$ *many nowhere dense sets. Let* $\mathbf{F}$ *be a family of at most* $\mathbf{c}$ *many uncountable subsets of the unit interval* $\mathbf{I}$. *Then there exists a subset* P *of* $\mathbf{I}$ *of cardinality* $\mathbf{c}$ *such that no element of* $\mathbf{F}$ *is a continuous image of* P.

Proof: By the previous remark, for each $Y \in \mathbf{F}$, there are at most $\mathbf{c}$ many continuous real-valued functions f defined on $\mathbf{G}_\delta$-subsets of $\mathbf{I}$ with $Y \subseteq \text{range}(f)$. Since $\mathbf{F}$ has cardinality $\mathbf{c}$, and $\mathbf{c}^2 = \mathbf{c}$, we can list all the pairs (Y, f) such that f is a real-valued continuous function defined on a $\mathbf{G}_\delta$-subset of $\mathbf{I}$ with $Y \subseteq \text{range}(f)$, in a list of length $\mathbf{c}$: $\mathbf{A} = \{(Y_\alpha, f_\alpha) : \alpha < \mathbf{c}\}$. We can also enumerate (possibly with repetitions) each set $Y_\alpha = \{y_{\alpha\beta} : \beta < \mathbf{c}\}$. Let $E_{\alpha\beta} = f_\alpha^{-1}(y_{\alpha\beta})$. Since f_α is continuous, it follows that $E_{\alpha\beta}$ is closed and hence has the Baire property. Furthermore, for $\beta < \beta'$, $E_{\alpha\beta} \cap E_{\alpha\beta'} = \emptyset$. Now we can apply Lemma 12 to the family $\{E_{\alpha\beta} : \alpha, \beta < \mathbf{c}\}$, to obtain a sequence $\{\gamma(\alpha) : \alpha < \mathbf{c}\}$ and a set P of cardinality $\mathbf{c}$ disjoint from every $E_{\alpha\gamma(\alpha)}$.

It remains to show that no $Y \in \mathbf{F}$ is a continuous image of P. Suppose (towards a contradiction) that g is a continuous function and $Y \subseteq g[P]$. As we noted after Lemma 12, there is a continuous extension g^* of the partial function $g|P$ to a $\mathbf{G}_\delta$-subset of $\mathbf{I}$. So $Y \subseteq g[P] \subseteq g*[P]$. Hence the pair $(Y, g*)$ must appear in the list $\mathbf{A}$ as some pair (Y_α, f_α) for some $\alpha < \mathbf{c}$. Since $P \cap E_{\alpha\gamma(\alpha)} = \emptyset$, we have

$$P \cap f_\alpha^{-1}(y_{\alpha\gamma(\alpha)}) = \emptyset,$$

so $y_{\alpha\gamma(\alpha)}$ does not belong to $f_\alpha[P]$, and so

$$y_{\alpha\gamma(\alpha)} \in Y_\alpha \setminus f_\alpha[P] \subseteq Y \setminus g[P],$$

which contradicts $Y \subseteq g[P]$. This completes the proof. $\blacksquare$

Corollary 14. *Assume that* $\mathbf{R}$ *is not the union of less than* $\mathbf{c}$ *many nowhere dense sets. Then there exists a subset* $P \subseteq \mathbf{I}$ *of cardinality* $\mathbf{c}$ *such that* $\mathbf{I}$ *is not a continuous image of* P.

Proof: Let $\mathbf{F} = \{\mathbf{I}\}$ in Theorem 13. $\blacksquare$

Corollary 15. *Martin's Axiom (or Martin's Axiom for countable partial orders) implies that there exists a subset* $P \subseteq \mathbf{I}$ *of cardinality* $\mathbf{c}$ *such that* $\mathbf{I}$ *is not a continuous image of* P.

In his paper [14], Miller constructed a model of ZFC using forcing in which every subset of power $\mathbf{c}$ of $\mathbf{I}$ can be mapped continuously onto $\mathbf{I}$. In his model, $\mathbf{c} = \aleph_2$. As of writing, it is an open question[3] whether there is a model of ZFC in which $\mathbf{c} > \aleph_2$ and every subset of power $\mathbf{c}$ of $\mathbf{I}$ can be mapped continuously onto $\mathbf{I}$.

Our last theorem concerns totally imperfect subsets of real numbers. A totally imperfect subset of $\mathbf{R}$ is one which contains no non-empty perfect set. A set $E \subseteq \mathbf{R}$ is perfect iff E is closed and contains no isolated points. In section 40 of [10], Kuratowski proved that there exists an uncountable (totally imperfect) set $P \subseteq \mathbf{I}$ each of whose continuous images (situated in $\mathbf{I}$) is a totally imperfect set.

Theorem 16. *Suppose that* $\mathbf{R}$ *is not the union of less than* $\mathbf{c}$ *many nowhere dense sets. Then there exists a (totally imperfect) set* $P \subseteq \mathbf{I}$ *of cardinality* $\mathbf{c}$ *each of whose continuous images is a totally imperfect set.*

[3] Information kindly supplied by Professor Arnie Miller. The interested reader can find some of his papers and a list of problems on his website at: http://math.wisc.edu/miller

Proof: Let $\mathbb{F}$ be the family of non-empty perfect subsets of $\mathbb{I}$. Recall that $\mathbb{F}$ and every uncountable perfect set have cardinality $\mathfrak{c}$. Now apply Theorem 13 to $\mathbb{F}$. ∎

Corollary 17. *Martin's Axiom (or Martin's Axiom for countable partial orders) implies that there exists a set $P \subseteq \mathbb{I}$ of cardinality $\mathfrak{c}$ each of whose continuous images is a totally imperfect set.*

Of course, in Miller's model of ZFC mentioned above, these conclusions are false. So the question whether there exists a set $P \subseteq \mathbb{I}$ of cardinality $\mathfrak{c}$ each of whose continuous images is a totally imperfect set is again independent of ordinary set theory.

Further generalizations of these results to second-countable complete metric spaces are also possible.

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LINEAR ENHANCEMENTS OF THE STREAMLINE DIFFUSION METHOD FOR CONVECTION-DIFFUSION PROBLEMS

Neil Madden and Martin Stynes

Several computationally simple modifications of the streamline diffusion finite element method are developed for linear convection-dominated convection-diffusion problems in two dimensions. Numerical experiments show that these modifications yield significantly more accurate results than are attainable from the basic streamline diffusion method. Full details appear in [1].

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A UNIFORMLY CONVERGENT GALERKIN METHOD ON A SHISHKIN MESH FOR A CONVECTION-DIFFUSION PROBLEM

Martin Stynes and Eugene O'Riordan

A Galerkin finite element method that uses piecewise bilinears on a simple piecewise equidistant mesh is applied to a linear convection-dominated convection-diffusion problem in two dimensions. The method is shown to be convergent, uniformly in the perturbation parameter, of order $N^{-1}\ln N$ in a global energy norm and of order $N^{-1/2}\ln^{3/2} N$ pointwise near the outflow boundary, where the total number of mesh points is $O(N^2)$. Full details appear in [1].

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Research Announcement

THE MIDPOINT UPWIND SCHEME

Martin Stynes and Hans-Görg Roos

A modified upwind scheme is considered for a singularly perturbed two-point boundary value problem whose solution has a single boundary layer. The scheme is analysed on an arbitrary mesh. It is then analysed on a Shishkin mesh and precise convergence bounds are obtained, which show that the scheme is superior to the standard upwind scheme. A variant of the scheme on the same Shishkin mesh is proved to achieve even better convergence behaviour. Full details appear in [1].

Reference

[1] M. Stynes and H.-G. Roos, *The midpoint upwind scheme* (1996), Appl. Numer. Math. (to appear).

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Book Review

Introduction to Coding Theory (second edition)

J. H. van Lint
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Reviewed by Pat Fitzpatrick

Coding Theory will soon be 50 years old: it dates precisely back to Claude Shannon's fundamental 1948 paper, [14]. For such a young subject it has achieved a great deal, particularly in establishing connections with fundamental mathematics in a wide variety of areas encompassing group theory, finite geometries, combinatorics, number theory, algebraic geometry, algebraic function fields, computational algebra, and complexity theory. These relationships are mainly in the sense that mathematics from other areas is applied to inform the coding theory, for instance in the development of the theory of geometric Goppa codes from curves over $\mathbf{F}_q$, but there have also been some notable applications in the opposite direction, such as in the proof of the non-existence of a projective plane of order 10, [8], and in classical sphere-packing problems, [2]. Coding theory is, in essence, an area of applied mathematics, although it makes use of mathematics which has, until recently, appeared only on the "pure" syllabus. Many researchers in coding theory are engineers and many of the fundamental concerns are with specifically engineering questions such as the implementation of finite field arithmetic in logic or the complexity of decoding algorithms.

Not so the present volume! This is a book about mathematics, written for mathematicians. The presentation is condensed almost to the point of terseness, but the writing is superb, reminiscent in style of what one finds in the poet's quintessential "slim volume." The book began life as a set of lecture notes, with the first published edition written in 1981 and one is immediately impressed by the obvious qualities of elegance and precision that must have imbued those lectures (given not only by the author but also by A. E. Brouwer, H. W. Lenstra, and H. C. A. van Tilborg, among others). It is also apparent that the audience required what the author refers to as a "fairly thorough mathematical background," in abstract algebra certainly, as well as in certain topics from number theory, probability, and combinatorics. Van Lint provides a whirlwind tour through the necessary background in the first chapter, setting up notation and quoting results without proof on algebraic structures, finite fields, combinatorics and probability, but giving a little more detail on the rather less well known theory of Krawtchouk polynomials (of which more later). He then sets out a basic five chapter course in coding theory followed by five further chapters on what he regards as important topics (and we have every reason to be convinced of the soundness of his judgement).

An $[n, M, d]$ block code C over the finite field $\mathbb{F}_q$ is a subset of size M of the n-dimensional vector space $\mathbb{F}_q^n$. In general the code is not required to have any structure, but if it forms a subspace of dimension k (so that $|M| = q^k$) then it is called an $[n, k, d]$ linear codes. The ambient space is equipped with the Hamming distance

$$d_H(u, v) = |\{i : u_i \neq v_i\}|$$

and the parameter d denotes the minimum distance between codewords in C. The value of $\log_q M/n$ (or k/n) is known as the rate of the code as it represents the rate at which information can be transmitted via an embedding $C \to \mathbb{F}_q^n$. It is easy to see that a codeword $c \in C$ sent over a noisy channel (in which errors are introduced independently of position) and received as $\hat{c}$ can be decoded uniquely to c with maximum likelihood provided that the number of errors $d_H(\hat{c}, c) \leq \lfloor \frac{d-1}{2} \rfloor$. This begs the question of whether any decoding algorithm can be carried out effectively—direct comparison of $\hat{c}$ with every codeword is of exponential complexity and therefore useless in practice. Consequently, two of the major themes in coding theory research are to define and analyse classes of good codes having relatively large minimum distance, and to find codes for which one can construct polynomial time decoding algorithms.

Shannon's main theorem establishes the existence of good random codes that for sufficiently large values of n can be used in principle to make decoding error probability arbitrarily small (at appropriate rates). This is clearly the cornerstone of the theory and van Lint makes sure that it has a prominent place in his treatment. However, since Shannon's codes are completely unstructured, considerations of practicality form a competing requirement and it has proved difficult to find codes with practical decoding algorithms that achieve anything like the error probability promised by Shannon's theorem.

After the introduction of some analytical tools such as weight enumerators (essentially generating functions for the numbers of codewords of given weights in a code), the dual code, and the fundamental MacWilliams identities relating a weight enumerator of a code with that of its dual, some specific classes of codes are described. The ubiquitous Golay codes are included, of course. Next come Reed–Muller codes which are not as good as some, but whose advantage is that they are easy to decode. More importantly from van Lint's mathematical perspective they link coding theory with finite geometries and Boolean functions and we are introduced to the automorphism group of a code (the coordinate permutations that preserve it). A short section (added in the second edition) on Kerdock codes, which are subcodes of certain Reed–Muller codes, confirms our belief in the author's instincts, since one of the major developments of the 1990's is the discovery by Hammons *et al*, [6], that the (nonlinear, binary) Kerdock codes can be represented as images of linear codes over $\mathbf{Z}_4$.

A restricted version of the decoding problem for a code with minimum distance d is to decode up to $t = \lfloor \frac{\delta-1}{2} \rfloor$ errors for some $\delta < d$ (and record a decoding error if any received word does not lie within t errors of a codeword). One type of code for which such a bounded distance, incomplete decoding algorithm is the class of BCH codes, a subset of which is formed by the Reed-Solomon codes that are widely used in terrestrial and satellite communications, compact disks and computer disk drives. A BCH code $\mathcal{C}$ over $\mathbf{F}_q$ may be conveniently defined as an ideal in the polynomial algebra $A = \mathbf{F}_q[x]/(x^n - 1)$, where n and q are relatively prime to avoid repeated factors in the decomposition of $x^n - 1$ into irreducibles. Thus, $\mathcal{C}$ is generated by a polynomial g dividing $x^n - 1$ and this means that the code is cyclic in the sense that every cyclic shift of a codeword is again a codeword. The theory of general cyclic codes is ultimately derived from the decomposition of the semisimple algebra A as a sum of minimal ideals based on a system of orthogonal primitive idempotents. Van Lint covers this—as do most coding theory books— from first principles, without appealing to general results. The BCH theorem says that if γ is a primitive n-th root of unity in an extension of $\mathbf{F}_q$ and if g contains the consecutive set $\gamma, \gamma^2, \ldots, \gamma^{\delta-1}$ among its roots then the code $\mathcal{C}$ has minimum distance at least δ. Extensions of this result, proved by Hartmann and Tzeng, [5], and Roos, [13], are based on the existence of *several* consecutive sets of powers of γ being among the roots of g. The best known bound of this type was proved by van Lint and Wilson, [10]; it is included in the section on BCH codes and the earlier results are derived as special cases.

In general the problem of finding the minimum distance of a given code or class of codes (and hence their exact error correcting capability) is difficult and the determination of upper and lower bounds is another principal theme of the theory. Linearity is not assumed and the function $A(n, d)$ is defined as the maximum value of M for which an $[n, M, d]$ code exists. A code $\mathcal{C}$ with $|\mathcal{C}| = A(n, d)$ is said to be optimal. The study of this function is the central problem of combinatorial coding theory and van Lint provides an overview of the known bounds. Of particular interest is the construction of classes of asymptotically good codes with parameters $[n_i, M_i, d_i]$ such that the rate k_i/n_i and the relative distance d_i/n_i are both bounded away from zero as $i \to \infty$. None of the classes used in practice (such as the BCH codes) have this property, but an outstanding example discovered by Justesen, [7], and representing a major achievement of the 1970's, has its place in van Lint's treatment. The introduction of Justesen codes requires the author to develop the notion of concatenation of codes which is valuable in itself, since this technique (in which the codewords of an inner code are used as information vectors to a second outer code) is widely used in practice (in compact disk and deep space telemetry, for example). Also, the recent generalization to what are known as turbo–codes, [1], [4], has produced some of the potentially best performing practical coding schemes known today.

Perhaps the most significant of the distance bounds, especially in terms of motivating new research, is the Gilbert (or Gilbert-Varshamov) bound. This concerns the asymptotic rate

$$\alpha(\tfrac{d}{n}) = \lim_{n\to\infty} \sup n^{-1} \log_q A(n, \tfrac{d}{n})$$

of an optimal code and establishes the existence for $1 \leq \frac{d}{n} \leq \frac{q-1}{q}$ of codes with

$$\alpha(\tfrac{d}{n}) \geq 1 - H_q(\tfrac{d}{n}),$$

where H is the entropy function

$$H_q(x) = x \log_q(q-1) - x \log_q x - (1-x)\log_q(1-x).$$

For many years this bound on $\alpha(\frac{d}{n})$ was thought to be best possible, until 1982, when in a remarkable development Tsfasman, Vlădut and Zink, [17], discovered a class of codes improving the bound for $q \geq 49$. Their codes, constructed from algebraic curves over $\mathbf{F}_q$, are based on the pioneering work of Goppa in the early 1980's (see [3]) and as a consequence of their discovery there has been an enormous amount of research over the past ten years in the development of new algebraic geometry (or AG) codes and the search for efficient decoding algorithms. An alternative function field approach to these codes is the subject of a book by Stichtenoth, [15], reviewed recently in these pages by Gary McGuire, [11]. Van Lint manages to give a good flavour of the geometric ideas in just a few pages appended to the original first edition section on Reed–Solomon codes, of which the algebraic geometry codes are a natural generalization.

A notable feature of van Lint's overall treatment of coding theory is the prominent position given to the Krawtchouk polynomials. For fixed values of n and q, this class of orthogonal polynomials is defined as

$$K_k(x;n,q) = \sum_{j=0}^{k} \binom{x}{j}\binom{n-x}{k-j}(q-1)^{k-1}$$

where

$$\binom{x}{j} = \frac{x(x-1)\cdots(x-j+1)}{j!}, \quad x \in \mathbf{R}.$$

Properties of these polynomials are used in several places, such as in the analysis of weight enumerators, referred to earlier, and in the classification of perfect codes. Defining a code over a general alphabet Q rather than just over $\mathbf{F}_q$ a perfect t-error correcting code C of length n has the the property that the Hamming spheres $S_t(x) = \{c \in C | d(c,x) \leq t\}$ are disjoint and completely fill the space Q^n. It was shown by Tietäväinen, [16], and van Lint, [9], that the only nontrivial t-error correcting perfect codes with $t > 1$ and $|Q|$ a prime power are the Golay codes. In the book van Lint proves the binary case using a remarkable sufficient condition, known as Lloyd's Theorem (see [9]), that if a binary perfect t-error correcting code of length n exists then the polynomial $\Psi_t(x) = K_t(x-1; n-1, 2)$ has t distinct zeros among the integers 1, 2, ..., n. This chapter also contains a study of binary uniformly packed codes (which generalize perfect codes) using certain sequences of numbers defined from linear functionals on the group algebra $\mathrm{C}\mathbf{F}_2^n$, as well as further properties of the Krawtchouk polynomials.

In the last two chapters of the book van Lint departs from the prevailing theme of block codes to introduce the reader to topics with radically different flavours. First there is a brief look at arithmetic codes which are used in the detection and correction of errors in ordinary arithmetic computations. Much more important from a practical point of view, convolutional codes are considered. In these codes the information sequence is potentially infinite and the encoded stream is formed by interleaving the convolutions of the input stream with two or more finite sequences (in practice of length no more than about 10). As van Lint notes in his introduction to this chapter "the mathematical theory of convolutional codes is not well developed ...[and this is] one of the reasons that mathematicians find it difficult to become interested [in them]." But convolutional codes are widely used in practice, often concatenated with an outer Reed–Solomon code, and moreover, the well known Viterbi decoding algorithm that is used for convolutional codes also plays a significant role in getting rid of intersymbol interference in the read–write channel for computer disk drives. (Permitting such intersymbol interference in a controlled manner is essentially what has led to the enormous increases over recent years in the density of data storage.) So these codes are not only very open to mathematical analysis but also very important in view of their applications. A particularly interesting and potentially fruitful avenue is in the investigation of the automorphism groups of convolutional codes pioneered by Piret, [12], and true to form van Lint hits the right note by dealing with that aspect in a short final section of this last chapter.

Van Lint's book might almost be regarded as a collection of "edited highlights" of coding theory, in many of which he has been personally involved. One wants to read and re-read in order to fully digest and savour their excellence. There is no doubt that the reader will have to work at this book, but the rewards are handsome.

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