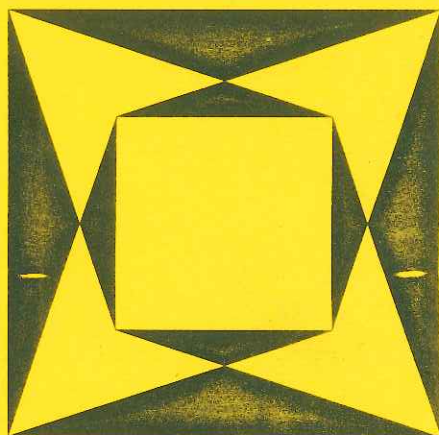


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IRISH MATHEMATICAL SOCIETY
BULLETIN

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The aim of the Bulletin is to inform Society members about the activities of the Society and about items of general mathematical interest. It appears twice each year, at Easter and at Christmas. The Bulletin is supplied free of charge to members; it is sent abroad by surface mail. Libraries may subscribe to the Bulletin for IR£20.00 per annum.

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NOTES ON APPLYING
FOR I.M.S. MEMBERSHIP

1. The Irish Mathematical Society has reciprocity agreements with the American Mathematical Society and the Irish Mathematics Teachers Association.
2. The current subscription fees are given below.

Institutional member	IR£50.00
Ordinary member	IR£15.00
Student member	IR£6.00
I.M.T.A. reciprocity member	IR£5.00

The subscription fees listed above should be paid in Irish pounds (puint) by means of a cheque drawn on a bank in the Irish Republic, a Eurocheque, or an international money-order.

3. The subscription fee for ordinary membership can also be paid in a currency other than Irish pounds using a cheque drawn on a foreign bank according to the following schedule:

If paid in United States currency then the subscription fee is US\$25.00.

If paid in sterling then the subscription fee is £15.00 stg.

If paid in any other currency then the subscription fee is the amount in that currency equivalent to US\$25.00.

The amounts given in the table above have been set for the current year to allow for bank charges and possible changes in exchange rates.

4. Any member with a bank account in the Irish Republic may pay his or her subscription by a bank standing order using the form supplied by the Society.
5. The subscription fee for reciprocity membership by members of the American Mathematical Society is US\$10.00.

6. Any ordinary member who has reached the age of 65 years and has been a fully paid up member for the previous five years may pay at the student membership rate of subscription.
7. Subscriptions normally fall due on 1 February each year.
8. Cheques should be made payable to the Irish Mathematical Society. If a Eurocheque is used then the card number should be written on the back of the cheque.
9. Any application for membership must be presented to the Committee of the I.M.S. before it can be accepted. This Committee meets twice each year.
10. Please send the completed application form with one year's subscription fee to

The Treasurer, I.M.S.
 Department of Math. Physics
 University College, Dublin
 Ireland

Minutes of the Meeting of the Irish Mathematical Society

Ordinary Meeting
 21st December 1995

The Irish Mathematical Society held an Ordinary Meeting at 12.15pm on Thursday 21st December 1995 in the Dublin Institute for Advanced Studies, 10 Burlington Road. There were 17 members present. The President, D. Hurley was in the chair. Apologies were received from P. Mellon (Secretary).

1. The **minutes** of the meeting of 11th April 1995 were approved and signed.

2. **Matters arising**

An amount of 120ECU has been paid to the E.M.S. towards the I.M.S.'s subscription. The reason for this reduced subscription was explained in a letter by the Treasurer to the E.M.S. committee, sent in October 1995. No reply has been received to date. E.M.S. individual membership is available for £12.

The Chair thanked the organizers of the 1995 September meeting at the University of Limerick: S. O'Brien, E. Gath and G. Lessells. The next invited mathematics education talk to be organized by the I.M.S. will be arranged for the morning before the start of the D.I.A.S. Christmas Symposium in 1996. It was reported that M. Tuite of UCG had been co-opted on to the committee.

3. **Treasurer's business**

The Treasurer reported that the I.M.S. had a surplus of £7.89 in the year 1995. There are at present 196 members in the I.M.S. There are 38 A.M.S. members and 5 institutional members. 32 members are in arrears. Local representatives will be approached to follow up on these. The annual fee will be £15, as from 1st January, 1996. The Treasurer will circulate a standing order form to all members, so that the new rate may be introduced. It was agreed that members may pay membership at any time for up to five years in advance.

The Treasurer, J. Pulé, proposed that any ordinary member, who has reached the age of 65 years or older, and has been an I.M.S. member for the previous five years, will then, and in all subsequent years, be eligible for the student membership rate. The motion, which was seconded by M. Tuite, was passed.

4. September Meeting 1996

The September Meeting 1996 will take place at the Queen's University, Belfast on Monday and Tuesday, 2nd and 3rd September, starting at 11am. A. Wickstead announced that the organization of the meeting is well underway. Some funding has already been raised, accommodation booked and tentative plans made for the banquet. He requested suggestions for speakers, and in particular anyone visiting the island at that time, who would not normally reside here. He may be reached by e-mail at a.wickstead@qub.ac.uk. The Chair urged I.M.S. members to make a special effort to attend this meeting.

5. Elections

It was noted that P. Mellon (Secretary), J. Pulé (Treasurer), E. Gath, R. Gow and R. Timoney have reached the end of their two-year terms (J. Pulé having taken up the position vacated by M. Van Dyck).

An election took place to fill the five vacant committee positions. The nominations were as follows:

Position	Nominee	Proposer	Seconder
Secretary	P. Mellon ††	D. Simms	D. Ó Mathúna
Treasurer	J. Pulé †	T. Murphy	R. Timoney
Ordinary Members	A. Wickstead	E. Gath	A. Wood
	E. Gath ††	R. Timoney	D. Hurley
	R. Gow †	G. Lessells	T. Laffey

Since there was only one candidate for each position all five nominees were elected. Each † denotes a previous term of office. It was noted that D. Hurley (President), C. Nash (Vice-President), M. Clancy, B. Goldsmith, K. Hutchinson, G. Lessells and M. Tuite each has one more year remaining in their current term of office.

6. Any other business

R. Timoney expressed concern about the proposed discontinuation of the graduate program in Mathematics at the University of Rochester (New York). He proposed that the I.M.S. should write letters the President, Provost, Vice-Provost and Chairman of the Board of University of Rochester to protest about this. A. Wood proposed an amendment (seconded by M. Stynes) to the letter drafted by R. Timoney. A vote was taken, resulting in 7 in favour, 7 opposed and 3 abstentions. R. Timoney then accepted the amendment and it was agreed that the amended form of the letter should be sent.

J. Pulé suggested that the I.M.S. establish a prize, such as a perpetual trophy, for the winner of the Irish/International Mathematical Olympiad contest(s). He noted that an annual prize, such as medals, might be too costly for the Society. Some concern was expressed about how this prize could affect the team aspect of the international contest. T. Laffey, who is an I.M.O. organizer, noted that the publicity associated with the prize-giving ceremony could be beneficial in terms of raising commercial sponsorship. It was agreed that J. Pulé, T. Laffey and G. Lessells will consider the various possibilities and report back to the next Ordinary Meeting of the I.M.S.

The President reported that formal contact had recently taken place with the Irish Mathematics Teachers' Association. He proposed that, with the I.M.T.A., the I.M.S. should sponsor and organize a lecture for transition (4th) year secondary school students at various venues around the country. This would involve two half-hour talks by people who could explain the uses of mathematics in industry *etc.*

R. Timoney proposed the formation of a World-Wide Web homepage for the I.M.S. There was some discussion about the merits

of using this to provide an electronic form of the Bulletin (*e.g.* as a postscript file). Concern was expressed about the impact this could have on membership. It was agreed that R. Timoney will initiate the process of establishing a home-page. The Mathematics Departments of the various institutions in Ireland will be requested to add a pointer to the I.M.S. page in their own home-pages.

The meeting closed at 1.20pm.

Eugene Gath
University of Limerick.

THE I.M.S. SEPTEMBER MEETING 1995

Eugene Gath

The eighth September meeting of the Irish Mathematical Society took place on 7th and 8th September 1995 on the University of Limerick campus. The lectures were held in the new Foundation Building, in a theatre adjacent to the University Concert Hall. The attendance at the meeting was approximately 60, of whom 35 travelled from outside of Limerick. Visitors were accommodated in the Plassey Village, adjacent to the campus.

The meeting was opened by Dr Edward Walsh, President of the University of Limerick. He talked briefly about the applicability of mathematics, embellished with an anecdote about his experiences of learning trigonometry. One of the guest speakers, Prof. Chris Budd from the University of Bath gave a lecture on impact oscillators, illustrating a model with a new type of bifurcation process, which has applications to the safety of fuel rods in nuclear reactors. Dr Andrew Fowler of Oxford University spoke on the dynamics of ice sheets, including an investigation of the effect of the melting of the ice caps during the last Ice Age. The standard of the talks was excellent and a wide variety of topics was covered: algebra, analysis, numerical analysis, fluid and solid mechanics, differential equations, theoretical computer science, dynamical systems, differential geometry, operator theory, mathematics education *etc.* Three speakers travelled from Britain and one from Northern Ireland. Serendipitously, there was, concurrent with the I.M.S. meeting, a *Z Users' Meeting* also taking place in the Foundation Building. One of their speakers Prof. David Gries, head of the Department of Computer Science at Cornell University, gave a short talk at the I.M.S. meeting.

The full program was:

Thursday 7th September, 1995

Dr E. Walsh (President, University of Limerick) <i>Welcoming address</i>
Dr D. Hurley (IMS President) <i>Opening Remarks</i>
Dr S. O'Brien (University of Limerick) <i>Industrial coating flows</i>
Prof. C. Budd (University of Bath) <i>Grazing bifurcations in impact oscillators</i>
Prof. M. Hayes (University College Dublin) <i>Directional ellipse method for solution of partial differential equations</i>
Prof. T. Laffey (University College Dublin) <i>Some combinatorial properties of matrices with nonnegative entries</i>
Prof. D. Gries (Cornell University) <i>The consequences of teaching proof to mathematics students using the calculational approach to logic</i>
Dr C. H. Chu (Goldsmiths' College, London) <i>Exponential functions</i>

Friday 8th September, 1995

Prof. John O'Donoghue (University of Limerick) <i>The mathematics of numeracy</i>
Dr M. Mac an Airchinnigh (Trinity College Dublin) <i>Applied constructive mathematics in computing-monoids and their morphisms</i>
Dr D. Hurley (University College Cork) <i>Symmetric Attractors</i>
Dr A. Fowler (Oxford University) <i>Large scale oscillations in pleistocene ice sheets</i>
Dr M. Stynes (University College Cork) <i>Efficient solution of convection-diffusion problems using PLTMG</i>
Prof. J. Miller (Trinity College Dublin) <i>Demonstration of Mathematica package for polynomial root magnitude determination</i>
Prof. S. Sen (Trinity College Dublin) <i>Abelian sandpiles</i>
Prof. A. Wickstead (Queen's University Belfast) <i>When do positive operators span the bounded ones?</i>
Dr A. Hegarty (University of Limerick) <i>Shishkin meshes for the numerical solution of convection-diffusion problems</i>
Dr D. Wraith (University of Notre Dame) <i>Exotic spheres with positive Ricci curvature</i>
Prof. F. Hodnett (University of Limerick) <i>Closing remarks</i>

The conference banquet was held in Goosers' restaurant, in Ballina, Co. Tipperary –the 'other side' of Killaloe. The I.M.S. sponsored a free bus service from the University to Killaloe and back. An optional cruise on the Shannon and the southern tip of Lough Derg was arranged for before dinner. As one may recall, the remnant of Hurricane Iris drenched the country on Wednesday 6th September, but luckily the weather was warm and sunny just in time for our cruise the following evening! Of those who didn't join in the cruise, some took a walking tour of the town of Killaloe, with its historic cathedral *etc.*

The meeting was sponsored mostly from a variety of sources within the University of Limerick. Bank of Ireland and Aer Rianta (in honour of their 50th year at Shannon) also gave generous donations. The Department of Mathematics and Statistics provided stationery, secretarial support and postage. The organizing committee would like to gratefully acknowledge these, and to thank all the speakers and everyone who contributed to the success of the September Meeting 1995.

Conference organizers:

Eugene Gath, Gordon Lessells, Stephen O'Brien

Department of Mathematics and Statistics,

University of Limerick.

SPACE-FILLING CURVES AND RELATED FUNCTIONS

Stephen M. Buckley

In this paper, we shall investigate several questions related to space-filling curves. We start with a question whose answer has been known (although not widely known, it would appear) for rather a long time.

Question 1. *Do there exist continuous functions $f : [0, 1] \rightarrow \mathbb{R}$ which take on each of uncountably many values uncountably often?*

The answer is "yes"; in fact the first component of any space-filling curve (Peano curve) is such a function. A recent rather simple example of such a curve can be found in [8]; for more information on space-filling curves, the reader should consult [7].

Here we shall give two rather different methods of constructing examples of functions answering our question. Some examples using the first construction have zero derivative almost everywhere, while the second construction always leads to nowhere-differentiable examples. We use the notation $f^{-1}(y)$ to denote the set of all pre-images of y .

We begin with the "digit coding" construction. The example we give maps the unit interval onto itself and takes on all its values uncountably frequently. First note that any function continuous on a closed subset S of $[0, 1]$ (with respect to the subspace topology) can be extended to a function continuous on the whole interval simply by filling in the omitted open intervals with continuous interpolating functions (for instance we can "join up the dots" in a linear fashion, and extend the function in a constant fashion at an omitted end-segment). Thus it suffices to find a continuous function $f : S \rightarrow [0, 1]$ such that $f^{-1}(y)$ is uncountable for all $y \in [0, 1]$.

Suppose $k \geq 3$ is an integer. Any x in $[0, 1]$ can be written in at least one base- k expansion $x = 0.x_1x_2x_3x_4x_5x_6 \dots$. Let S be the (closed) set of all $x \in [0, 1]$ whose even-subscripted digits are all less than $k - 1$. If $x \in S$, then $f(x)$ is defined as the number whose base- k expansion is obtained by removing the even-subscripted digits of x , i.e. $f(x) = 0.x_1x_3x_5 \dots$. Clearly $f^{-1}(y)$ is uncountable (it has the cardinality of the continuum) for every $y \in [0, 1]$. We are left with showing that f is continuous on S . This is also easy but, as we shall use similar arguments several times later, let us give a little more detail this first time. Suppose $f(x) \neq f(y)$ and the first difference occurs at the m -th digit in their base- k expansion, and so $|f(x) - f(y)| \leq k^{-m+1}$. It follows that the $(2m - 1)$ -st base- k digit of x and y must differ, and the fact that $x, y \in S$ now implies that $|x - y| \geq k^{-2m}$. Thus we are done.

The function f constructed above does not have the bonus property of having zero derivative almost everywhere, but a small adjustment fixes this: one simply uses singular continuous functions to interpolate on the omitted segments rather than linear ones. For instance if (a, b) is one of the omitted intervals, define

$$f(a + t(b - a)) = f(a) + (f(b) - f(a))h(t), \quad \text{for all } 0 < t < 1,$$

where $h : [0, 1] \rightarrow [0, 1]$ is any increasing continuous function such that $h(0) = 0$, $h(1) = 1$, $h'(x) = 0$ for almost all $x \in (0, 1)$ (a basic example of such a function, due to Lebesgue, is described in [4, p.113]).

A slight variant of the above construction leads to a Peano curve. For example in two dimensions, we first define S' to be the (closed) set of all $x \in [0, 1]$ all of whose digits are less than $k - 1$. If $x \in S'$, then $f_1(x)$ and $f_2(x)$ are defined as the numbers whose base- $(k - 1)$ expansions are given by $f_1(x) = 0.x_1x_3x_5 \dots$ and $f_2(x) = 0.x_2x_4x_6 \dots$. We extend f_1 and f_2 to all of $[0, 1]$ as before. It follows readily that $F = (f_1, f_2)$ is continuous from $[0, 1]$ onto $[0, 1] \times [0, 1]$.

Our second construction uses lacunary functions. Examples of this type are easy to write down, but proving that they have

the required properties requires some effort. A typical example is $f(x) = \sum_{j=0}^{\infty} 4^{-j} \cos(100^j x)$. More generally we have the following result:

Theorem 1. Suppose $f(x) = \sum_{j=0}^{\infty} a_j \cos(b_j x)$, where $a_j, b_j > 0$, and $a_{j+1} < a_j/4$, for all $j \geq 0$. Suppose further that there exists an integer j_0 such that if $j \geq j_0$ then $a_{j+1}b_{j+1} > 6\pi a_j b_j$. Then f is continuous and takes on all values in the interior of its range uncountably frequently.

Before proving Theorem 1, let us make a few remarks. First of all, f is clearly uniformly continuous on \mathbb{R} since $a_{j+1} < a_j/4$. For the same reason, f takes on all values between $-2a_0/3$ and $2a_0/3$, so Theorem 1 asserts that f takes on uncountably many values uncountably frequently. The numbers b_{j+1}/b_j do not have to be integers, so f may not be periodic.

Proof of Theorem 1: Let us write $s_n(x) = \sum_{j=0}^n a_j \cos b_j x$, $r_j = 4a_{j+1}/3$, and $L_j = \pi/b_j$. We fix an arbitrary point c in the interior of the range of f and let n_0 be chosen so large that $n_0 \geq j_0$ and that f takes on all values in the interval $[c - 2r_{n_0}, c + 2r_{n_0}]$. We also assume n_0 is chosen so large that $\sum_{j=0}^n a_j b_j \leq 2a_n b_n$ for all $n \geq n_0$; this is possible by the geometric growth assumption on $a_j b_j$.

Given $n \geq 0$, we call an open interval I a *level- n trap* if the values of $s_n - c$ at the endpoints of I are of opposite sign and larger than r_n in absolute value. Note that a level- n trap contains roots of $f - c$ and $s_m - c$ for all $m \geq n$.

There exists a level- n_0 trap, call it (u, v) , since we can solve the equations $f(u) = c - 2r_{n_0}$ and $f(v) = c + 2r_{n_0}$. Writing $I_{n_0,1} = (u, v)$, we shall construct a nested binary tree of level- n traps $I_{n,k}$, $n \geq n_0$, $1 \leq k \leq 2^{n-n_0}$, all contained in $I_{n_0,1}$. In fact, each $I_{n,k}$ will contain the closure of two disjoint traps $I_{n+1,j}$ and the length of $I_{n,k}$ will tend to 0 as $n \rightarrow \infty$. By continuity, any nested sequence of intervals $(I_{n,k_n})_{n=n_0}^{\infty}$ extracted from this tree specifies a unique root of $f - c$ (the single point which is in the intersection of the I_{n,k_n} 's). This root cannot be at the endpoint of any of the containing traps (since they are compactly nested) and

so different sequences of intervals lead to different roots. Thus $f^{-1}(c)$ is uncountable as required.

Assume inductively that we have already defined $I = I_{n,k}$. We must prove the existence of two disjoint level- $(n+1)$ traps contained within it. Suppose $x \in I$ is a root of $s_n - c$. $|s'_n|$ is bounded by $2a_nb_n$ and so, if $d_n = a_{n+1}/(4a_nb_n)$, then s_n differs from c by less than $a_{n+1}/2$ on the interval $[x - d_n, x + d_n]$. Since I is a level- n trap, it must contain $[x - d_n, x + d_n]$.

By hypothesis, $3L_{n+1} < 2d_n$. Thus, $[x - d_n, x + d_n]$ contains subintervals of the form $[mL_n, (m+1)L_n]$ for two consecutive values of m . Since $a_{n+1} \cos(b_{n+1}x)$ takes on the values $\pm a_{n+1}$ at the endpoints of such subintervals, it follows that the interiors of these subintervals are the required level- $(n+1)$ traps, and so we are done. ■

The constants $1/4$ and 6π are only convenient values for the proof and are far from sharp. If one examines the proof one sees that the choice of the former constant affects the latter but, even if we leave $1/4$ unchanged, 6π can be replaced by, say, $18\pi/7$ if we estimate things a little more carefully. In fact, since $18\pi/7$ is a little larger than 8, we can choose n_0 so large that $|s'_n|$ is less than sa_nb_n , where $s = 1/(1 - 7/18\pi) < 8/7$. Thus if $d_n = ta_{n+1}/(sa_nb_n)$ for any $t < 2/3$, then s_n differs from c by less than ta_{n+1} on the interval $[x - d_n, x + d_n]$. Choosing t close enough to $2/3$, we have $t/s > 7/12$, and so $3L_{n+1} < 2d_n$ as before.

Incidentally, it follows from the proof of Theorem 1 that any f considered here exhibits the Weierstrass property of being nowhere differentiable. In fact, the variation of f on $I_j = [x - L_j, x + L_j]$ is at least $2a_j$, ensuring that for some $y \in I_j$,

$$|f(y) - f(x)|/|y - x| \geq a_j b_j / \pi \rightarrow \infty \quad (j \rightarrow \infty).$$

This non-differentiability result is much less sharp, however, than that of Hardy [3], who proved that $\sum a^n \cos b^n \pi x$ is a continuous nowhere-differentiable function whenever $0 < a < 1$, $ab \geq 1$. This suggests the following question:

Question 2. *Do all continuous nowhere-differentiable lacunary series take on uncountably many values uncountably frequently?*

I have no answer to this question, but I would be rather surprised if it were true; perhaps more likely to be true is the conjecture that $\sum a^n \cos b^n \pi x$ takes on a whole interval of values uncountably frequently whenever $0 < a < 1$, $ab > 1$ (since here we have some "room to manoeuvre").

Before going on to our next question, let us introduce some terminology that we need here and later. Given $0 < t < 1$, we say $f : [0, 1] \rightarrow \mathbb{R}^n$ is *t-Hölder continuous* if, for some $C > 0$,

$$|f(x) - f(y)| \leq C|x - y|^t, \quad \forall 0 \leq x, y \leq 1. \quad (1)$$

In the case $t = 1$, we instead say that f is *Lipschitz (continuous)*, or *C-Lipschitz* if we wish to specify the constant.

Question 3. *Do there exist Lipschitz functions $f : [0, 1] \rightarrow \mathbb{R}$ which take on each of uncountably many values uncountably often?*

The answer to Question 3 is again "yes," although examples like the previous ones fail because $f^{-1}(x)$ must be finite almost everywhere (see Theorem 2 below). Instead we first define f on $S \subset [0, 1]$, the closed set of numbers whose decimal expansion can be written using only the digits 0, 2, 7, and 9. For these numbers, the decimal expansion of $f(x)$ is calculated from that of x by changing all 2's to 0's, and all 7's to 9's (and so $f^{-1}(x)$ is uncountable for every x whose decimal expansion involves only 0's and 9's). We define f at all other values by linear interpolation. We leave to the reader the task of verifying that the resulting function f satisfies the Lipschitz condition $|f(y) - f(x)|/|y - x| \leq 2$ on S (and hence on $[0, 1]$).

For any exponent $t < 1$, one can construct a *t-Hölder continuous* $f : [0, 1] \rightarrow \mathbb{R}$ which takes on all values in an interval uncountably frequently. Our first digit-coding example f is an example for $t = 1/2$. This construction is easily modified to handle any $t < 1$. First let S be the set of $x \in [0, 1]$ for which the base- k expansion has no digit equal to $k - 1$ in any position whose subscript is divisible by a fixed integer $m > 1$. We define f on S by deleting all digits whose subscript is divisible by m , and extend f using linear interpolation. Then f is *t-Hölder continuous* for $t = (m - 1)/m$, and $f^{-1}(x)$ is uncountable for all $x \in [0, 1]$.

The following theorem shows how different things are for Lipschitz functions. This result is a special case¹ of a more general result concerning Lipschitz maps between metric spaces (see [2, Corollary 2.10.11]), but we give a short proof here for completeness.

Proposition 2. *If $f : [0, 1] \rightarrow \mathbf{R}$ is C -Lipschitz and $N : \mathbf{R} \rightarrow [0, \infty]$ is the cardinality of $f^{-1}(x)$, then $\int_{\mathbf{R}} N(x) dx \leq C$. Consequently, $N(x)$ is finite almost everywhere.*

Proof: For all $j > 0$, let Δ_j be the collection of dyadic intervals of the form $[(k-1)2^{-j}, k2^{-j}]$, for $1 \leq k < 2^j$, and $[1-2^{-j}, 1]$. Note that $(\Delta_j)_{j=1}^{\infty}$ is a nested sequence of partitions of $[0, 1]$. Let $N_j(x)$ be the number of intervals $f(I)$, $I \in \Delta_j$, which contain x . Using the properties of Δ_j , we see that for each $x \in \mathbf{R}$, $N_j(x)$ is an increasing function of j which tends to $N(x)$ as $j \rightarrow \infty$. Furthermore, it is clear that

$$\int_0^1 N_j(x) dx = \sum_{I \in \Delta_j} |f(I)| \leq \sum_{I \in \Delta_j} C|I| = C,$$

where $|I|$ and $|f(I)|$ denote the lengths of the intervals I and $f(I)$. An appeal to Lebesgue's Monotone Convergence Theorem finishes the proof. ■

Question 4. *Does there exist a function f from $[0, 1]$ onto $U \equiv [0, 1] \times [0, 1]$ which is t -Hölder continuous for some $t \geq 1/2$?*

Question 4, like Question 3, is inspired by a shortcoming in the earlier examples: our base- k Peano curve F is t -Hölder continuous for $t = (\log(k-1))/(2 \log k)$, thus providing examples for all $t < 1/2$. The following theorem answers Question 4.

Theorem 3. *There exist Peano curves $F : [0, 1] \rightarrow U$ which are t -Hölder continuous for $t = 1/2$, but no such curve is t -Hölder continuous for $t > 1/2$.*

Proof: We first examine the case $t > 1/2$. We define the *Minkowski dimension* of a compact subset E of \mathbf{R}^n by

$$\mathcal{M}\text{-dim } E = \sup\{s \geq 0 : \limsup_{r \rightarrow 0} H_s(E, r) = \infty\},$$

¹ I would like to thank P. Hajlasz for pointing this out to me.

where $H_s(E, r)$ is the α -dimensional Minkowski precontent, defined as kr^s , where k is the minimum number of balls of radius r required to cover E . These concepts, and the related concept of Hausdorff dimension, are discussed at greater length in [5] and [1]. We shall need only the easily proven fact that any compact $E \subset \mathbf{R}^n$ of positive measure has Minkowski dimension n . Also noteworthy, although not needed by us, is the obvious fact that the Minkowski dimension of a set is greater than or equal to its Hausdorff dimension.

Suppose that $F : [0, 1] \rightarrow U$ satisfies (1) for some $t > 1/2$. We claim that the Minkowski dimension of $F([0, 1])$ is at most $1/t$ (and hence the range of F cannot be all of U). To see this note that the image of any interval $[i/k, (i+1)/k]$ is contained in a ball of radius C/k^t about $f(i/k)$. Thus $H_{1/t}(F([0, 1]), C/k^t) \leq C^{1/t}$, and our claim follows easily.

We next construct the required $1/2$ -Hölder continuous Peano curve. The base-3 example I shall give is the same as Peano's original example of a space-filling curve² [6]. The basic idea is simple: we can "almost" get the solution by "chopping" x into its base- k digits, allocating them one at a time to be the next base- k digit of either $f_1(x)$ or $f_2(x)$. This certainly gives a space-filling function but it is not $1/2$ -Hölder continuous (or even continuous) because of the following phenomenon: if $y = 0.y_1y_2 \dots y_n \dots$ in base- k , where $y_m \neq 0$ and $y_n = 0$ for all $n > m$, and if m is odd (even) then the left- and right-hand limits for f_2 (respectively f_1) at y are different. The way out of this problem is fairly clear: we allocate digits one at a time to $f_1(x)$ and $f_2(x)$ but introduce a parity effect to compensate for these discontinuities. We describe this process for base 3 where it is most easily described.

To avoid problems caused by non-unique expansions, we define functions $A : [0, 1] \rightarrow S$ and $B : S \rightarrow [0, 1]$, where S is the set of infinite sequences whose terms are restricted to the set $\{0, 1, 2\}$. A maps numbers to (one of) their base-3 expansions, and B maps (x_1, x_2, x_3, \dots) to the number with base-3 expansion

² I would like to thank the editor for sending me a copy of this paper.

$0.x_1x_2x_3\cdots$. We shall write values of these functions in the form Ay and Bx . Whenever $x \in S$, we denote its i -th term by x_i .

Let $G = (g_1, g_2) : S \rightarrow S \times S$ be defined by $G(x) = (u, v)$ where

$$u_k = \begin{cases} x_{2k-1}, & \text{if } \sum_{i=1}^{k-1} x_{2i} \text{ is even} \\ 2 - x_{2k-1}, & \text{if } \sum_{i=1}^{k-1} x_{2i} \text{ is odd,} \end{cases}$$

$$v_k = \begin{cases} x_{2k}, & \text{if } \sum_{i=1}^k x_{2i-1} \text{ is even} \\ 2 - x_{2k}, & \text{if } \sum_{i=1}^k x_{2i-1} \text{ is odd.} \end{cases}$$

We now define $F(y) = (Bg_1(Ay), Bg_2(Ay))$ whenever $y \in [0, 1]$.

Clearly F has range U . We are left with showing that F is $1/2$ -Hölder continuous. A simple case-by-case argument reveals that F is independent of the choice of A (for example, $G(0, 2, 2, 2, \dots) = G(1, 0, 0, 0, \dots)$). Whenever $x \in S$, $G(x) = (u, v)$, let us call $u_1, v_1, u_2, v_2, u_3, v_3, \dots$ the *standard order* of the terms of $G(x)$.

Suppose $x, y \in S$ and $Bx < By$. Let us assume that the first term of $G(x)$ which differs from the corresponding digit of $G(y)$, using the standard order, is the m -th term of the $g_2(y)$ (if the first difference is in $g_1(y)$, a similar argument applies). Then $|F(Bx) - F(By)| \leq 3^{-m+1}$. If $|Bx - By| \geq 3^{-2m}$, we are done, so we may assume $|Bx - By| < 3^{-2m}$. But then, if there is some $0 < j < 2m$ such that $x_i = y_i$ if $i < j$ and $x_j \neq y_j$, we must have $x_j + 1 = y_j$ and, whenever $j < i \leq 2m$, $x_i = 2$ and $y_i = 0$. This forces the m -th digit of the second coordinates of $F(Bx)$ and $F(By)$ to be equal, contrary to assumption. The only remaining possibility is that $x_i = y_i$ if $i < 2m$ and $y_{2m} = x_{2m} + 1$. In this case, $Bx \leq Bz \leq By$, where $z_i = y_i$ for $i \leq 2m$ and $z_i = 0$ if $i > 2m$. If $z \neq y$ and the j -th term is the first term where they differ, then it is clear that

$$By - Bx \geq By - Bz \geq 3^{-j}, \quad |F(y) - F(z)| \leq 3^{1-j/2}. \quad (2)$$

Next let z' be the sequence defined by $z_i = x_i$ for $i \leq 2m$ and $z_i = 2$ if $i > 2m$, so that $Bz = Bz'$. If $z' \neq x$ and the k -th term is the first term where they differ, then it is again clear that

$$By - Bx \geq Bz - Bx \geq 3^{-k}, \quad |F(z) - F(x)| \leq 3^{1-k/2}. \quad (3)$$

Putting (2) and (3) together, we get the desired Hölder continuity. ■

Our previous argument actually implies that there are no Peano curves $f : [0, 1] \rightarrow [0, 1]^n$, $n \geq 2$, which are t -Hölder continuous for $t > 1/n$. The construction for $t = 1/2$ also generalizes to give an n -dimensional Peano curve which is $1/n$ -Hölder continuous in the higher dimensional setting: again using a base-3 expansion, we “deal out” the digits one at a time to each of the n coordinates, replacing each “0” by “2” and vice versa whenever the sum of the digits previously dealt to the other coordinates is odd. We leave the verification of $1/n$ -Hölder continuity to the reader.

Question 5. Does there exist a map G from the unit square $U = [0, 1] \times [0, 1]$ to U such that the image of any non-trivial line segment in U has non-empty interior?

We give a couple of methods for constructing such a map G . The map $A(x, y) = F(x)$, where $F = (f, g)$ is the Peano curve defined earlier, has this property on all non-vertical lines. Defining $B : U \rightarrow U$ by $B(x, y) = ((x + y^2)/2, y)$, $G = A \circ B$ has the desired property (since if $L \subset U$ is a non-trivial line segment, the x -projection of $B(L)$ is also a non-trivial line segment).

One might feel that the previous method is not completely satisfactory since we have simply “hidden” the straight lines. Our second method, has the advantage that it produces a function G for which the image of $G \circ \gamma$ has non-empty interior whenever γ is a non-trivial C^1 path in U . First let $F_k = (f_k, g_k)$ to be our old base- k Peano curve F . F_k is t -Hölder continuous for $t = (\log(k-1))/(2 \log k)$ but not for any larger t ; in fact, it is easily seen that for any n , the image of any interval of length $1/k^{2n}$ under F_k is contained in a square of length $(k-1)^{-n+1}$ and contains a square of length $(k-1)^{-n-1}$.

We claim that $G(x, y) = F_i(x) + F_j(y)$ is a function of the type we require for any $3 \leq i < j$. We shall content ourselves here with sketching the proof. Clearly images of vertical line segments have non-empty interior. If γ is not a vertical line, then we need only look in the vicinity of a single point (x_0, y_0) on γ where the tangent line is non-vertical. In this case, one expects everything to work

out since on any sufficiently small neighbourhood of γ (dependent on the slope of the tangent line), the variation in F_i is much larger than the variation in F_j . To make this idea rigorous, assume $G(x_0, y_0) = (u_0, v_0)$. We solve the equation $G(x, y) = (u, v)$ for all (u, v) sufficiently near (u_0, v_0) by an iterative method. Having found the approximate solution (x_k, y_k) , we find $(x_{k+1}, y_{k+1}) \in I$, a nearby point on the curve for which $F_i(x_{k+1}) + F_j(y_k) = (u, v)$. With this hint, we leave the details to the reader.

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BERNSTEIN'S POLYNOMIAL INEQUALITIES AND FUNCTIONAL ANALYSIS

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1. Introduction

This expository article shows how classical inequalities for the derivative of polynomials can be proved in real and complex Hilbert spaces using only elementary arguments from functional analysis. As we shall see, there is a surprising interconnection between an equality of norms for symmetric multilinear mappings due to Banach and an inequality for the derivative of trigonometric polynomials due to van der Corput and Schaake. We encounter little extra difficulty in establishing our inequalities in several or infinite dimensions.

After giving the definitions of polynomials and derivatives in normed linear spaces, we establish a lemma of Hörmander, which is an extension of a theorem of Laguerre to complex vector spaces. This powerful lemma is the key to the proofs of the polynomial inequalities we discuss; however, its proof is a simple argument relying only on the fundamental theorem of algebra. Following de Bruijn (who considered only the case of the complex plane), we deduce a theorem which obtains discs inside the range of a complex-valued polynomial on the closed unit ball of a complex Hilbert space. Here the size of the disc is determined by the value of the derivative.

An easy consequence is an extension to complex Hilbert spaces of an estimate of Malik on the derivative of polynomials whose roots lie outside a given disc. (Malik's estimate generalized a conjecture of Erdős that was proved by Lax.) Another consequence is an extension to complex Hilbert spaces of the classical complex form of Bernstein's inequality. Still another consequence

is an inequality for the derivative of a polynomial on a complex Hilbert space whose real part has a known bound on the closed unit ball. When the Hilbert space is the complex plane, this inequality contains an inequality of Szegő and leads to an inequality of van der Corput and Schaake for trigonometric polynomials, which is a strengthened form of the Bernstein inequality.

Using methods of van der Corput and Schaake, we deduce an inequality for the derivative of homogeneous polynomials on real Hilbert spaces that extends a result of O. D. Kellogg for \mathbf{R}^n . A slight extension of a result of Banach is an immediate consequence. Specifically, the norm of a continuous symmetric multilinear mapping is the same as the norm of the associated homogeneous polynomial on any real Hilbert space. From this, we deduce an estimate on the derivative of polynomials which satisfy an ℓ^2 -growth condition on real Hilbert spaces. Finally, we give an argument which shows how to derive the inequality of van der Corput and Schaake for trigonometric polynomials from the two dimensional case of Banach's result.

A source for this approach to polynomial inequalities is [8].

2. Definitions and notation

The reader who wishes may take all vector spaces below to be finite dimensional so that the definition of polynomials is already familiar. To give the general definition, let X and Y be any real or complex normed linear spaces and let

$$F : X \times \cdots \times X \rightarrow Y$$

be a continuous symmetric m -linear mapping with respect to the chosen scalar field, where m is a positive integer. Define

$$\hat{F}(x) = F(x, \dots, x)$$

for $x \in X$. We say that a mapping $P : X \rightarrow Y$ is a *homogeneous polynomial of degree m* if $P = \hat{F}$ for some continuous symmetric m -linear mapping F as above. Define a mapping $P : X \rightarrow Y$ to be a *polynomial of degree $\leq m$* if

$$P = P_0 + P_1 + \cdots + P_m,$$

where $P_k : X \rightarrow Y$ is a homogeneous polynomial of degree k for $k = 1, \dots, m$ and a constant function when $k = 0$. (Note that a constant polynomial is not a homogeneous polynomial by our definition unless it is the zero polynomial.)

This definition of polynomials agrees with the classical definition when $X = \mathbf{R}^n$ and $Y = \mathbf{R}$ and when $X = \mathbf{C}^n$ and $Y = \mathbf{C}$. In either case,

$$P(x_1, \dots, x_n) = \sum_{k=0}^m \sum_{k_1 + \cdots + k_n = k} a_{k_1 \dots k_n} x_1^{k_1} \cdots x_n^{k_n},$$

where k_1, \dots, k_n are restricted to the non-negative integers and the coefficients $a_{k_1 \dots k_n}$ are in the appropriate scalar field, i.e., Y . As expected, with our definitions, if a polynomial P satisfies $P(tx) = t^m P(x)$ for all $x \in X$ and $t \in \mathbf{R}$, then P is a homogeneous polynomial of degree m . When F is as above, for convenience we will write $F(x^j y^k)$ for $F(\underbrace{x, \dots, x}_j, \underbrace{y, \dots, y}_k)$. Thus, the binomial

theorem for F can be written as

$$\hat{F}(x + y) = \sum_{k=0}^m \binom{m}{k} F(x^{m-k} y^k). \quad (1)$$

It is not difficult to show, [10, §26.2], that a weaker definition suffices. Specifically, a continuous mapping $P : X \rightarrow Y$ is a polynomial of degree $\leq m$ if and only if

$$Q(\lambda) = \ell(P(x + \lambda y)), \lambda \text{ scalar}$$

is a polynomial of degree $\leq m$ (in the classical sense) for every $x, y \in X$ and every $\ell \in Y^*$, where Y^* denotes the space of all continuous linear functionals on Y .

Let $\mathcal{L}(X, Y)$ denote the space of all continuous linear mappings $L : X \rightarrow Y$ with the operator norm, i.e.

$$\|L\| = \sup_{\|x\| \leq 1} \|Lx\|.$$

If P is a mapping of a domain \mathcal{D} in X into Y and if $x \in \mathcal{D}$, we say that an $L \in \mathcal{L}(X, Y)$ is the Fréchet derivative of P at x if

$$\lim_{y \rightarrow 0} \frac{\|P(x+y) - P(x) - L(y)\|}{\|y\|} = 0.$$

We denote the Fréchet derivative of P at x by $DP(x)$. Clearly

$$DP(x)y = \left. \frac{d}{dt} P(x+ty) \right|_{t=0} \quad (2)$$

when $DP(x)$ exists. If P is a polynomial of degree $\leq m$, then $DP(x)$ exists for all $x \in X$ and $x \rightarrow DP(x)$ is a polynomial mapping of X into $\mathcal{L}(X, Y)$ of degree $\leq m-1$. Indeed, it suffices to show this for homogeneous polynomials \hat{F} of degree m and here the Fréchet differentiability of \hat{F} follows easily from (1) with

$$D\hat{F}(x)y = mF(x^{m-1}y). \quad (3)$$

For example, if $P(x) = (P_1(x), \dots, P_m(x))$ is a polynomial mapping of \mathbf{R}^n into \mathbf{R}^m , then the matrix of $DP(x)$ is the $m \times n$ Jacobian matrix $[\partial P_i(x)/\partial x_j]$. The same formula also holds when \mathbf{R} is replaced by \mathbf{C} except that $\partial P_i(x)/\partial x_j$ now denotes a complex derivative. (The proof in both cases follows easily from (2) and the chain rule.) See [10] and [7] for further discussion of polynomials and Fréchet differentiability.

3. Polynomials on complex spaces

The lemma below is the key to the proofs of the polynomial inequalities we wish to give. Let X be a complex normed linear space. Recall that a function $f : X \times X \rightarrow \mathbf{C}$ is called a *Hermitian form* on X if $f(x, y)$ is linear in x for each $y \in X$ and $\overline{f(x, y)} = f(y, x)$ for all $x, y \in X$. For example, when $X = \mathbf{C}^n$, the Hermitian forms f on X are given by

$$f(x, y) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i \overline{y_j},$$

where $\overline{a_{ij}} = a_{ji}$ for all $1 \leq i, j \leq n$.

Lemma 1. (Hörmander [11].) Put

$$A = \{x \in X : f(x, x) \geq 0, x \neq 0\}.$$

If $P : X \rightarrow \mathbf{C}$ is a (non-constant) homogeneous polynomial with $P(x) \neq 0$ for all $x \in A$, then $DP(x)y \neq 0$ for all $x, y \in A$.

Proof: By definition, $P = \hat{F}$ for some continuous symmetric m -linear mapping F on X . If the lemma is false, there exist $x, y \in A$ with $DP(y)x = 0$, so $F(y^{m-1}x) = 0$ by (3). Then by the binomial theorem (1), the coefficient of λ^{m-1} in the polynomial $\lambda \rightarrow P(x + \lambda y)$ is 0. By the fundamental theorem of algebra,

$$P(x + \lambda y) = c(\lambda - \lambda_1) \cdots (\lambda - \lambda_m),$$

where $c \neq 0$, and hence $\sum_{k=1}^m \lambda_k = 0$. None of the roots λ_k is 0 since $P(x) \neq 0$. Then $x + \lambda_k y \neq 0$ since otherwise $y = \alpha x$, where $\alpha = -1/\lambda_k$, and this gives

$$P(y) = F(y^{m-1}(\alpha x)) = \alpha F(y^{m-1}x) = 0.$$

Hence by hypothesis, $f(x + \lambda_k y, x + \lambda_k y) < 0$ since $P(x + \lambda_k y) = 0$. This inequality expands to

$$f(x, x) + 2 \operatorname{Re} \lambda_k f(y, x) + |\lambda_k|^2 f(y, y) < 0,$$

so $\operatorname{Re} \lambda_k f(y, x) < 0$. Therefore,

$$0 = \operatorname{Re} \left[\left(\sum_{k=1}^m \lambda_k \right) f(y, x) \right] = \sum_{k=1}^m \operatorname{Re} \lambda_k f(y, x) < 0,$$

the desired contradiction. ■

Note that the above lemma holds for any complex vector space and without any continuity assumptions if formula (3) is taken as a definition. See [15] for further discussion of Hörmander's results and related references.

We now apply the above lemma to obtain an extension of a theorem of de Bruijn, [5], who considered the case $X = \mathbf{C}$ and deduced the Erdős-Lax theorem, [13]. We carry his argument further to obtain an extension (Corollary 4 below) of Malik's generalization, [14], of the Erdős-Lax theorem. We also deduce an extension (Corollary 3 below) of an inequality of Szegő, [18], which we apply to trigonometric polynomials in the next section.

Theorem 2. Let X be a complex Hilbert space and let $P : X \rightarrow \mathbb{C}$ be a polynomial of degree $\leq m$. Define

$$S(x) = mP(x) - DP(x)x$$

for $x \in X$ and let

$$X_1 = \{x \in X : \|x\| \leq 1\}.$$

Then

$$DP(x)y + S(x) \in mP(X_1)$$

for all $x, y \in X_1$.

Corollary 3. If $|\operatorname{Re} P(x)| \leq 1$ for all $x \in X_1$, then

$$|DP(x)y| + |\operatorname{Re} S(x)| \leq m$$

for all $x, y \in X_1$.

Corollary 4. Suppose that $r \geq 1$. If $|P(x)| \leq 1$ for all $x \in X_1$ and if P has no zeros in the closed ball in X about 0 with radius r , then

$$\|DP(x)\| \leq \frac{m}{1+r}$$

for all $x \in X_1$.

Proofs: Our approach to the proof of Theorem 2 is to add an additional dimension to X and use the extra variable to make P into a homogeneous polynomial. Let $X' = X \times \mathbb{C}$ and write the elements of X' as ordered pairs $\langle x, \lambda \rangle$. Define a Hermitian form f on X by

$$f(\langle x, \lambda \rangle, \langle y, \mu \rangle) = \lambda \bar{\mu} - (x, y)$$

and note that

$$A = \{\langle x, \lambda \rangle \in X' : \|x\| \leq |\lambda|, \lambda \neq 0\}.$$

Suppose $\alpha \in \mathbb{C}$ with $\alpha \notin P(X_1)$. Define

$$Q(\langle x, \lambda \rangle) = \lambda^m [\alpha - P(x/\lambda)]$$

for $\lambda \neq 0$ and note that Q extends to all of X' . Then $Q : X' \rightarrow \mathbb{C}$ is a homogeneous polynomial of degree m (by the equivalent weaker definition mentioned in the introduction) and $Q(\langle x, \lambda \rangle) \neq 0$ for all $\langle x, \lambda \rangle \in A$. Hence by Lemma 1, $DQ(\langle x, 1 \rangle)(y, 1) \neq 0$ for all $x, y \in X_1$. Now by (2) and the rules of differentiation,

$$\begin{aligned} DQ(\langle x, 1 \rangle)(y, 1) &= \frac{d}{dt} Q(\langle x + ty, 1 + t \rangle) \Big|_{t=0} \\ &= \frac{d}{dt} (1+t)^m \left[\alpha - P\left(\frac{x+ty}{1+t}\right) \right] \Big|_{t=0} \\ &= m\alpha - [DP(x)y + S(x)]. \end{aligned}$$

Thus $DP(x)y + S(x) \neq m\alpha$, which proves Theorem 2.

To deduce Corollary 3, observe that

$$|\operatorname{Re} [DP(x)y + S(x)]| \leq m$$

for all $x, y \in X_1$ by Theorem 2. Here y can be replaced by λy where λ is a complex number with $|\lambda| = 1$ and λ can be chosen so that the left-hand side of the inequality above is the required expression.

To prove Corollary 4, note that by Theorem 2, for each $x, y \in X_1$, the closed (possibly degenerate) disc with center $S(x)$ and radius $|DP(x)y|$ is contained in the closed disc about 0 with radius m . Hence

$$|DP(x)y| + |S(x)| \leq m. \quad (4)$$

Define $P_r(x) = P(rx)$ and put $S_r(x) = mP_r(x) - DP_r(x)x$. By hypothesis and Theorem 2, for each $x, y \in X_1$, the closed disc with center $S_r(x)$ and radius $|DP_r(x)y|$ does not contain 0 so $|DP_r(x)y| \leq |S_r(x)|$. Since $DP_r(x)y = rDP(rx)y$, $S_r(x) = S(rx)$ and $r \geq 1$, it follows that

$$r|DP(x)y| \leq |S(x)| \quad (5)$$

for all $x, y \in X_1$. Combining (4) and (5), we have

$$(1+r)|DP(x)y| \leq m,$$

which gives Corollary 4. ■

The extended Erdős-Lax theorem is the case $r = 1$ of Corollary 4. Note that this follows immediately from Theorem 2 since for $x, y \in X_1$, the closed disc Δ with center $S(x)$ and radius $|DP(x)y|$ is contained in the closed disc about 0 with radius m but Δ does not contain the point 0. The largest possible diameter of Δ is m and hence $|DP(x)y| \leq m/2$.

To state our extensions of Bernstein's theorem, define the norm of a polynomial $P : X \rightarrow Y$ by

$$\|P\| = \sup\{\|P(x)\| : x \in X_1\} \quad (6)$$

and define the norm of a continuous m -linear mapping

$$F : X \times \cdots \times X \rightarrow Y$$

by

$$\|F\| = \sup\{\|F(x_1, \dots, x_m)\| : x_1, \dots, x_m \in X_1\}.$$

Obviously, $\|\hat{F}\| \leq \|F\|$. (If X is a complex normed linear space, by the maximum principle, [10, Th. 3.18.4], the value of $\|P\|$ does not change when the supremum in (6) is taken over only the unit vectors in X .) Suppose Y is any complex normed linear space.

Theorem 5. *If X is a complex Hilbert space and if $P : X \rightarrow Y$ is a polynomial of degree $\leq m$, then $\|DP\| \leq m\|P\|$.*

Corollary 6. *If X is a complex Hilbert space and if $F : X \times \cdots \times X \rightarrow Y$ is a continuous symmetric m -linear mapping, then $\|F\| = \|\hat{F}\|$.*

The corollary above will be generalized later (Theorem 9) to real Hilbert spaces. See [9] for inequalities between $\|F\|$ and $\|\hat{F}\|$ for other spaces.

Proofs: Without loss of generality, we may assume that $\|P\| = 1$. We first apply linear functionals to reduce to the case where $Y = \mathbb{C}$. Specifically, let $\ell \in Y^*$ with $\|\ell\| = 1$ and define $Q(x) = \ell(P(x))$ for $x \in X$. Then $Q : X \rightarrow \mathbb{C}$ is a polynomial of degree $\leq m$ satisfying $|Q(x)| \leq 1$ for all $x \in X_1$ and $DQ(x)y = \ell(DP(x)y)$.

Let $x, y \in X_1$. Then $|\ell(DP(x)y)| \leq m$ by Corollary 3 and by the Hahn-Banach theorem, [10, Th. 2.7.4], we may choose ℓ so that $\ell(DP(x)y) = \|DP(x)y\|$. Hence $\|DP(x)y\| \leq m$ for all $x, y \in X_1$ and Theorem 5 follows.

We deduce Corollary 6 from the theorem above by induction. The equality is obviously true when $m = 1$. Suppose it is true for $m - 1$. Then holding x_m fixed, we have that

$$\|F(x_1, \dots, x_m)\| \leq \sup\{\|F(x^{m-1}x_m)\| : x \in X_1\}$$

for all $x_1, \dots, x_{m-1} \in X_1$. Since $D\hat{F}(x)x_m = mF(x^{m-1}x_m)$, by Theorem 5, $\|F(x^{m-1}x_m)\| \leq \|\hat{F}\|$ for all $x, x_m \in X_1$. Therefore, $\|F\| \leq \|\hat{F}\|$, as required. ■

4. Polynomials on real spaces

The results of this section depend on an inequality for trigonometric polynomials which we will deduce from our previous results. By definition, a *trigonometric polynomial* $T(\theta)$ of degree $\leq m$ is given by

$$T(\theta) = \sum_{k=0}^m (a_k \cos k\theta + b_k \sin k\theta), \quad (7)$$

where the coefficients a_0, \dots, a_m and b_0, \dots, b_m are complex numbers. If all the coefficients are real numbers, we say that $T(\theta)$ is a *real trigonometric polynomial*. It is not difficult to show using the addition formulae for the sine and cosine functions that the product of a trigonometric polynomial of degree $\leq m$ with a trigonometric polynomial of degree $\leq n$ is a trigonometric polynomial of degree $\leq m + n$. Hence any sum

$$\sum_{j+k \leq m} c_{jk} \cos^j \theta \sin^k \theta, \quad (8)$$

where j and k are non-negative integers and each c_{jk} is a real number, is a real trigonometric polynomial of degree $\leq m$.

Theorem 7. (van der Corput and Schaake [6].) *If $T(\theta)$ is a real trigonometric polynomial of degree $\leq m$ satisfying $|T(\theta)| \leq 1$ for all real θ , then*

$$T'(\theta)^2 + m^2 T(\theta)^2 \leq m^2 \quad (9)$$

for all real θ .

Corollary 8. (Bernstein [2, p.39].) *If $T(\theta)$ is a trigonometric polynomial of degree $\leq m$ satisfying $|T(\theta)| \leq 1$ for all real θ , then $|T'(\theta)| \leq m$ for all real θ .*

Note that (9) holds with equality for all real θ when $T(\theta) = \cos m\theta$ and when $T(\theta) = \sin m\theta$. Bernstein's original statement of Corollary 8 had the bound of $2m$ in place of m . (See [17, p.569] for a discussion of priorities.)

Proofs: Our method of proof is to express $T(\theta)$ as the real part of a polynomial on the unit circle and apply Corollary 3 in the case $X = \mathbb{C}$. Let T be given by (7) and define the conjugate \bar{T} of T by

$$\bar{T}(\theta) = \sum_{k=0}^m (-b_k \cos k\theta + a_k \sin k\theta).$$

Define a polynomial $P : \mathbb{C} \rightarrow \mathbb{C}$ by

$$P(z) = \sum_{k=0}^m c_k z^k,$$

where $c_k = a_k - ib_k$ for $k = 0, \dots, m$. Then

$$P(e^{i\theta}) = T(\theta) + i\bar{T}(\theta) \quad (10)$$

for all real θ . By hypothesis and the maximum principle for harmonic functions, $|\operatorname{Re} P(z)| \leq 1$ for all $|z| \leq 1$ and hence $|P'(z)| + |\operatorname{Re} S(z)| \leq m$ for all $|z| \leq 1$ by Corollary 3.

Let $z = e^{i\theta}$. Differentiating (10) with respect to θ , we see that $izP'(z) = T'(\theta) + i\bar{T}'(\theta)$ and hence $\operatorname{Re} S(z) = mT(\theta) - \bar{T}'(\theta)$.

Now if t_1 and t_2 are any real numbers with $t_1^2 + t_2^2 = 1$, then by the Cauchy-Schwarz inequality,

$$\begin{aligned} |T'(\theta)t_1 + mT(\theta)t_2| &= |T'(\theta)t_1 + \bar{T}'(\theta)t_2 + \operatorname{Re} S(z)t_2| \\ &\leq |T'(\theta)t_1 + \bar{T}'(\theta)t_2| + |\operatorname{Re} S(z)| \\ &\leq \sqrt{T'(\theta)^2 + \bar{T}'(\theta)^2} + |\operatorname{Re} S(z)| \\ &= |P'(z)| + |\operatorname{Re} S(z)| \leq m. \end{aligned}$$

The maximum of the left-hand side of the above is

$$r = \sqrt{T'(\theta)^2 + m^2 T(\theta)^2}$$

and it is attained when $t_1 = T'(\theta)/r$ and $t_2 = mT(\theta)/r$ if $r \neq 0$. Thus (9) holds.

One can deduce Corollary 8 easily by letting λ be a complex number with $|\lambda| = 1$ and applying Theorem 7 to $S(\theta) = \operatorname{Re}[\lambda T(\theta)]$. ■

Let Y be any real normed linear space.

Theorem 9. (Banach [1].) *If X is a real Hilbert space and if $F : X \times \dots \times X \rightarrow Y$ is a continuous symmetric m -linear mapping, then $\|F\| = \|\hat{F}\|$.*

Lemma 10. *If $P : X \rightarrow Y$ is a homogeneous polynomial of degree m , then $\|DP\| \leq m\|P\|$.*

Note that Lemma 10 is an analogue of Theorem 5 for the case of real scalars. It was proved for the case $X = \mathbb{R}^n$ by Kellogg, [12]. See [3, p.62] for a direct proof of Theorem 9 using only Hilbert space techniques.

Proofs: To prove the lemma, we may suppose that $\|P\| = 1$. As in the proof of Theorem 5, we may apply linear functionals to reduce to the case $Y = \mathbb{R}$. Let x and y be unit vectors in X and let $\{x, w\}$ be an orthonormal basis for the space spanned by x and y . Then $y = t_1 x + t_2 w$, where $t_1^2 + t_2^2 = 1$. Define

$$T(\theta) = P((\cos \theta)x + (\sin \theta)w)$$

and note that $T(\theta)$ is a real trigonometric polynomial of degree $\leq m$ since it is of the form (8) by the binomial theorem (1). Clearly $T(0) = P(x)$ and $T'(0) = DP(x)w$. By (3), we have $DP(x)x = mP(x)$, and hence

$$DP(x)y = t_1 DP(x)x + t_2 DP(x)w = t_1 mT(0) + t_2 T'(0).$$

Since $\|P\| = 1$, it follows that $|T(\theta)| \leq 1$ for all real θ and hence $|DP(x)y| \leq m$ by the Cauchy-Schwarz inequality and (9). In fact, this inequality holds for all $x, y \in X_1$ since these vectors can be written as scalar multiples of unit vectors.

Theorem 9 follows easily from the lemma above by induction as in the proof of Corollary 6. ■

The case $X = \mathbf{R}$ of our next theorem is a sharpening given in [6] of a theorem of Bernstein.

Theorem 11. *If X is a real Hilbert space and if $P : X \rightarrow \mathbf{R}$ is a polynomial of degree $\leq m$ satisfying*

$$|P(x)|^2 \leq (1 + \|x\|^2)^m$$

for all $x \in X$, then

$$\|DP(x)\|^2 + S(x)^2 \leq m^2(1 + \|x\|^2)^{m-1}$$

for all $x \in X$.

Proof: Our approach is similar to that of the proof of Theorem 2. Let $X' = X \times \mathbf{R}$ and note that X' is a real Hilbert space in the norm $\|\langle x, t \rangle\| = (\|x\|^2 + t^2)^{1/2}$. Define a homogeneous polynomial $Q : X' \rightarrow \mathbf{R}$ of degree m by $Q(\langle x, t \rangle) = t^m P(x/t)$ for $t \neq 0$. By hypothesis,

$$|Q(\langle x, t \rangle)|^2 \leq |t|^{2m} \left(1 + \left\|\frac{x}{t}\right\|^2\right)^m = \|\langle x, t \rangle\|^{2m}$$

so $\|Q\| \leq 1$. By a differentiation as in the proof of Theorem 2,

$$DQ(\langle x, 1 \rangle)\langle y, t \rangle = DP(x)y + tS(x).$$

Hence by Lemma 10,

$$|DP(x)y + tS(x)| \leq m\|\langle x, 1 \rangle\|^{m-1}\|\langle y, t \rangle\|.$$

By replacing y by ry in the above, where $y \in X_1$, and maximizing the left-hand side over all r and t satisfying $r^2 + t^2 = 1$, we obtain

$$\sqrt{[DP(x)y]^2 + S(x)^2} \leq m(1 + \|x\|^2)^{\frac{m-1}{2}}$$

for all $y \in X_1$, and Theorem 11 follows. ■

We will show that Theorem 7 can be derived from the preceding theorem using only the case $X = \mathbf{R}$. Thus any of the results of this section can be derived from any of the others (except Corollary 8) by arguments given here.

Suppose $T(\theta)$ is a real trigonometric polynomial of degree $\leq m$ satisfying $|T(\theta)| \leq 1$ for all real θ . It suffices to prove (9) for the case $\theta = 0$ since this case can be applied to the trigonometric polynomial $S(\phi) = T(\theta + \phi)$ for fixed θ . Let P be the polynomial defined in the proof of Theorem 7 and define

$$Q(t) = (1 + t^2)^m \operatorname{Re} P(z(t)),$$

where $z(t) = (1 + it)^2 / (1 + t^2)$. Then $Q(t)$ is a polynomial of degree $\leq 2m$ on \mathbf{R} . If $t = \tan \theta$, then $z(t) = e^{2i\theta}$ so $Q(t) = (1 + t^2)^m T(2\theta)$ by (10). Hence, $|Q(t)| \leq (1 + t^2)^m$ for all $t \in \mathbf{R}$ and therefore

$$Q'(0)^2 + [2mQ(0)]^2 \leq (2m)^2$$

by Theorem 11. Clearly $Q(0) = T(0)$ and by differentiating $Q(\tan \theta)$ at $\theta = 0$, we obtain $Q'(0) = 2T'(0)$. Thus (9) holds at $\theta = 0$, proving Theorem 7.

Bernstein theorems for arbitrary normed linear spaces are given in [8] and [16]. In fact, an elementary argument is given in [16] to show that Markov's theorem for the first derivative holds in any normed linear space. For a discussion of connections between Bernstein's inequality for entire functions and functional analysis, see [4].

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JONSSON GROUPS, RINGS AND ALGEBRAS

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A Jonsson group G is one all of whose proper subgroups have smaller cardinality than G . Jonsson rings and Jonsson algebras are defined in a similar fashion. In this paper, we present an introductory account of Jonsson algebras in the light of pcf theory, a recent development within set theory. In section 1, we give some examples and summarize what is known about Jonsson groups and rings. In section 2, we prove the basic results on Jonsson algebras. Most of this section is self-contained, and the reader will need to know little more than some naive set theory and first-order model theory [H, HS or ChK]. Section 3 contains the elements of pcf theory, deals with the most recent results on Jonsson algebras, and summarizes the impact of additional set-theoretic axioms in this area.

1. Jonsson groups, rings, algebras and cardinals

To start matters off, we define Jonsson groups, algebras and cardinals.

Definition

1. A group G is a **Jonsson group** iff G has no proper subgroup H of the same cardinality as G , i.e. every proper subgroup of G has fewer elements than G .
2. Suppose that F is a countable set of finitary operations on a set A . The algebra $\mathbf{A} = (A, F)$ is a **Jonsson algebra** iff \mathbf{A} has no proper subalgebra $\mathbf{B} = (B, F|B)$ of the same cardinality as \mathbf{A} .

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3. A cardinal λ is a **Jonsson cardinal** iff there is no Jonsson algebra of cardinality λ .

In writing $F|B$, we mean the family of operations in F , each restricted to B^n for the appropriate number n of arguments. Since the nature of the underlying set A is irrelevant, we shall often assume without comment that it is a cardinal, and also say that there is a Jonsson algebra on λ meaning that there is one on a set of power λ .

It seems that Jonsson algebras were identified by B. Jonsson in the fifties, [EH]. Relatively little was known about them (at least in ordinary set theory) until the early eighties. Devlin surveys the state of play up to 1973 in section 3 of his paper [D].

Every Jonsson group is a Jonsson algebra (treating the identity element as a 0-ary operation). It is obvious that every finite algebra is a Jonsson algebra. So the first natural question is whether there are any (infinite) Jonsson cardinals at all.

Example 1: Let $A = (\omega, \{m\})$, where ω (the first infinite ordinal) is the set of natural numbers and $m(x) := x - 1$ for all $x > 0$, $m(0) := 0$. The algebra A is a Jonsson algebra of cardinality \aleph_0 (the first infinite cardinal), so \aleph_0 is not a Jonsson cardinal.

Example 2 [F]: Let p be prime number and let $C(p^n)$ be the cyclic group of order p^n . Then the p -quasicyclic group,

$$C(p^\infty) = \bigcup_{n=1}^{\infty} C(p^n),$$

is a countably infinite abelian group all of whose proper subgroups are finite. In accordance with the convention in abelian group theory, we shall write $C(p^\infty)$ additively. It is generated by elements $c_1, c_2, \dots, c_n, \dots$, such that

$$pc_1 = 0, pc_2 = c_1, \dots, pc_{n+1} = c_n, \dots$$

and $o(c_n) = p^n$. It is easy to check that $C(p^\infty)$ is a Jonsson group of cardinality \aleph_0 . For, if H is a proper subgroup, then there exists a least n such that c_{n+1} does not belong to H . Now $\langle c_n \rangle \leq H$. Conversely, if $h \in H$, then $h \in C(p^\infty)$, and so there exists k such

that $h = kc_m$, where $(p, k) = 1$. Since $(p^m, k) = 1$, there exist r and s such that $rk + sp^m = 1$, and hence

$$c_m = (rk + sp^m)c_m = rh \in H.$$

So $c_m \in H$ and $m \leq n$, so that $h \in \langle c_n \rangle$ and $H = \langle c_n \rangle$.

The family of p -quasicyclic groups contains all the countable Jonsson abelian groups, since if G is infinite abelian with all its proper subgroups finite, then G is a p -quasicyclic group for some prime p .

In 1979, Ol'shanskii, [O], proved the existence of an infinite non-abelian group all of whose proper subgroups are finite, solving Schmidt's problem².

Jonsson rings are rings all of whose proper subrings have smaller cardinality.

Example 3 [L]: From the p -quasicyclic group $C(p^\infty)$, it is easy to construct a countable Jonsson ring by defining all products to be zero. Since every proper subring of $C(p^\infty)$ is also a proper subgroup of $C(p^\infty)$, it must be finite.

We know all about countable Jonsson rings. Laffey classified them in a slightly different terminology, proving in his paper [L]:

Theorem. *If R is a countable Jonsson ring, then either*

(i) $R^2 = \{0\}$ and $R = C(p^\infty)$ for some prime p ,

or

(ii) $R = G_{p,q}$ for some primes p and q , where

$$G_{p,q} = \cup_{n=0}^{\infty} \text{GF}(p^{q^n})$$

and $\text{GF}(p^{q^n})$ is the finite field of order p^{q^n} .

In the early sixties, Kurosh conjectured that uncountable Jonsson groups exist. This conjecture was settled positively by Shelah in 1980. The group that Shelah built had even stronger properties:

Theorem. [Sh80] *There is a Jonsson group S of cardinality \aleph_1 . This group is simple.*

² Professor Wilfrid Hodges kindly supplied this reference.

It follows that \aleph_1 is not a Jonsson cardinal. We shall derive a more general version of this consequence in section 2. Shelah noted too that his group S has no maximal proper subgroup. In particular, the operation of taking the Frattini subgroup does not commute with direct products. For if $\tau(S)$ is the Frattini subgroup of S , i.e. the intersection of all the maximal subgroups of S , then $\tau(S) = S$, but $\tau(S \times S) = \{(a, a) : a \in S\}$.

To close this section, let us note a rather surprising connection between Jonsson semigroups and Jonsson groups.

Theorem. [McK] *If a semigroup G of cardinality λ is a Jonsson semigroup, then either G is a group, or else λ has countable cofinality and $(\exists \kappa < \lambda) (\lambda < 2^\kappa)$.*

In particular, if λ has uncountable cofinality or if $\kappa < \lambda$ implies $2^\kappa \leq \lambda$, then every Jonsson semigroup of power λ is a Jonsson group. This means for example that if one wishes to construct a Jonsson group of power λ , where λ has uncountable cofinality (e.g. λ a successor cardinal), then it is enough (or as hard) to build a Jonsson semigroup.

Within ordinary set theory, there seems to be no easy way to climb from Jonsson groups of power \aleph_1 to ones of power \aleph_2 . The theory of Jonsson algebras is in this respect a good deal smoother.

2. Jonsson algebras

In this section we prove the easiest results on Jonsson algebras in ordinary set theory. Although some of these can be established using combinatorial arguments and historically were first obtained in this way, one can abbreviate the arguments by employing elementary submodels. A Jonsson model is a model $\mathbf{A} = (A, R, F)$ where R and F are countable sets of finitary relations and operations on the set A such that every elementary submodel of \mathbf{A} has smaller cardinality. So every Jonsson algebra is a Jonsson model, since it has no relations and its elementary submodels are therefore subalgebras. The reader can find all relevant basic information about elementary submodels in the appendix [pp.165-176] of the monograph by Heindorf and Shapiro, [HS], or in the standard texts [H, ChK].

Proposition 1. *There is a Jonsson algebra of power λ iff there is a Jonsson model of power λ .*

Proof: For the non-trivial direction, add Skolem functions to the Jonsson model; the Skolem hull is the required Jonsson algebra. This is a standard and very useful technique for building algebras from models. We give a brief sketch. Fix a well-order \leq of the universe A of the Jonsson model $\mathbf{A} = (A, R, F)$. For each formula $\psi(x_1, \dots, x_n, y)$ in the language $L = R \cup F$ of the model, define a new n -ary operation f_ψ on A^n as follows: $f_\psi(a_1, \dots, a_n)$ is the \leq -least element $b \in A$ such that $\psi(a_1, \dots, a_n, b)$ is true in \mathbf{A} if such an element exists, otherwise $f_\psi(a_1, \dots, a_n)$ is any element of A . Repeat this process for the new (countable) language

$$L_1 = L \cup \{f_\psi : \psi \text{ is an } L\text{-formula}\},$$

and so on (countably many times) to get the expanded (still countable) language

$$L^* = \cup L_n.$$

The required algebra is

$$\mathbf{A}^{sk} = (A, \{f_\psi : \psi \text{ is an } L^*\text{-formula}\}),$$

since any proper subalgebra of \mathbf{A}^{sk} will give rise to a proper elementary submodel of the Jonsson model \mathbf{A} of the same cardinality. ■

We combine Proposition 1 with some set theory to obtain a necessary and sufficient criterion for the existence of Jonsson algebras.

Recall that for a cardinal θ , the collection of sets which are hereditarily of cardinality less than θ is denoted $H(\theta)$: a set x belongs to $H(\theta)$ iff $|x| < \theta$ and if $y \in x$, then $|y| < \theta$, and so on. For reference, we summarize the main features of $H(\theta)$ in a theorem:

Theorem. *If θ is a regular uncountable cardinal, then $H(\theta)$ is a transitive model of ZFC with the possible exception of the power set axiom. If α is an ordinal, then $\alpha < \theta$ iff $\alpha \in H(\theta)$.*

Intuitively, $H(\theta)$ is a reasonably small universe of most of the axioms of ordinary set theory. The main properties of elementary submodels of $H(\theta)$ are given in detail in [EM, pp.151-152].

Lemma 2. [BM] *Suppose that λ is an infinite cardinal. There is a Jonsson algebra on λ iff:*
for some (all) regular cardinal(s) $\theta > \lambda$ and for all elementary submodels $\mathbf{M} \leq H(\theta)$:

(*) if $\lambda \in M$ and $|\lambda \cap M| = \lambda$, then $\lambda \subseteq M$.

Proof: For the forward direction, note that since $\mathbf{M} \leq H(\theta)$, there is a Jonsson algebra $\mathbf{A} \in M$ on λ , say $A = (\lambda, \{f_n : n \in \omega\})$. Let $B = M \cap \lambda$ (by hypothesis unbounded in λ). So B has cardinality λ (λ is regular), and $\mathbf{B} = (B, \{f_n|_B : n \in \omega\})$ is a subalgebra of \mathbf{A} . However, \mathbf{A} is Jonsson, hence $B = A = \lambda$, i.e. $M \cap \lambda = \lambda$, and so $\lambda \subseteq M$.

For the reverse implication, fix $\mathbf{M} \leq H(\theta)$, $|M| = \lambda$, $\lambda \subseteq M$. Let $h : \lambda \rightarrow M$ be a bijection. Then $\mathbf{M}^+ = (M, \in, h)$ is a Jonsson model of power λ . For, if $\mathbf{N} \leq \mathbf{M}^+$ is an elementary submodel of power λ , then $\lambda \in N$ (by elementarity, since λ is the least ordinal which does not belong to $\text{dom}(h)$). Also $|N \cap \lambda| = \lambda$, and hence by (*) $\lambda \subseteq N$, since $\mathbf{N} \leq H(\theta)$. Therefore $\text{range}(h) \subseteq N$. But $\text{range}(h) = M$, so $N = M^+$, and \mathbf{M}^+ is a Jonsson model of power λ . We appeal now to Proposition 1 to complete the proof. ■

Theorem 3. [Sh, BM] *If there is a Jonsson algebra on λ , then there is one on λ^+ , where λ^+ is the least cardinal greater than λ .*

Proof: We use the lemma. Suppose that $\mathbf{M} \leq H(\lambda^{++})$, $\lambda^+ \in M$, $|M \cap \lambda^+| = \lambda^+$. We must show $\lambda^+ \subseteq M$. Suppose that $\beta < \lambda^+$. Note that $\lambda \in M$ (since $\lambda^+ \in M$ by elementarity), and $\exists \alpha \in M \cap \lambda^+$, $\alpha > \beta$, such that $|M \cap \alpha| = \lambda$, because $|M \cap \lambda^+| = \lambda^+$. M contains a bijection g from λ onto α , hence $|M \cap \lambda| = \lambda$, and so $\lambda \subseteq M$ (applying the lemma to the hypothesis that there is a Jonsson algebra on λ). Therefore $\alpha = \text{range}(g) \subseteq M$. In particular, $\beta \in M$. Since β was arbitrary, it follows that $\lambda^+ \subseteq M$. ■

In fact, for a little more effort and terminology, a stronger result is provable:

Theorem. (Tryba [T], Woodin) *If λ is regular and there is non-*

reflecting stationary subset of λ , then there is a Jonsson algebra on λ .

The short proof can be found in [BM]. In particular, there is a Jonsson algebra on λ^+ whenever λ is regular, since the set $\{\alpha < \lambda^+ : \text{cf}(\alpha) = \lambda\}$ is non-reflecting and stationary in λ^+ .

Example 1 and Theorem 3 yield a corollary:

Corollary 4. $(\forall n \in \mathbb{N})(\text{There is a Jonsson algebra on } \aleph_n).$

The simplest unanswered questions (at least to formulate) thus far are whether there are Jonsson algebras on \aleph_ω , on $\aleph_{\omega+1}$, and more generally on the successors of singular cardinals. We shall discuss these questions in section 3.

For completeness, let me mention two other equivalent conditions for the existence of a Jonsson algebra. The first is based on results of Los and Sierpiński (see [D]):

Theorem. *There is a Jonsson algebra of cardinality λ iff there is a Jonsson algebra of cardinality λ with exactly one commutative binary operation.*

The second characterization is related to a question of Mycielski about locally finite algebras:

Definition An algebra $A = (A, \{f_n : n \in \mathbb{N}\})$ is locally finite iff whenever X is a finite subset of A , then $A|X$ (the subalgebra of A generated by X) is finite.

What can one say about locally finite Jonsson algebras? Improving a theorem of Erdős and Hajnal, Devlin proved:

Theorem. [D] *There is a locally finite Jonsson algebra of cardinality λ iff there is a Jonsson algebra of cardinality λ .*

3. pcf theory

Possible cofinality (pcf) theory is the study of the cofinalities of ultraproducts of sets of cardinals. It was discovered (invented) by Shelah, and developed in its fullest form in his work on cardinal arithmetic, [Sh]. The theory has found applications in set theory, infinitary combinatorics (partition calculus), model theory, algebra (infinite abelian groups), set-theoretic topology, Boolean algebras (productivity of chain conditions) and Jonsson algebras. We select

just the definitions and result that are necessary to prove that there is a Jonsson algebra on $\aleph_{\omega+1}$. A lucid introduction to pcf theory is available in the paper by Burke and Magidor, [BM], which serves also as an excellent entry-point to Shelah's treatise.

Suppose that a is a set of regular cardinals and $\min(a) > |a|$. Let D be an ultrafilter on a . The elements of Πa are functions f such that $\text{dom}(f) = a$ and $(\forall \alpha \in a)(f(\alpha) < \alpha)$. We can define an equivalence relation $=_D$ on Πa by

$$f =_D g \text{ iff } \{\alpha \in a : f(\alpha) = g(\alpha)\} \in D,$$

and use the notation f/D ($\Pi a/D$) for the equivalence class of f (the set $\{f/D : f \in \Pi a\}$). The ultraproduct $(\Pi a/D, \leq_D)$, where

$$f \leq_D g \text{ iff } \{\alpha \in a : f(\alpha) \leq g(\alpha)\} \in D,$$

is a linear order since D is an ultrafilter. Hence it has a **true cofinality**:

Definition Suppose that λ is a cardinal. We say that λ is the **true cofinality** of $\Pi a/D$, and write $\lambda = \text{tcf}(\Pi a/D)$, iff:

- (1) λ is regular;
- (2) \exists a strictly increasing cofinal sequence $\{f_\zeta \in \Pi a : \zeta < \lambda\}$ in $\Pi a/D$, i.e.

$$(2.1) \quad \zeta < \xi < \lambda \text{ implies } f_\zeta <_D f_\xi$$

$$\text{and } (2.2) \quad (\forall h \in \Pi a)(\exists \zeta < \lambda)(h <_D f_\zeta).$$

To illustrate the idea, we compute some true cofinalities.

Example 4: If D is a principal ultrafilter on a (so D is generated by a singleton subset $\{\alpha\}$ of a say), then $\text{tcf}(\Pi a/D) = \alpha$.

Example 5: Suppose that $a = \{\aleph_n : 1 < n < \omega\}$. If D is a non-principal ultrafilter on a , then $\text{tcf}(\Pi a/D) > \aleph_\omega$. Why? Well, if $\{f_i \in \Pi a : i < \aleph_k\}$ is $<_D$ -increasing and $k < \omega$, then $(\forall n > k)(\{f_i(\aleph_n) : i < \aleph_k\})$ is bounded in \aleph_n by β_i say, and so the function

$$g(\aleph_m) = \sup\{\beta_i : i < \aleph_k\} + 1, \text{ if } m > k (= 0, \text{ if } 1 < m < k)$$

is an element of Πa and $(\forall i < \aleph_k)(f_i <_D g)$, since D contains the co-finite filter on a . In other words, $\{f_i \in \Pi a : i < \aleph_k\}$ is not cofinal in $\Pi a/D$.

One of the fundamental tasks in pcf theory is to determine which cardinals are the true cofinalities of the ultraproducts $\Pi a/D$, or how many possible cofinalities the set a supports.

Definition We define $\text{pcf}(a)$, the possible cofinalities of the set a , to be the collection

$$\{\lambda : \text{for some ultrafilter } D \text{ on } a, \text{tcf}(\Pi a/D) = \lambda\}.$$

Example 4 tells us that $a \subseteq \text{pcf}(a)$. We know too that there are $2^{2^{|a|}}$ ultrafilters on a (since every ultrafilter belongs to $P(P(a))$). Thus trivially

$$|a| \leq |\text{pcf}(a)| \leq 2^{2^{|a|}}.$$

It can be shown that $|\text{pcf}(a)| \leq 2^{|a|}$ and, for more money, $|\text{pcf}(a)| \leq |a|^{+3}$. The major open question in pcf theory is whether $|\text{pcf}(a)| = |a|$.

For our purposes, we shall need a special case of one of Shelah's theorems:

Theorem. [Sh] Suppose that $a = \{\aleph_n : 1 < n < \omega\}$. Then $\aleph_{\omega+1} \in \text{pcf}(a)$.

So one can represent $\aleph_{\omega+1}$ as the true cofinality of $\Pi a/D$ for some ultrafilter D on a . Note that this does not tell us anything about \aleph_ω or other singular cardinals, since they can never appear in a set of possible cofinalities (which are by definition regular). Shelah demonstrated the power which this representation provides in his proof of the existence of a Jonsson algebra of cardinality $\aleph_{\omega+1}$:

Theorem. [Sh] There is a Jonsson algebra on $\aleph_{\omega+1}$.

Proof: Let $\mu = \aleph_\omega$, $\theta = (2^\mu)^+$, and fix $M \leq H(\theta)$, $\mu^+ \in M$, $|M \cap \mu^+| = \mu^+$. We show that $\mu^+ \subseteq M$ (and appeal to Lemma 2).

We know that $\mu^+ \in \text{pcf}(a)$, where $a = \{\aleph_n : 1 < n < \omega\}$. By elementarity, it follows that:

- (1) $a \in M$;
- (2) $a \subseteq M$;
- (3) there is an ultrafilter D on a , $D \in M$, and a sequence

$$\{f_i : i < \mu^+\} \in M \cap \Pi a$$

such that $\{f_i/D : i < \mu^+\}$ is increasing and cofinal in $\Pi a/D$.

Claim: $\{\alpha \in a : |M \cap \alpha| = \alpha\}$ is cofinal in a .

Proof of claim: Otherwise, let $g(\alpha) = \sup(M \cap \alpha)$ ($g(\alpha) = 0$ if $\sup(M \cap \alpha) = \alpha$). So $g \in \Pi a$, and hence by (3) there is $\kappa < \mu^+$ such that $g/D < f_\kappa/D$. So for some $\alpha \in a$, $0 < g(\alpha) < f_\kappa(\alpha)$ (D contains the co-finite filter). But $f_\kappa(\alpha) \in M \cap \alpha$, and $g(\alpha) = \sup(M \cap \alpha)$, contradiction. Hence $\{\alpha \in a : |M \cap \alpha| = \alpha\}$ is cofinal in a . By Corollary 4, there is a Jonsson algebra on a for each $\alpha \in a$, and so $\alpha \subseteq M$ (by (2) and Lemma 2). Thus: $\mu \subseteq M$. Finally, for $\xi \in M \cap [\mu, \mu^+)$, there is a bijection ϕ from μ onto ξ , and hence $\xi \subseteq M$. But $|M \cap \mu^+| = \mu^+$, hence $\mu^+ \subseteq M$. By Lemma 2, this establishes that there is a Jonsson algebra on $\mu^+ = \aleph_{\omega+1}$. ■

Shelah has extended this result to cover a wide class of successors of singular cardinals and also the class of inaccessible cardinals which are not in some degree Mahlo or have a stationary subset not reflecting in any inaccessible cardinals. These results are presented in [Sh]. Their broad import is to make it progressively more difficult for Jonsson cardinals to exist. And indeed, if one increases one's axiomatic commitments beyond ordinary set theory (ZFC), this difficulty becomes an impossibility:

Theorem [Erdős-Hajnal-Rado, Keisler-Rowbottom]

- (1) If $2^\lambda = \lambda^+$, then there is a Jonsson algebra on λ^+ .
- (2) If $V = L$, then for every infinite cardinal λ , there is a Jonsson algebra on λ .

The relatively easy proofs of these can be found in [BM] (or [J] or [EHMR]). Thus it is consistent that there are no Jonsson cardinals at all. For Jonsson groups, additional set-theoretic hypotheses also have decisive implications:

Theorem. [Sh80] Suppose that λ is an uncountable cardinal and $2^\lambda = \lambda^+$.

- (1) There is a Jonsson group of cardinality λ^+ .
- (2) Moreover this group is a Jonsson semigroup, is simple, and there is a natural number n such that for any subset X of the group of cardinality λ^+ , any element of the group is equal to the product of n elements of X .

Whether there can be a Jonsson algebra of singular cardin-

ality (e.g. \aleph_ω) appears more difficult to resolve and different in character from the regular case. In 1988, Koepke [K], building on the work of Jensen on inner models of set theory, proved results which indicate that the non-existence of Jonsson algebras of singular cardinality is essentially connected with large cardinal axioms: if there is a Jonsson cardinal \aleph_ξ such that $\xi < \aleph_\xi$, then for each α there is a model of ZFC whose set of uncountable measurable cardinals has order type α . He also showed that if there is a singular Jonsson cardinal of uncountable cofinality κ , then there is an inner model of ZFC with κ measurable cardinals. These results establish that the assumption of the non-existence of a Jonsson algebra of singular cardinality is much stronger than the assumption that ZFC is consistent.

To conclude this brief survey, let me mention perhaps the most attractive open question about Jonsson algebras: can one prove in ordinary set theory (ZFC) that every successor cardinal carries a Jonsson algebra?

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INTEGRAL INEQUALITY ESTIMATES FOR P.D.E.s IN UNBOUNDED DOMAINS

J. N. Flavin

1. Introduction

In the context of problems involving P.D.E.s (boundary value problems, initial boundary value problems etc.), inequality estimates for certain, non-negative, L^2 integral measures of the solution are of interest: typically, they yield, *inter alia*, uniqueness of solution and continuous dependence upon data. The purpose of this note is to show how such estimates for *weighted* measures may be obtained in the context of *unbounded* domains where (prescribed) growth of the solution at infinity is allowed. Three results are proved which represent perhaps the *simplest* cases of such; they are believed to be new, or, at least, not well known.

In the matter of notation, subscripts denote partial differentiation, and *it will be convenient occasionally to write* (for the weight function)

$$g(\xi) = e^{-\lambda\xi},$$

where λ is a constant, ξ being the appropriate independent variable.

2. An Hyperbolic Equation

Let us commence with (arguably) the simplest example of the (linear) *wave equation* in an unbounded region, where growth at infinity is allowed which is not faster than exponential. A weighted energy inequality is derived therefor which has obvious analogues in all the common cases of wave-like equations in unbounded media (of whatever type); moreover, it has analogues in *nonlinear* elastodynamics and electrodynamics for certain classes of constitutive

equations. It is an easy matter to modify the proof of the proposition given hereunder in order to include a source term (as in subsequent Propositions).

Consider C^2 solutions of the initial boundary value problem: $u(x, t)$ satisfies

$$u_{tt} = u_{xx}, \quad 0 < x < \infty, \quad t > 0, \quad (1)$$

$$u(x, 0), \quad u_t(x, 0) \text{ specified}, \quad 0 < x < \infty, \quad (2)$$

$$u(0, t) = 0, \quad (3)$$

$$u_x, \quad u_t = O(e^{\mu x}) \text{ as } x \rightarrow \infty, \quad (4)$$

where μ is a given positive constant (here and subsequently).

Proposition 1. *The weighted energy associated with the foregoing, namely*

$$E(t) = \int_0^\infty e^{-\lambda x} \frac{1}{2} (u_t^2 + u_x^2) dx \quad (5)$$

where λ is a constant such that

$$\lambda > 2\mu, \quad (6)$$

satisfies the inequality

$$E(t) \leq E(0) \exp(\lambda t). \quad (7)$$

Proof: Differentiation, use of (1), followed by integration by parts using (3)-(6) yields

$$\frac{dE}{dt} = \int_0^\infty g(x) (u_t u_x)_x dx = \lambda \int_0^\infty g(x) u_x u_t dx.$$

Use of the arithmetic-geometric inequality gives

$$\frac{dE}{dt} \leq \lambda E$$

and the required estimate (7) follows on integration.

Remark 1. Uniqueness of solution, and continuous dependence (in a certain sense) on data, for the problem (1)-(4) follow by standard means.

Remark 2. The inequality (7) is *sharp* in the sense that

$$E(t)/\{E(0)\exp(\lambda t)\} \sim 1 \text{ as } 2\mu/\lambda \uparrow 1$$

when

$$u(x, t) = \sinh \mu x e^{\mu t}.$$

Remark 3. An estimate for $u(x, t)$, in terms of the initial data, follows from (7) via Schwarz's inequality:

$$\begin{aligned} u^2(x, t) &= \left(\int_0^x u_x dx \right)^2 \leq \left(\int_0^x e^{\lambda x} dx \right) \left(\int_0^x e^{-\lambda x} u_x^2 dx \right) \\ &\leq 2\lambda^{-1} (e^{\lambda x} - 1) E(t). \end{aligned}$$

3. A Parabolic Equation

Consider smooth solutions of the I.B.V.P. for the heat equation (with source) in a semi-infinite rod: $u(x, t)$ satisfies

$$u_t = u_{xx} + f(x), \quad 0 < x < \infty, \quad t > 0, \quad (8)$$

$$u(x, 0) = \text{specified}, \quad (9)$$

$$u(0, t) = 0, \quad (10)$$

$$u, u_x, u_t = O(e^{\mu x}) \text{ as } x \rightarrow \infty, \quad (11)$$

where μ is a positive constant, and f is a given function such that

$$f = O(e^{\mu x}) \text{ as } x \rightarrow \infty. \quad (12)$$

Proposition 2. The weighted L^2 measure of solution associated with the foregoing, namely

$$F(t) = \int_0^\infty e^{-\lambda x} u^2 dx \quad (13)$$

where λ is a constant such that

$$\lambda > 2\mu, \quad (14)$$

satisfies

$$F(t) \leq [\{F^{1/2}(0) + \sigma\}e^{\lambda^2 t/4} - \sigma]^2, \quad (15)$$

where

$$\sigma = 4S\lambda^{-2} \quad (16)$$

with

$$S = \left\{ \int_0^\infty g(x) f^2 dx \right\}^{\frac{1}{2}}. \quad (17)$$

Proof: Differentiation, use of (8), integration by parts (twice) using (10), (11), (13), (14) yields

$$\begin{aligned} \frac{dF}{dt} &= -2 \int_0^\infty g(x) u_x^2 dx + \lambda \int_0^\infty g(x) (u^2)_x dx + 2 \int_0^\infty g(x) u f dx \\ &= -2 \int_0^\infty g(x) u_x^2 dx + \lambda^2 F + 2 \int_0^\infty g(x) u f dx. \end{aligned} \quad (18)$$

Applying the inequality of Appendix 1 to the first term, and Schwarz's inequality to the last, we obtain

$$\frac{dF}{dt} \leq \frac{1}{2} \lambda^2 F + 2SF^{1/2}, \quad (19)$$

whence the proposition follows by straightforward integration ($F^{1/2} = \sqrt{F}$).

Remark 4. Similar to Remark 1.

Remark 5. The inequality (15) is *sharp* in the sense that both sides are asymptotically equivalent as $2\mu/\lambda \uparrow 1$ when u has the form

$$u(x, t) = U(t) \sinh \mu x$$

(the associated $f(x)$, $u(x, 0)$ also being proportional to $\sinh \mu x$).

Remark 6. A similar—though more involved—analysis may be carried out, *mutatis mutandis*, for the measure

$$F(t) = \int_0^\infty e^{-\lambda x} u_x^2 dx.$$

Moreover, one may deduce therefrom pointwise bounds, in terms of data, for $|u(x, t)|$ (cf. Remark 3).

4. An Elliptic Equation

Consider smooth solutions of Poisson's equation in a semi-infinite strip: $u(x, y)$ satisfies

$$u_{xx} + u_{yy} = f(x, y), \quad 0 < x < 1, \quad 0 < y < \infty, \quad (20)$$

$$u(x, 0) = 0, \quad (21)$$

$$u(0, y), u(1, y) \text{ specified}, \quad (22)$$

$$u, u_x, u_y = O(e^{\mu y}) \text{ as } y \rightarrow \infty, \quad (23)$$

where μ is a given positive constant, and $f(x, y)$ is a given function such that

$$f(x, y) = O(e^{\mu y}) \text{ as } y \rightarrow \infty. \quad (24)$$

Proposition 3. The weighted L^2 (cross-sectional) measure of solution of the foregoing, namely

$$F(x) = \int_0^\infty e^{-\lambda y} u^2 dy \quad (25)$$

where λ is a constant such that

$$\lambda > 2\mu, \quad (26)$$

satisfies the estimate

$$F^{1/2}(x) \leq G(x), \quad (27)$$

where $G(x)$ satisfies the (simple) boundary value problem

$$G'' + \left(\frac{\lambda^2}{4}\right) G = -\phi(x), \quad (28)$$

wherein

$$\phi(x) = \left\{ \int_0^\infty e^{-\lambda y} f^2(x, y) dy \right\}^{1/2}, \quad (29)$$

subject to

$$G(0) = F^{1/2}(0), \quad G(1) = F^{1/2}(1), \quad (30)$$

(both of which are available from data) PROVIDED

$$\lambda < 2\pi. \quad (31)$$

Proof: Successive differentiations, use of (20) and integration by parts using (21), (23), (25), (26), together with the inequality in Appendix 1, yield

$$F'(x) = \int_0^\infty 2g(y) u u_x dy, \quad (32)$$

$$F''(x) \geq 2 \int_0^\infty g u_x^2 dy - \frac{\lambda^2}{2} F + \int_0^\infty 2g f u dx. \quad (33)$$

Assume that $F > 0$ strictly. Noting that

$$(F^{1/2})'' = \frac{1}{2} F^{-3/2} [F F'' - \frac{1}{2} F'^2],$$

(25), (26), (32), (33), together with Schwarz's inequality (used twice) lead to

$$(F^{1/2})'' + \left(\frac{\lambda^2}{4}\right) F^{1/2} \geq -\phi(x).$$

The proof is completed on invoking Appendix 2 with h identified with $F^{1/2} - G$. The restriction that $F > 0$ strictly may be removed without difficulty, but this is not pursued here.

Remark 7. The restriction (31) is irremovable as the example $u = \sinh \pi y \sin \pi x$ shows.

Remark 8. It is verifiable that the inequality (27) is *sharp* in the sense that both sides are asymptotically equivalent as $2\mu/\lambda \uparrow 1$ when u has the form

$$u = U(x) \sinh \mu y$$

(the associated $f(x, y)$, $u(0, y)$, $u(1, y)$ being suitable constant multiples of $\sinh \mu y$).

Inequality estimates for integral measures of P.D.E.s in many contexts are treated in [1]. They include ones for unbounded regions which allow (prescribed) growth at infinity, of which the ones given here are perhaps the simplest examples: one of the earliest - perhaps even the earliest - examples of these latter techniques in mechanics occur in [2] and in the papers cited therein.

We conclude with the remark that where estimates for unbounded media of the type considered in this paper are concerned, the exponential weight function used throughout is by no means essential: growth conditions other than the exponential ones together with complementary weight functions, can equally well be contemplated.

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Appendix 1

Proposition. If $\Phi(x) \in C^1(0 \leq x \leq \infty)$ is such that $\Phi(0) = 0$ and

$$\int_0^\infty e^{-\lambda x} (\Phi_x^2 + \Phi^2) dx < \infty$$

for some positive constant λ , but is otherwise arbitrary, then

$$\frac{\lambda^2}{4} \int_0^\infty e^{-\lambda x} \Phi^2 dx \leq \int_0^\infty e^{-\lambda x} \Phi_x^2 dx.$$

Proof: The hypotheses imply that

$$\int_0^\infty (e^{-\lambda x} \Phi^2)_x dx = 0,$$

whence

$$\begin{aligned} \left(\frac{\lambda^2}{4}\right) \left(\int_0^\infty e^{-\lambda x} \Phi^2 dx\right)^2 &= \left(\int_0^\infty e^{-\lambda x} \Phi \Phi_x dx\right)^2 \\ &\leq \int_0^\infty e^{-\lambda x} \Phi^2 dx \int_0^\infty e^{-\lambda x} \Phi_x^2 dx \end{aligned}$$

by Schwarz's inequality. (For a proof under weaker hypotheses, see [2].)

The inequality above is *sharp* in the sense that both sides are asymptotically equivalent when

$$\Phi = \sinh \mu x,$$

μ being a constant such that $\lambda > 2\mu$, and $2\mu/\lambda \uparrow 1$.

Appendix 2

Proposition. Suppose that $h(x) \in C(0 \leq x \leq 1)$ satisfies

$$h'' + \Lambda h \geq 0, \quad x \in (0, 1),$$

$$h(0) = h(1) = 0,$$

where Λ is any positive constant such that $\Lambda < \pi^2$, then

$$h \leq 0.$$

The case $\Lambda = 0$ is geometrically obvious (curve under chord property for convex functions). Different proofs of the proposition may be found in [1A] by means of maximum principles, and in [2A] by means of Wirtinger's inequality.

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SOME NEW RESULTS IN RICCI CURVATURE

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The purpose of this note is to announce some new results (see [10]) concerning manifolds of positive Ricci curvature.

The motivation for this line of study is ultimately to understand the relationship between curvature and topology. The earliest result in this direction was the Gauss-Bonnet Theorem, which states that for a closed Riemannian 2-manifold, the Euler characteristic is proportional (by a factor of 2π) to the integral of the Gaussian curvature. As a consequence we have for example that any 2-manifold admitting a metric of everywhere positive curvature must have a positive Euler characteristic.

In dimensions greater than two, there are various competing notions of curvature: the sectional, the Ricci and the scalar. The scalar curvature, being the weakest, has proved the easiest to analyse, and much work has been carried out into understanding its topological implications.

It turns out that there are in fact no topological restrictions for negative scalar curvature in dimensions ≥ 3 . In other words any closed manifold of dimension ≥ 3 can be equipped with a metric of negative scalar curvature - even a sphere!

The case of positive scalar curvature is more interesting. The topological implications are not fully known, but partial results include the following (see [9]):

Theorem. (Stolz) *Let M^n be a smooth, closed, simply-connected manifold with $n \geq 5$. If M is non-spin, then M admits a metric of positive scalar curvature. If M is spin, then M has a positive scalar curvature metric if and only if $\alpha(M) = 0$, where α is a*

certain (topologically defined) homomorphism of spin bordism into connective K -theory.

$$\alpha : \Omega_n^{spin} \longrightarrow ko_n.$$

The key to much of the progress with scalar curvature involves the concept of 'surgery', which we describe presently.

Given an embedding

$$\iota : D^n \times S^m \longrightarrow M,$$

where M is a manifold of dimension $n+m$, we form a new manifold \hat{M} in the following way:

$$\hat{M} = (M \setminus \iota(D^n \overset{\circ}{\times} S^m) \amalg (S^{n-1} \times D^{m+1})) / \sim$$

where \circ denotes the interior and \sim denotes identification of the boundaries via the map ι . (Note that $\partial(D^n \times S^m) = \partial(S^{n-1} \times D^{m+1}) = S^{n-1} \times S^m$.) This process is known as performing an m -surgery. If we wish to be more precise we speak of performing a surgery on the embedded sphere $\iota(* \times S^m) \subset M$. The number n is called the codimension of the surgery, and \hat{M} is the result of the surgery.

The relevance of surgery to questions of curvature arises from the following theorem, which is due to Gromov and Lawson [2] and independently to Schoen and Yau [7].

Theorem. (Gromov, Lawson, Schoen, Yau) *Suppose M is a manifold of dimension ≥ 5 with a positive scalar curvature metric. Let \hat{M} be the result of performing a surgery of codimension ≥ 3 on M . Then \hat{M} has a positive scalar curvature metric.*

We turn our attention now to Ricci curvature. It has been established that there are no topological restrictions also for negative Ricci curvature in dimensions ≥ 3 (see [6]). Since a positive Ricci curvature metric is also a positive scalar curvature metric, any obstruction to admitting a positive scalar curvature metric is also an obstruction to admitting a positive Ricci curvature metric. The only known restriction to positive Ricci curvature which does not arise from positive scalar curvature is stated in the following theorem which is due to Myers:

Theorem. (Myers) *If M is compact and has a positive Ricci curvature metric then the fundamental group $\pi_1 M$ is finite.*

Although there are manifolds (such as $S^2 \times S^1$) which admit metrics of positive scalar curvature but not of positive Ricci curvature, none of the known examples are simply connected.

In the light of the progress made for positive scalar curvature, it is reasonable to look for surgery results in the realm of Ricci curvature. In [8], Sha and Yang prove such a result, though it only applies in very special circumstances.

Note that the normal bundle of the sphere S^m in the product $S^n \times S^m$ has a canonical trivialization, i.e. there is a canonical embedding

$$\iota : D^n \times S^m \longrightarrow \nu(* \times S^m) \subset S^n \times S^m.$$

Suppose that ι is actually an isometry, where the metric on $S^n \times S^m$ is a product of round metrics and the normal bundle fibres have constant radius. Sha and Yang show that provided $m+1, n \geq 2$, the result of performing any number of surgeries using such embeddings yields a manifold which admits a positive Ricci curvature metric.

Our first result is a surgery theorem with more flexibility than that of Sha and Yang. It can be shown that Sha and Yang's conclusion remains true if the embedding ι is replaced by an arbitrary one, provided $m+1 \geq n \geq 3$. More generally we have the following:

THEOREM A. *Suppose we have a manifold M of positive Ricci curvature together with an isometric embedding*

$$\iota : D_R^n(N) \times S^m(\rho) \longrightarrow M,$$

where $D_R^n(N)$ denotes a geodesic ball of radius R in the n -sphere with the round metric of radius N , and where $S^m(\rho)$ is the m -sphere with the round metric of radius ρ . Suppose further that $m+1 \geq n \geq 3$. We can twist ι to a non-isometric embedding by composing with a map

$$\tau : D^n \times S^m \longrightarrow D^n \times S^m$$

$$(x, y) \longmapsto (T(y)x, y)$$

where $T : S^m \rightarrow SO(n)$. Let $\{0\}$ denote the centre point of D^n . Performing surgery on $\iota(\{0\} \times S^n)$ using the map $\iota \circ \tau$, we again obtain a manifold of positive Ricci curvature provided the ratio $\frac{\rho}{N}$ is suitably small.

Using this result we can prove the following:

THEOREM B. *Homotopy spheres which bound parallelizable manifolds admit metrics of positive Ricci curvature.*

(Note that a homotopy sphere is a manifold homotopy equivalent to a sphere, and that a parallelizable manifold is a manifold with trivial tangent bundle.)

Many examples of Ricci positive manifolds are homotopy spheres. However, by a result of Hitchin, [4], this is not true for all homotopy spheres. Indeed some admit no metric of positive scalar curvature. One would like to find criteria for deciding whether a given homotopy sphere admits a Ricci positive metric or not.

We can divide the set of homotopy spheres into those which bound a parallelizable manifold and those which do not. It is reasonable to ask if this division mirrors the division by positive/negative Ricci curvature.

The diffeomorphism classes of homotopy spheres bounding parallelizable manifolds of dimension m form an abelian group under the connected sum operation. This group is denoted bP_m . It was shown by Kervaire and Milnor in [5] that $bP_{odd} = 0$, bP_{4k+2} is either 0 or \mathbb{Z}_2 (depending on k), and bP_{4k} is cyclic. In [3] Hernandez showed that a certain class of Brieskorn manifolds carry Ricci positive metrics. This class includes homotopy spheres representing the non-trivial element of those groups bP_{4k+2} which are isomorphic to \mathbb{Z}_2 (a case previously covered by Cheeger in [1]), as well as many elements in bP_{4k} . Until now, however, it was an open question whether in fact all such homotopy spheres admit Ricci positive metrics.

On the other hand, using these methods it can be shown that there is a homotopy sphere of dimension 8 which admits a Ricci

positive metric, although it is not the boundary of a parallelizable manifold. Hence the converse to Theorem B is false.

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THE 36TH INTERNATIONAL MATHEMATICAL OLYMPIAD

Fergus Gaines

The 36th International Mathematical Olympiad took place in York University, Ontario, Canada, on 19th and 20th July 1995. 412 students participated from 73 countries and Ireland was represented by Robert Hayes (St Thomas Community College, Bray), Gavin Hurley (Coláiste an Spioraid Naoimh, Cork), Brian Jones (Gonzaga College, Dublin), Peter McNamara (East Glendalough School, Wicklow), Deirdre O'Brien (Mount Mercy College, Cork) and Gregory Wall (St Mary's Academy, Carlow). The team consisted of the top six performers in the Irish Mathematical Olympiad, which took place on 6th May 1995. The University of Limerick hosted a three-day intensive training session for the team from 5th to 7th July. The training involved staff from UL, UCC and UCD and some previous Irish Olympiad team members. The team leader was Gordon Lessells of UL and the deputy leader was Eugene Gath, also of UL. I accompanied Gordon as an "observer".

Gordon and I flew to Toronto on 12th July, whence we were taken to the University of Waterloo. The first job of the team leaders was to select six problems, from a short list of 28, which would form the two papers of the IMO exam. A local committee had chosen the short-listed questions from a total of about 200 problems which were submitted by the participating countries. The leaders and observers were free to work on the problems for 24 hours, without the distraction (?) of having the official solutions! I felt a bit disappointed because I only managed to solve four of the problems in that time, but felt a little less chastened on discovering that many of the team leaders had similar success! One

of the questions submitted by Ireland was shortlisted and was in contention for a long time, before being eliminated from consideration. The problem subsequently appeared on this year's Iranian Mathematical Olympiad. The problem, which was composed by Tom Laffey, is as follows:

At a meeting of $12k$ people, each person exchanges greetings with exactly $3k + 6$ others. For any two people, the number who exchange greetings with both is the same. How many people are at the meeting? Prove that such a meeting is possible.

The team, accompanied by Eugene Gath, arrived in Toronto on 15th July and were taken to York University. A lot of social activities were organized for them and there was plenty of time to establish friendships with students from other countries. For two days after the exams the leaders and deputy leaders were fully involved in marking the students' work and agreeing marks with the Canadian problem coordinators – this was an acrimonious business, at times. Gold, silver and bronze medals were awarded to the top performers – not more than 50% of the students can get medals. The performance of the Irish team was a bit disappointing. They won no medals, although Deirdre O'Brien and Peter McNamara got an "honourable mention" for their solutions to Question 1. Overall the team came 61st out of 73 countries.

Before the exam took place, some team leaders were vociferous in their claim that the exam was too easy – one leader going so far as to claim that each member of his team would get full marks. However, once the exam had taken place, no more such remarks were to be heard, although it was generally agreed that Question 1 was a bit too easy. China took first place, with Romania second and Russia third. The United States was very disappointed with 11th place, particularly after their unique performance in Hong Kong in 1994, when all their students got full marks. Questions 2 and 6 turned out to be very difficult for most students, although one Bulgarian student was given a special prize for the elegance of his solution to Question 6 (the second solution given on p.75).

Here are the questions.

First Day

1. Let A, B, C and D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at the points X and Y . The line XY meets BC at the point Z . Let P be a point on the line XY different from Z . The line CP intersects the circle with diameter AC at the points C and M , and the line BP intersects the circle with diameter BD at the points B and N . Prove that the lines AM, DN and XY are concurrent.

2. Let a, b and c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

3. Determine all integers $n > 3$ for which there exist n points A_1, A_2, \dots, A_n in the plane, and real numbers r_1, r_2, \dots, r_n satisfying the following two conditions:

- (i) no three of the points A_1, A_2, \dots, A_n lie on a line;
- (ii) for each triple i, j, k ($1 \leq i < j < k \leq n$) the triangle $A_i A_j A_k$ has area equal to $r_i + r_j + r_k$.

Time Allowed – $4\frac{1}{2}$ hours.

Second Day

4. Find the maximum value of x_0 for which there exists a sequence of positive real numbers $x_0, x_1, \dots, x_{1995}$ satisfying the two conditions:

$$(i) x_0 = x_{1995} \quad (ii) x_{i-1} + \frac{2}{x_{i-1}} = 2x_i + \frac{1}{x_i} \text{ for } i = 1, 2, \dots, 1995.$$

5. Let $ABCDEF$ be a convex hexagon with

$$AB = BC = CD$$

$$DE = EF = FA$$

and

$$\angle BCD = \angle EFA = 60^\circ.$$

Let G and H be two points in the interior of the hexagon such that $\angle AGB = \angle DHE = 120^\circ$. Prove that

$$AG + GB + GH + DH + HE \geq CF.$$

6. Let p be an odd prime number. Find the number of subsets A of the set $\{1, 2, \dots, 2p\}$ such that

- (i) A has exactly p elements, and
- (ii) the sum of all the elements of A is divisible by p .

Time Allowed— $4\frac{1}{2}$ hours.

The solutions to these problems are on pp.69-76.

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Research Announcement

OPTIMAL APPROXIMABILITY OF SOLUTIONS OF SINGULARLY PERTURBED DIFFERENTIAL EQUATIONS

R. Bruce Kellogg and Martin Stynes

Using the theory of n -widths, the approximability of solutions of singularly perturbed reaction-diffusion and convection-diffusion problems in one dimension is quantified. Full details appear in [1].

Reference

- [1] R. B. Kellogg and M. Stynes, *Optimal approximability of solutions of singularly perturbed differential equations* (1995) (Preprint 1995-3, Mathematics Department, University College Cork).

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Research Announcement

FINITE ELEMENT METHODS FOR CONVECTION-DIFFUSION PROBLEMS USING EXPONENTIAL SPLINES ON TRIANGLES

Riccardo Sacco and Martin Stynes

A new family of Petrov-Galerkin finite element methods on triangular grids is constructed for singularly perturbed elliptic problems in two dimensions. It uses divergence-free trial functions that form a natural generalization of one-dimensional exponential trial functions. This family includes an improved version of the divergence-free finite element method used in the PLTMG code. Numerical results show that the new method is able to compute strikingly accurate solutions on coarse meshes. An analysis of the use of Slotboom variables shows that they are theoretically unsatisfactory and explains why certain Petrov-Galerkin methods lose their stability when generalized from one to two dimensions. Full details appear in [1].

Reference

- [1] R. Sacco and M. Stynes, *Finite element methods for convection-diffusion problems using exponential splines on triangles* (1995) (Preprint 173/P, Mathematics Department, Politecnico di Milano, Milan).

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Research Announcement

NECESSARY CONDITIONS FOR UNIFORM CONVERGENCE OF FINITE DIFFERENCE SCHEMES FOR CONVECTION-DIFFUSION PROBLEMS WITH EXPONENTIAL AND PARABOLIC LAYERS

Hans-Görg Roos and Martin Stynes

A difference scheme for a convection-dominated problem is said to be uniformly convergent when its convergence behaviour is essentially independent of the diffusion parameter. In this paper we discuss necessary conditions that uniformly convergent schemes must satisfy in the presence of exponential and parabolic boundary layers. Full details appear in [1].

Reference

- [1] H.-G. Roos and M. Stynes, *Necessary conditions for uniform convergence of finite difference schemes for convection-diffusion problems with exponential and boundary layers* (1995) (Math. Appl. (to appear)).

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Research Announcement

EFFICIENT GENERATION OF SHISHKIN MESHES IN SOLVING CONVECTION-DIFFUSION PROBLEMS

Neil Madden and Martin Stynes

A description of Shishkin meshes for resolving boundary and interior layers is given. It is shown how PLTMG can be used to construct such meshes with minimal effort. Several types of singularly perturbed convection-diffusion problems are solved on these meshes. These solutions are compared with those obtained on uniform meshes and on meshes adaptively refined by PLTMG; it is seen that Shishkin meshes yield much more accurate solutions with little additional computational effort. Full details appear in [1].

Reference

- [1] N. Madden and M. Stynes, *Efficient generation of Shishkin meshes in solving convection-diffusion problems* (1995) (Preprint 1995-2, Mathematics Department, University College Cork).

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Research Announcement

ANALYSIS OF A CELL-VERTEX FINITE VOLUME METHOD FOR CONVECTION-DIFFUSION PROBLEMS

K.W. Morton, Martin Stynes and Endre Süli

The cell-vertex finite volume approximation of an elliptic convection-dominated diffusion equation is considered in two dimensions. The scheme is shown to be stable and second-order convergent in a mesh-dependent L_2 norm. Full details appear in [1].

Reference

- [1] K. W. Morton, M. Stynes and E. Süli, *Analysis of a cell-vertex finite volume method for convection-diffusion problems* (1994) (Report Number 94/6, Oxford University Computing Laboratory).

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Book Review

Mathematics—The Music of Reason

Translated from the French by J. Dales and H. G. Dales

J. Dieudonné

Springer-Verlag 1992, 287 pp.

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Price DM 71.00.

Reviewed by Robin Harte

Here is a master of exposition at the peak of his form - the interpreter and expositor of Grothendieck's theories offers us a dancing run over the surface of modern mathematics, carrying us from astronomy in the ancient world to Gödel, "independence" and Cohen "forcing". As we might expect, the perspective throughout is very "Bourbaki". After two more or less introductory chapters on "Mathematics and Mathematicians" and "The Nature of Mathematical Problems", each chapter is addressed to non-specialists and then furnished with an Appendix for the professionals. Thus we have "Objects and Methods in Classical Mathematics", with an Appendix ranging from ratios à la Euclid to limits via exhaustion, "Some Problems of Classical Mathematics" with an Appendix covering prime numbers and the Riemann zeta function, "New Objects and New Methods", whose Appendix is about Galois Theory and the foundations of metric spaces, and finally "Problems and Pseudo-problems about Foundations", with an Appendix about surface geometry and models of the real numbers. The translation, by J. and H. G. Dales, is uniformly excellent: only the typeface seems to have an old fashioned air about it.

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Solutions to the Problems of the 36th IMO

1. First solution. Let DN meet XY at the point R . The triangles RZD and BZP are similar and hence $RZ/ZD = BZ/ZP$. Thus $RZ = BZ \cdot ZD/ZP = ZX^2/ZP$. If S is the point of intersection of AM and XY , then a similar argument proves that $SZ = ZX^2/ZP$. Thus the points R and S coincide and the result follows.

Second solution. Choose coordinates so that the line $ABCD$ is the x -axis with Z as origin and XY is the y -axis. Let the coordinates of A, B, C, D and P be $(a, 0), (b, 0), (c, 0), (d, 0)$ and $(0, p)$, respectively. The problem can now be solved using routine calculations.

2. The expression on the left hand side of the inequality can be made a little more friendly by letting $a = 1/x, = 1/y$ and $c = 1/z$. Then $xyz = 1$ and the inequality to be proved is:

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{3}{2}.$$

If S denotes the left hand side, then

$$\begin{aligned} 2(x+y+z)S &= [(x+y) + (y+z) + (z+x)]S = \\ &= [(\sqrt{x+y})^2 + (\sqrt{y+z})^2 + (\sqrt{z+x})^2] \times \\ &\quad \left[\left(\frac{z}{\sqrt{x+y}} \right)^2 + \left(\frac{x}{\sqrt{y+z}} \right)^2 + \left(\frac{y}{\sqrt{z+x}} \right)^2 \right] \geq (z+x+y)^2 \end{aligned}$$

by Cauchy's inequality. But the arithmetic-geometric mean inequality gives $x+y+z \geq 3$, since $xyz = 1$. Thus

$$2(x+y+z)S \geq 3(x+y+z).$$

Hence $S \geq 3/2$ and the result is proved.

3. If A_1, A_2, A_3, A_4 are the vertices of a square of unit area and if $r_i = 1/6$ for $i = 1, 2, 3, 4$, then the triangle $A_i A_j A_k$ has area $r_i + r_j + r_k$ for each triple i, j, k ($1 \leq i < j < k \leq 4$). So the result holds for $n = 4$.

Suppose that $B_1B_2B_3B_4$ is a convex quadrilateral and that there exist real numbers r_1, r_2, r_3, r_4 such that the area of the triangle $B_iB_jB_k$ is $r_i + r_j + r_k$ for all integers i, j, k with $1 \leq i < j < k \leq 4$. Let B be the point of intersection of B_1B_3 and B_2B_4 . Then, by considering the areas of the triangles BB_3B_4 , BB_2B_3 , B_1B_2B , B_1BB_4 , it is not difficult to prove that

$$(*) \quad r_1 + r_3 = r_2 + r_4.$$

Let $n \geq 5$ and suppose there exist n points A_1, A_2, \dots, A_n and n real numbers r_1, r_2, \dots, r_n satisfying the conditions of the problem. Form the smallest convex set C containing all the points A_1, A_2, \dots, A_5 .

Case 1. Suppose C is a convex pentagon. Apply $(*)$ to the convex quadrilaterals $A_1A_2A_3A_4$ and $A_1A_2A_3A_5$ to get

$$r_1 + r_3 = r_2 + r_4 \text{ and } r_1 + r_3 = r_2 + r_5.$$

Thus $r_4 = r_5$. Repeating this argument for the appropriate pairs of quadrilaterals we get

$$r_1 = r_2 = r_3 = r_4 = r_5 = r,$$

say. Then the area of the triangle $A_2A_3A_4 = 3r =$ the area of the triangle $A_2A_3A_5$ and thus A_2A_3 is parallel to A_4A_5 . Also, the area of $A_1A_2A_3 =$ the area of $A_2A_3A_5$ and hence A_2A_3 is parallel to A_1A_5 . Thus A_1, A_4 and A_5 are collinear. This contradiction proves that C is not a convex pentagon.

Case 2. Suppose C is a convex quadrilateral. Without loss of generality, let C be $A_1A_2A_3A_4$. Since no three of the points A_1, A_2, \dots, A_5 are collinear we may also suppose, without loss of generality, that A_5 is strictly inside the triangle $A_1A_2A_4$. The equation

$$\begin{aligned} \text{area } A_1A_2A_4 + \text{area } A_2A_3A_4 = \\ \text{area } A_2A_3A_5 + \text{area } A_3A_4A_5 + \\ \text{area } A_4A_1A_5 + \text{area } A_1A_2A_5 \end{aligned}$$

implies that $r_1 + r_3 + 4r_5 = 0$ and, hence, $r_2 + r_4 + 4r_5 = 0$ (using $(*)$). The equation

$$\text{area } A_1A_2A_4 = \text{area } A_1A_2A_5 + \text{area } A_1A_4A_5 + \text{area } A_2A_4A_5$$

implies that $r_1 + r_2 + r_4 + 3r_5 = 0$. Thus $r_1 = r_5$. Thus the triangles $A_2A_3A_5$ and $A_2A_3A_1$ have the same area. Hence the triangle $A_1A_2A_5$ has zero area. This implies that A_1, A_2 and A_5 are collinear, which is impossible. Hence C is not a quadrilateral.

Case 3. Suppose C is a triangle. Then, without loss of generality $C = A_1A_2A_3$. The points A_4 and A_5 are inside the triangle. The equation

$$\text{area } A_1A_2A_4 + \text{area } A_2A_3A_4 + \text{area } A_3A_1A_4 = \text{area } A_1A_2A_3$$

implies that $r_1 + r_2 + r_3 + 3r_4 = 0$. Similarly, on replacing A_4 by A_5 , we get $r_1 + r_2 + r_3 + 3r_5 = 0$. Thus $r_4 = r_5$. Then

$$r_1 + r_2 + r_4 = r_1 + r_2 + r_5$$

implies that A_4A_5 is parallel to A_1A_2 and

$$r_1 + r_3 + r_4 = r_1 + r_3 + r_5$$

implies that A_4A_5 is parallel to A_1A_3 . Thus A_1, A_2, A_3 are collinear. This contradiction implies that C is not a triangle.

Since we get a contradiction in all cases we must have $n < 5$. Hence $n = 4$ is the only integer greater than 3 satisfying the conditions of the problem.

4. It is easy to deduce from condition (ii) that, for each integer $i \geq 1$,

$$\text{either } x_i = \frac{1}{2}x_{i-1} \text{ or } x_i = \frac{1}{x_{i-1}}.$$

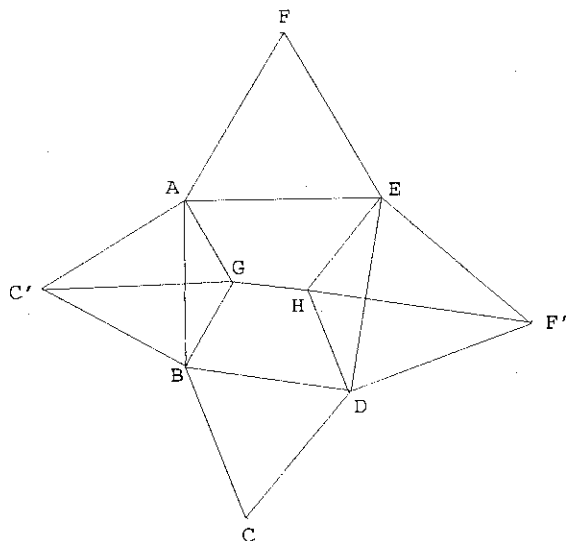
For each integer $i \geq 1$, induction can be used to prove that $x_i = 2^r x_0^s$, for some integer r with $-i \leq r < i$ and $s = (-1)^{i-t}$, where $t = |r|$.

Let $x_{1995} = 2^r x_0^s$. Then $x_0 = x_{1995}$ gives $x_0^{1-s} = 2^r$. If $s = 1$ then $r = 0$. But this gives the contradiction $1 = s = (-1)^{1995-0} = -1$. Hence $s = -1$ and $x_0^2 = 2^r$. So the largest value x_0 can have is attained when r here has its largest possible value. Now $-1995 \leq r < 1995$. The value $r = 1994$ is attained for the sequence which satisfies

$$x_{i+1} = \frac{1}{2} x_i, \text{ for } i = 0, 1, \dots, 1993 \text{ and } x_{1995} = \frac{1}{x_{1994}}.$$

Then $x_{1995} = 2^{1994} x_0^{-1}$. So, in this case, $x_0^2 = 2^{1994}$. Thus $x_0 = 2^{997}$ is the maximum value of x_0 for which a sequence with the required properties exists.

5.



Form the quadrilateral $ABDE$. The triangles BCD and FAE are clearly equilateral. On the line segment AB construct the (exterior) equilateral triangle $AC'B$ and on the line segment DE

construct the (exterior) equilateral triangle EDF' . In the quadrilaterals $CBAF$ and $C'BDF'$ we have $CB = C'B$, $BA = BD$ and $AF = DF'$. Also $\angle CBA = \angle C'BD$ and $\angle BAF = \angle BDF'$ because $\angle BAE = \angle BDE$, since the triangles ABD and AED are isosceles. Thus the quadrilaterals $CBAF$ and $C'BDF'$ are congruent. So $CF = C'F'$.

Since $\angle AC'B = 60^\circ$ and $\angle AGB = 120^\circ$, the quadrilateral $AC'BG$ is cyclic. Ptolemy's theorem then says that

$$AC' \cdot BG + BC' \cdot AG = AB \cdot C'G$$

and thus,

$$BG + GA = GC',$$

since the triangle $AC'B$ is equilateral.

Similarly

$$EH + HD = HF'.$$

Thus

$$AG + GB + GH + HD + HE = C'G + GH + HF' \geq C'F' = CF,$$

and the result is proved.

Note. The result is still true without the condition $\angle AGB = \angle DHE = 120^\circ$. This follows from the fact that the extended version of Ptolemy's theorem (applied to the quadrilateral $AC'BG$):

$$AC' \cdot BG + BC' \cdot AG \geq AB \cdot C'G$$

applies when the angle $\angle AGB$ is arbitrary ($< 180^\circ$). The inequalities

$$BG + GA \geq C'G \text{ and } DH + HE \geq HF'$$

can then be used to prove the result.

6. First solution. Set $\Omega = \{1, 2, \dots, 2p\}$ and let S be the collection of all the subsets of Ω each of which contains p elements. Then S contains $\binom{2p}{p}$ sets.

For each set $X \in S$ let $s(X)$ denote the sum of the elements of X . Let $B = \{1, 2, \dots, p\}$ and $C = \{p+1, p+2, \dots, 2p\}$. Then $B, C \in S$ and $s(B) \equiv s(C) \equiv 0 \pmod{p}$. If $A \in S$ and $A \neq B, C$ then $A \cap B \neq \emptyset$ and $A \cap C \neq \emptyset$. Let T be the collection of sets obtained by excluding B and C from S . Then T contains $\binom{2p}{p} - 2$ sets. Partition T into collections of sets as follows: two sets A and A' are in the same collection if and only if $A \cap C = A' \cap C$ and there exists an integer m with $0 \leq m < p$ such that

$$A' \cap B = \{x + m \pmod{p} : x \in A \cap B\}.$$

Then each such collection contains p sets. Let A and A' be distinct sets in the same collection and suppose $A \cap B$ has n elements. Then $0 < n < p$ and there exists an integer m with $0 < m < p$ such that

$$A' \cap B = X \cup Y,$$

where

$$X = \{x + m : x \in A \cap B, x + m \leq p\}$$

and

$$Y = \{x + m - p : x \in A \cap B, x + m > p\}.$$

Then $s(A') - s(A) \equiv mn \pmod{p}$. But p does not divide mn . Thus, if we calculate $s(A) \pmod{p}$ for each of the p sets in any of the collections, we get all of the residues $0, 1, \dots, p-1$. In particular, each collection contains exactly one set A satisfying $s(A) \equiv 0 \pmod{p}$. Hence T contains

$$\frac{1}{p} \left(\binom{2p}{p} - 2 \right)$$

sets such that the sum of the elements in each set is divisible by p . Hence the number of p -element subsets of Ω such that the sum of the elements in each subset is divisible by p is

$$\frac{1}{p} \left(\binom{2p}{p} - 2 \right) + 2.$$

Second Solution. Set $\Omega = \{1, 2, \dots, 2p\}$ as before and let n_j denote the number of p -element subsets of Ω such that the sum of the elements of each subset is congruent to $j \pmod{p}$, for $j = 0, 1, \dots, p-1$. Form the generating function

$$f(x) = \sum_{j=0}^{p-1} n_j x^j$$

of the sequence n_0, n_1, \dots, n_{p-1} . Let ω be a primitive p -th root of 1. If $A = \{i_1, i_2, \dots, i_p\}$ is a p -subset of Ω such that

$$i_1 + i_2 + \dots + i_p \equiv j \pmod{p},$$

then

$$\omega^{i_1 + i_2 + \dots + i_p} = \omega^j.$$

Thus

$$f(\omega) = \sum \omega^{i_1 + i_2 + \dots + i_p},$$

where the sum is taken over all p -subsets A of Ω as above. The coefficient of x^p in the product

$$(x - \omega)(x - \omega^2) \dots (x - \omega^{2p})$$

is

$$(-1)^p \sum \omega^{i_1 + i_2 + \dots + i_p} = -f(\omega).$$

But the product equals

$$\{(x - \omega)(x - \omega^2) \dots (x - \omega^p)\}^2 = (x^p - 1)^2 = x^{2p} - 2x^p + 1.$$

Thus $f(\omega) = 2$. So

$$n_0 - 2 + n_1\omega + n_2\omega^2 + \dots + n_{p-1}\omega^{p-1} = 0.$$

But ω is any primitive p -th root of 1. So, if

$$g(x) = n_0 - 2 + n_1x + n_2x^2 + \dots + n_{p-1}x^{p-1},$$

then

$$g(\omega) = g(\omega^2) = \dots = g(\omega^{p-1}) = 0,$$

because $\omega, \omega^2, \dots, \omega^{p-1}$ are all the primitive p -th roots of 1. Thus

$$g(x) = (x - \omega)(x - \omega^2) \dots (x - \omega^{p-1})h(x),$$

for some polynomial $h(x)$. By comparing degrees we see that $h(x) = k$, a constant. Thus

$$g(x) = k(1 + x + x^2 + \dots + x^{p-1}).$$

Thus

$$n_0 - 2 = n_1 = \dots = n_{p-1} = k.$$

But

$$n_0 + n_1 + \dots + n_{p-1} = \binom{2p}{p}.$$

Hence

$$n_j = \frac{1}{p} \left(\binom{2p}{p} - 2 \right)$$

for $j = 1, 2, \dots, p-1$ and

$$n_0 = \frac{1}{p} \left(\binom{2p}{p} - 2 \right) + 2.$$

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