each being given in alternate years. The first J. L. Synge Prize in Mathematics was shared by John Callan and Raymond Russell in 1993, and the second was awarded to Conal Kennedy in 1995. The first J. L. Synge Public Lecture was given by Professor Sir Hermann Bondi in 1992, and the second by Professor Werner Israel, a student of Professor Synge. The third lecture was given by Professor Sir Roger Penrose on May 7, 1996.

Professor Synge was a kind and generous man. He encouraged and inspired several generations of students who will always remember him with gratitude, fondness and the deepest respect.

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A CONIC AND A PASCAL LINE AS CUBIC LOCUS

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1. Statement of results

This material arose out of an effort to generalize a result of William Wallace in 1797, to the effect that the feet of the perpendiculars from a point on the circumcircle of a triangle onto the side-lines are collinear. Through historical mis-attribution, the lines of collinearity have been widely known as *Simson lines*.

Our most general result is Theorem 3. A reduced case of that is Theorem 1. A converse of the latter is Theorem 2, and this constitutes an enhancement of the configuration in the celebrated Pascal's theorem.

Theorem 1. *In a projective plane, let A_1, A_2, A_3 be non-collinear points and B_1, B_2, B_3 distinct collinear points such that*

$$B_1 \neq A_2,\ A_3,\ B_2 \neq A_3,\ A_1,\ B_3 \neq A_1,\ A_2,$$

$$A_2B_3 \neq A_3B_2,\ A_3B_1 \neq A_1B_3,\ A_1B_2 \neq A_2B_1 \tag{1}$$

Let C_1, C_2, C_3 be the points specified by

$$C_1 = A_2B_3 \cap A_3B_2,\ C_2 = A_3B_1 \cap A_1B_3,\ C_3 = A_1B_2 \cap A_2B_1. \tag{2}$$

For a variable point P, take points $Q_1 \in A_2A_3$, $Q_2 \in A_3A_1$, $Q_3 \in A_1A_2$, such that $Q_1 \in PB_1$, $Q_2 \in PB_2$, $Q_3 \in PB_3$. Then the set $\mathcal{E}_1$ of points P for which Q_1, Q_2, Q_3 are collinear, contains the points A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3. It is either the whole plane or else a conic through A_1, A_2, A_3, C_1, C_2, C_3, and *the line $B_1B_2B_3$. The degenerate case of the plane occurs when $B_1 \in A_2A_3$, $B_2 \in A_3A_1$, $B_3 \in A_1A_2$.*

Now by (2) we also have

$$A_2C_3 \cap A_3C_2 = B_1,\ A_3C_1 \cap A_1C_3 = B_2,\ A_1C_2 \cap A_2C_1 = B_3, \quad (3)$$

so we have the conic through A_1, A_2, A_3, C_1, C_2 and C_3, and the Pascal line $B_1B_2B_3$. This is the configuration of Pascal's theorem.

Working somewhat in reverse and starting differently, we can also state the following, which is a converse of Theorem 1.

Theorem 2. *In a projective plane, let $\mathcal{C}_1$ be a proper point conic, and A_1, A_2, A_3, C_1, C_2, C_3 distinct points on $\mathcal{C}_1$. Let*

$$A_2C_3 \cap A_3C_2 = B_1,\ A_3C_1 \cap A_1C_3 = B_2,\ A_1C_2 \cap A_2C_1 = B_3,$$

so that B_1, B_2, B_3 are collinear. If for any point P, PB_1 meets A_2A_3 at Q_1, PB_2 meets A_3A_1 at Q_2 and PB_3 meets A_1A_2 at Q_3, then Q_1, Q_2 and Q_3 are collinear if and only P is on $\mathcal{C}_1$ or on the line $B_1B_2B_3$.

Figure 1 refers to Theorems 1 and 2.

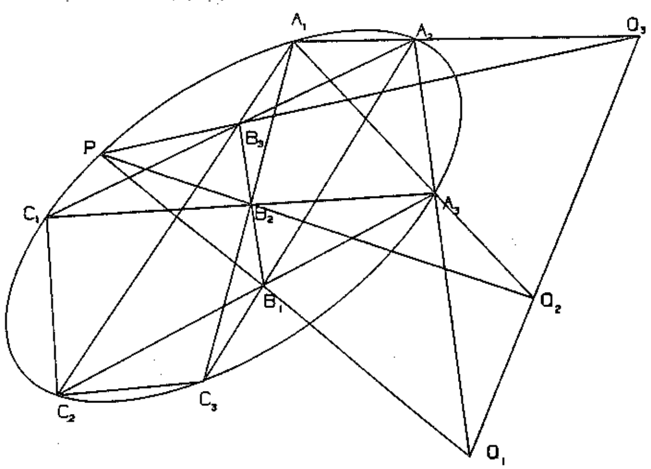

Figure 1

A conic and a line constitute a reducible cubic and our locus is essentially a cubic. Our approach has caused us to take B_1, B_2 and B_3 to be collinear, and if we take them to be non-collinear we find that we obtain a cubic which passes through A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, and C_3.

Theorem 3. *In a projective plane, let a_1, a_2, a_3 be distinct lines and write*

$$A_1 = a_2 \cap a_3,\ A_2 = a_3 \cap a_1,\ A_3 = a_1 \cap a_2.$$

Let B_1, B_2, B_3 be distinct points such that (1) is satisfied, and let C_1, C_2, C_3 be defined by (2). For a variable point P, let

$$PB_1 \cap a_1 = Q_1,\ PB_2 \cap a_2 = Q_2,\ PB_3 \cap a_3 = Q_3.$$

Then the set $\mathcal{E}_1$ of points P such that Q_1, Q_2, Q_3 are collinear is either a point cubic or the whole plane. The set $\mathcal{E}_1$ contains each of the points A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3, and it degenerates to the plane if and only if B_1, B_2, B_3 are collinear and $B_1 \in a_1$, $B_2 \in a_2$, $B_3 \in a_3$.

2. Proofs

To start on our proofs, in a projective plane we let a_1, a_2, a_3 be distinct lines and write

$$A_1 = a_2 \cap a_3,\ A_2 = a_3 \cap a_1,\ A_3 = a_1 \cap a_2.$$

Let B_1, B_2, B_3 be distinct points satisfying (1). We then introduce the points C_1, C_2, C_3 in (2). For a variable point P, let

$$PB_1 \cap a_1 = Q_1,\ PB_2 \cap a_2 = Q_2,\ PB_3 \cap a_3 = Q_3.$$

We seek the set $\mathcal{E}_1$ of points P such that Q_1, Q_2, Q_3 are collinear. It can be checked directly from the definition that A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3 are all in $\mathcal{E}_1$, and indeed that if B_1, B_2, B_3 are collinear, then every point P of the line $B_1B_2B_3$ is in $\mathcal{E}_1$.

Supposing first that a_1, a_2, a_3 are not concurrent, as in [1] we use homogeneous coordinates and take a triangle of reference so that

$$A_1 = (1,0,0),\ A_2 = (0,1,0),\ A_3 = (0,0,1).$$

Suppose that

$$B_1 = (a,b,c),\ B_2 = (d,e,f),\ B_3 = (g,h,k),\ P = (x,y,z). \quad (4)$$

Then

$$\begin{aligned} Q_1 &= (0, -bx+ay, -cx+az),\\ Q_2 &= (ex-dy, 0, ez-fy),\\ Q_3 &= (kx-gz, ky-hz, 0). \end{aligned} \quad (5)$$

Hence for Q_1, Q_2, Q_3 to be collinear it is necessary and sufficient that

$$\det\begin{pmatrix} 0 & -bx+ay & -cx+az \\ ex-dy & 0 & ez-fy \\ kx-gz & ky-hz & 0 \end{pmatrix} = 0,$$

which expands to

$$(ay-bx)(ez-fy)(kx-gz) + (az-cx)(ex-dy)(ky-hz) = 0,$$

and then to

$$\begin{aligned} &a(fg-dk)y^2z + a(dh-eg)yz^2 + e(bg-ah)z^2x + e(ch-bk)zx^2 \\ &\quad + k(bf-ce)x^2y + k(cd-af)xy^2 + (2aek-bfg-cdh)xyz = 0. \end{aligned} \quad (6)$$

Turning now specifically to Theorem 1, we note that a condition that B_1, B_2, B_3 be collinear is that

$$\Delta = \det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$$

satisfy $\Delta = 0$. As A_1, A_2, A_3 are not collinear, at least one of them is not on B_1B_2. If $A_3 \notin B_1B_2$ we can solve $\Delta = 0$ for k and insert in (6) to obtain the product of

$$(bf-ce)x + (cd-af)y + (ae-bd)z \quad (7)$$

which gives the equation of $B_1B_2B_3$, and

$$a(dh-eg)yz + e(bg-ah)zx + [f(ah-bg) + c(eg-dh)]xy. \quad (8)$$

This last yields a conic unless all its coefficients are equal to 0, in which case the locus is degenerate. The other cases are treated similarly. This establishes Theorem 1, apart from analysing fully the degenerate case which we shall return to later.

For Theorem 2, we start by supposing that A_1, A_2, A_3, C_1, C_2, C_3 are on a proper conic $\mathcal{C}_1$. We define B_1, B_2, B_3 by (3) and then (2) holds. Here $A_3 \notin B_1B_2$, so we obtain (7) and (8). Now (8) cannot degenerate to having all its coefficients equal to 0, as *e.g.* $B_1 \notin A_2A_3$, $A_3 \notin B_2B_3$ imply

$$a \neq 0,\ dh - eg \neq 0.$$

Thus (8) gives the equation of a conic through A_1, A_2, A_3, C_1, C_2, C_3, and hence of $\mathcal{C}_1$. This establishes Theorem 2. We note that in it, the roles of (A_1, A_2, A_3) and (C_1, C_2, C_3) are interchangeable.

Continuing so as to cover the case where B_1, B_2, B_3 are not collinear, we suppose that a_1, a_2, a_3, A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3, P, Q_1, Q_2, Q_3 are as before, except that now we take a_1, a_2, a_3 to be any three distinct lines (so that they may be concurrent and then $A_3 = A_2 = A_1$), and B_1, B_2, B_3 to be any distinct points (and thus not confined to being collinear), such that (1) is satisfied and so C_1, C_2, C_3 are well-defined.

When a_1, a_2, a_3 are not concurrent, we choose coordinates as before and the calculations above show that $\mathcal{E}_1$ has the equation (6). When a_1, a_2, a_3 are concurrent we take the triangle of reference so that these lines have the equations

$$y + z = 0,\ y = 0,\ z = 0,$$

respectively. With (4) as before, instead of (5) we find that

$$
\begin{aligned}
Q_1 &= ((b+c)x - ay - az, cy - bz, -cy + bz),\\
Q_2 &= (ex - dy, 0, ez - fy),\\
Q_3 &= (kx - gz, ky - hz, 0).
\end{aligned}
$$

Then these points are collinear if and only if

$$
\det \begin{pmatrix} (b+c)x - ay - az & cy - bz & -cy + bz \\ ex - dy & 0 & ez - fy \\ kx - gz & ky - hz & 0 \end{pmatrix} = 0,
$$

which expands to

$$
\begin{aligned}
&k(cd - af)y^3 + e(bg - ah)z^3 + e(ch - bk)z^2x\\
&+ \{a[ek + f(h-k)] - bdk + c(fg - dh)\}y^2z\\
&- \{a[e(h-k) - fh] + b(fg - dh) + ceg\}yz^2\\
&+ k(bf - ce)xy^2 - [bf(h-k) + ch(f-e)]xyz = 0. \qquad (9)
\end{aligned}
$$

Thus $\mathcal{E}_1$ has this as equation.

Checking the cases in which $\mathcal{E}_1$ degenerates to the whole plane is rather detailed. It is convenient to denote by capital letters the cofactors of the elements in Δ. When a_1, a_2, a_3 are non-concurrent, by (6) degeneracy occurs only if all of

$$
\begin{aligned}
&aB = 0, aC = 0, eF = 0, eD = 0, kG = 0, kH = 0,\\
&aA + eE + kK - \Delta = 0, \qquad (10)
\end{aligned}
$$

hold. We divide into the cases

(i) all three of a, e, k are equal to 0;
(ii) exactly two of a, e, k are equal to 0, and by symmetry we can take $e = k = 0, a \neq 0$;
(iii) exactly one of a, e, k is equal to 0, and we can take $a = 0, e \neq 0, k \neq 0$;
(iv) none of a, e, k is equal to 0.

In (i), as $a = 0$, we have $B_1 \in A_2A_3$ and similarly $B_2 \in A_3A_1$, $B_3 \in A_1A_2$. As $\Delta = 0$, B_1, B_2 and B_3 are collinear. In this case $\mathcal{E}_1$ degenerates. In (ii), as $e = k = 0$, we have $B_2 \in A_3A_1$, $B_3 \in A_1A_2$. As $B = C = 0$ we have $A_2 \in B_2B_3$, $A_3 \in B_2B_3$. Thus $B_2 = A_3, B_3 = A_2$, which is incompatible with (1). Similarly we find that (iii) and (iv) are incompatible with (1).

Similarly when a_1, a_2, a_3 are concurrent, by (9) $\mathcal{E}_1$ can degenerate to being the whole plane only when

$$
\begin{aligned}
&eD = 0, eF = 0, kG = 0, kH = 0, fD + hG = 0,\\
&- fE - hH + kK = 0, -eE + fF + hK = 0. \qquad (11)
\end{aligned}
$$

Now $C_1 = C_2 = C_3 = A_1$ and (1) implies that none of the triples

$$\{A_1, B_2, B_3\}, \{A_1, B_3, B_1\}, \{A_1, B_1, B_2\} \text{ is collinear.} \qquad (12)$$

We divide into the cases

(v) $e = k = 0$;
(vi) $e = 0, k \neq 0$;
(vii) $e \neq 0, k \neq 0$.

In (v), as $e = k = 0$, we have $B_2 \in a_2$, $B_3 \in a_3$ and so

$$fD + hG = 0, \; fE + hH = 0, \; fF + hK = 0.$$

If we had $f = g = 0$, then we would have $B_2 \in a_3$, $B_3 \in a_2$ and so

$$B_2 = B_3 = A_1,$$

which is ruled out as $B_2 \neq B_3$. We then have $(f, h) \neq (0, 0)$ and so

$$DH - GE = 0, \; EK - FH = 0, \; FG - DK = 0,$$

that is $c\Delta = a\Delta = b\Delta = 0$. Now $\Delta \neq 0$ would imply that $(a, b, c) = (0, 0, 0)$, which is impossible as these are homogeneous coordinates for B_1. Thus $\Delta = 0$, and so B_1, B_2, B_3 are collinear. Here $Q_2 = B_2$, $Q_3 = B_3$ and so we need $Q_1 \in B_2B_3$; this makes $Q_1 = B_1$ and so $B_1 \in a_1$. In this case $\mathcal{E}_1$ degenerates. In (vi) $e = 0$ implies $B_2 \in a_2$, and $k \neq 0$ implies $G = 0$ and $A_1 \in B_1B_2$.

These conflict with (12). Similarly (vii) conflicts with (12). These combined cases establish Theorem 3.

By considering the dual of Theorem 3, it can be deduced that the set $\mathcal{E}_2$ of lines p that are a line of collinearity $Q_1Q_2Q_3$ in Theorem 3, is either a line cubic or the set of all lines in the plane. If a_1, a_2, a_3 are concurrent, then the lines on A_1 form part of $\mathcal{E}_2$, and in the non-degenerate case $\mathcal{E}_2$ consists of a line conic and the lines on one of its Brianchon points.

It is evident that we do not obtain all cubics in Theorem 3, as $\mathcal{E}_1$ there is determined by the six points A_1, A_2, A_3, B_1, B_2 and B_3. Nonetheless, it yields a large class of cubics with a geometrical property. This class is closed under projective transformations.

An example

The equation (6) does not suit taking $z = 1$ to obtain Cartesian coordinates, as A_1 and A_2 would be points at infinity. Because of this we introduce Cartesian coordinates (X, Y) for P by applying the transformation

$$x = 1 - X - Y,\ y = X,\ z = Y.$$

In this way A_1, A_2, A_3 have Cartesian coordinates (0,0), (1,0), (0,1), respectively. Taking for an example, B_1, B_2, B_3 to have Cartesian coordinates (3,1), (3,2), (2,2), respectively, we have

$$B_1 = (-3, 3, 1),\ B_2 = (-4, 3, 2),\ B_3 = (-3, 2, 2).$$

Then we find that (6) becomes

$$2X^3 - 8X^2 - 2X(Y^2 - Y - 3) - 3Y(Y-1)(Y-4) = 0.$$

The graph of this is shown in Figure 2.

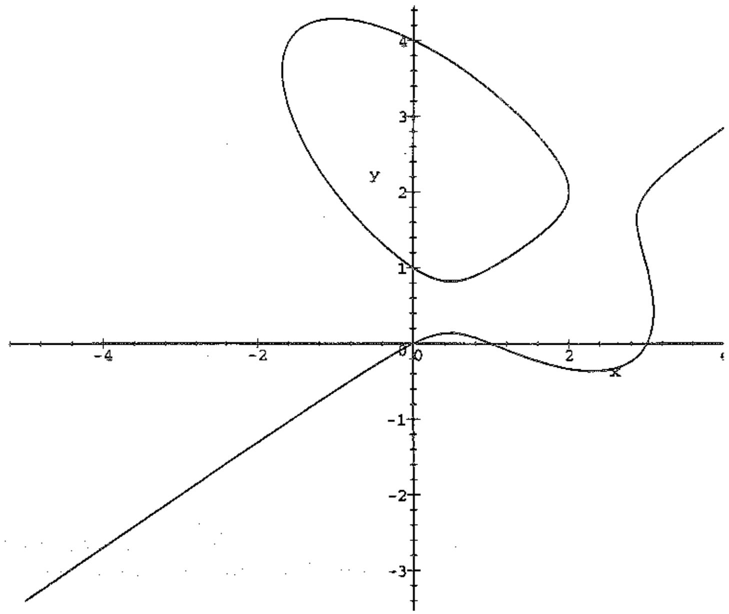

Figure 2

References

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