TENSOR PRODUCTS AND PROJECTIONS

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Summary In this article, using a conjecture of Grothendieck as focal point, we give a display of the interaction between various concepts from the geometry of Banach spaces. These concepts include tensor norms, the Banach-Mazur distance and uniformly complemented subspaces. The interaction is achieved with the aid of three powerful results:
(a) an inequality on bilinear forms due to Hardy and Littlewood,
(b) F. John's upper bound for projection norms,
and
(c) Dvoretzky's spherical sections theorem.

1. Tensor Products

For a vector space E, the tensor product $E \otimes E$ of E with itself consists of all finite sums of the form $\sum_i x_i \otimes y_i$. A. Grothendieck was mainly responsible for the development of a theory of tensor products in Banach spaces. He investigated norms on $E \otimes E$ satisfying

$$||x \otimes y|| = ||x|| \cdot ||y||. \qquad (*)$$

He observed that there is a largest norm π (and a smallest norm ε) which satisfies $(*)$, where, taking $z = \sum_i x_i \otimes y_i$,

$$||z||_\pi = \inf\{\sum_i ||x_i|| \cdot ||y_i||\}$$

and

$$||z||_\varepsilon = \sup\{|\sum_i \varphi(x_i)\psi(y_i)| : \varphi, \psi \in E', ||\varphi|| \leq 1, ||\psi|| \leq 1\}.$$

Since

$$|\sum_i \varphi(x_i)\psi(y_i)| \leq \sum_i ||\varphi|| \cdot ||x_i|| \cdot ||\psi|| \cdot ||y_i|| \leq \sum_i ||x_i|| \cdot ||y_i||,$$

we have

$$|| \cdot ||_\varepsilon \leq || \cdot ||_\pi. \qquad (**)$$

Grothendieck conjectured that $|| \cdot ||_\varepsilon$ and $|| \cdot ||_\pi$ were equivalent norms on E if and only if $\dim E < \infty$. Over the years it has been shown that Grothendieck's conjecture is true for large (and important) collections of Banach spaces. However, in 1983, G. Pisier, [2], showed that the conjecture is false in general. We show that the conjecture is true for Banach spaces which contain uniformly complemented l_p^n's and that any counterexample must contain a large number of badly located almost Euclidean subspaces.

A comprehensive study of Grothendieck's conjecture is given in G. Pisier, [3], and our results are special cases of results given there. The monograph [3] is extremely well written but technically demanding. Our aim in this article is to provide some insight for the non-expert.

2. An inequality of Hardy and Littlewood

Let j, k and n denote positive integers and let $\alpha_{j,k} = e^{2\pi ijk/n}$, where $i = \sqrt{-1}$. Let A denote the $n \times n$ matrix $(\alpha_{j,k})_{1\leq j,k \leq n}$. With A we can associate, in a canonical fashion, a bilinear form $\tilde{A}$ as follows:

$$\tilde{A}((x_j)_{j=1}^n, (y_j)_{j=1}^n) = \sum_{j,k} \alpha_{jk} x_j y_k.$$

Let

$$||\tilde{A}||_p = \sup\{|\sum_{j,k=1}^n \alpha_{jk} x_j y_k| : \sum_{j=1}^n |x_j|^p \leq 1, \sum_{k=1}^n |y_k|^p \leq 1\}.$$

Hardy and Littlewood, [1], proved that if

$$a(p) = \begin{cases} 3/2 - 2/p & \text{for } p \geq 2 \\ 1 - 1/p & \text{for } 1 \leq p \leq 2, \end{cases}$$

then there exists $c > 0$ such that

$$c \cdot n^{a(p)} \leq ||\tilde{A}||_p \leq n^{a(p)}$$

for all n and p.

If $z \in E \otimes E$, with $z = \sum_i x_i \otimes y_i$, we associate with z the bilinear form $\tilde{z}$ on $E' \otimes E'$ where E' denotes the space of all continuous real valued linear forms on E, by the formula

$$\tilde{z}(\varphi, \psi) = \sum_i < \varphi, x_i >< \psi, y_i >$$

and with this identification we have

$$||z||_\varepsilon = \sup\{|\tilde{z}(\varphi,\psi)| : \; ||\varphi|| \leq 1, ||\psi|| \leq 1\}.$$

For a positive integer n and $1 \leq p < \infty$, we let l_p^n denote $\mathbf{C}^n$ endowed with the norm

$$||(z_i)_{i=1}^n||_p = \left(\sum_{j=1}^n |z_j|^p\right)^{1/p}$$

and for $p = \infty$, we let l_∞^n denote $\mathbf{C}^n$ endowed with the supremum norm

$$||(z_i)_{i=1}^n||_\infty = \sup_i |z_i|.$$

For $1 \leq p < \infty$, let $\frac{1}{p'} = 1 - \frac{1}{p}$ and let $p' = 1$ when $p = \infty$. Translated into the language of tensor products the Hardy-Littlewood inequality says that

$$||\sum_{j,k=1}^n \alpha_{j,k} e_j \otimes e_k||_{l_{p'}^n \otimes_\varepsilon l_{p'}^n} \sim n^{a(p)},$$

where $e_j = (0, \cdots, 0, 1, 0, \cdots)$, the entry 1 occurring in the j-th position.

For all finite dimensional Banach spaces (and many infinite dimensional spaces) we have

$$(E \otimes_\pi E)' = E' \hat{\otimes}_\varepsilon E'$$

where $E'\hat{\otimes}_\varepsilon E'$ is the completion of $E' \otimes_\varepsilon E'$.

If $E = l_p^n$, $E' = l_{p'}^n$, and $1/p + 1/p' = 1$, then this duality is given by

$$< \sum_{j,k=1}^n a_{j,k} e_j \otimes e_k, \sum_{j,k=1}^n b_{j,k} e'_j \otimes e'_k >= \sum_{j,k=1}^n a_{j,k} b_{j,k},$$

where $(e'_j)_{j=1}^n$ is the standard dual basis to $(e_j)_{j=1}^n$, that is,

$$< e'_j, e_k >= \delta_{jk} \quad \text{(the Kronecker } \delta \text{ function)}.$$

Hence

$$n^2 = |< \sum_{j,k=1}^n \alpha_{j,k} e_j \otimes e_k, \sum_{j,k=1}^n \bar{\alpha}_{j,k} e'_j \otimes e'_k >|$$
$$\leq n^{a(p)} || \sum_{j,k=1}^n \bar{\alpha}_{j,k} e'_j \otimes e'_k ||_{l_{p'}^n \otimes_\pi l_{p'}^n}$$

and

$$|| \sum_{j,k=1}^n \alpha_{j,k} e_j \otimes e_k ||_{l_p^n \otimes_\pi l_p^n} \geq n^{2-a(p')}. \qquad (***)$$

To simplify our notation, we introduce the concept of tensorial diameter (td). For a Banach space E, the tensorial diameter of E, $td(E)$, is defined by

$$td(E) = \sup \frac{||z||_\pi}{||z||_\varepsilon}, \text{ where } z \in E \otimes E, \, z \neq 0.$$

By (**), $td(E) \geq 1$ and an infinite dimensional Banach space E is a counterexample to Grothendieck's conjecture if and only if $td(E) < \infty$. By $(***)$,

$$td(l_p^n) \geq \frac{||\sum_{j,k} \alpha_{j,k} e_j \otimes e_k||_{(l_p^n \otimes l_p^n, \pi)}}{||\sum_{j,k} \alpha_{j,k} e_j \otimes e_k||_{(l_p^n \otimes l_p^n, \varepsilon)}}$$
$$\geq n^{2-a(p)-a(p')} = n^{b(p)},$$

where

$$b(p) = \begin{cases} 3/2 - 1/p & \text{for } 1 \leq p \leq 2 \\ 1/2 + 1/p & \text{for } p \geq 2. \end{cases}$$

3. The Banach-Mazur distance

When two Banach spaces are isomorphic, the Banach-Mazur distance d measures how close they are isometrically. For isomorphic Banach spaces E and F

$$d(E,F) = \inf\{||T||\cdot||T^{-1}|| : T : E \to F \text{ is a linear isomorphism}\}.$$

The function $\log d$ is symmetric and obeys the triangle inequality. If E (and hence F) is finite dimensional then $d(E,F) = 1$ if and only if E and F are isometrically isomorphic.

Lemma 1. *If E and F are isomorphic Banach spaces then*

$$td(E) \leq (d(E,F))^2 \cdot td(F).$$

Proof: Every linear mapping $T : E \to F$ gives rise to a canonical linear mapping $T \otimes T : E \otimes E \to F \otimes F$, where

$$(T \otimes T)(x \otimes y) = Tx \otimes Ty$$

and moreover,

$$||T \otimes T||_\pi = ||T \otimes T||_\varepsilon = ||T||^2.$$

In addition, if T is a linear isomorphism then so also is $T \otimes T$ for both ε and π and

$$||(T \otimes T)^{-1}||_\pi = ||(T \otimes T)^{-1}||_\varepsilon = ||T^{-1}||^2.$$

Now suppose $T : E \to F$ is a linear isomorphism. For $z \in E \otimes E$

$$\begin{aligned} ||z||_\pi &= ||(T \otimes T)^{-1}(T \otimes T)(z)||_\pi \\ &\leq ||(T \otimes T)^{-1}||_\pi \cdot ||(T \otimes T)(z)||_\pi \\ &= ||T^{-1}||^2 \cdot ||(T \otimes T)(z)||_\pi. \end{aligned}$$

Moreover,

$$||(T \otimes T)(z)||_\varepsilon \leq ||T||^2 \cdot ||z||_\varepsilon$$

and hence, if $z \neq 0$,

$$\frac{1}{||z||_\varepsilon} \leq \frac{||T||^2}{||(T \otimes T)(z)||_\varepsilon}.$$

This implies

$$\frac{||z||_\pi}{||z||_\varepsilon} \leq \frac{||T||^2}{||(T \otimes T)(z)||_\varepsilon} \cdot ||T^{-1}||^2 \cdot ||(T \otimes T)(z)||_\pi.$$

By first taking the supremum with respect to z and then the infimum with respect to T we get the required result.

Lemma 2. *If E and F are Banach spaces and P is a projection of E onto F then*

$$td(F) \leq ||P||^2 \cdot td(E).$$

Proof: This is similar to the the proof of Lemma 1, using the easily verifiable fact that $||P \otimes P||_\pi \leq ||P||^2$.

We also require the following result of F. John: if F is a finite dimensional subspace of a Banach space E then there exists a projection P of E onto F such that

$$||P|| \leq \sqrt{\dim(F)}.$$

4. Local theory of Banach spaces

The study of the properties of the finite dimensional subspaces of a Banach space is known as the local theory of Banach spaces. This often leads to global results. For instance, if all the finite dimensional subspaces of a Banach space E are isometric to a Hilbert space, then E itself is a Hilbert space. The Dvoretzky spherical sections theorem says that for every $\varepsilon > 0$, every positive integer n and every infinite dimensional Banach space E, there exists an n-dimensional subspace F of E such that

$$d(F, l_2^n) \leq 1 + \varepsilon.$$

We say that a Banach space E contains l_p^n's uniformly if for every $\varepsilon > 0$ there exists $F_n \subset E$ such that

$$d(F_n, l_p^n) \leq 1 + \varepsilon.$$

E is said to contain l_p^n's uniformly complemented if, in addition, for each n there exists a projection $P_n : E \to E$ such that $P_n(E) = F_n$ and $||P_n|| \leq 1 + \varepsilon$. The infinite dimensional Banach space l_p contains l_p^n's uniformly complemented.

Proposition 3. *If for some p the Banach space E contains uniformly complemented l_p^n's, then E satisfies Grothendieck's conjecture.*

Proof: We have

$$td(E) \geq \limsup_n \frac{td(F_n)}{||P_n||^2} \geq \limsup_n \frac{td(l_p^n)}{||P_n||^2 d^2(F_n, l_p^n)}$$
$$\geq \limsup_n n^{b(p)} = \infty,$$

the first inequality holding by Lemma 2, the second by Lemma 1. This proves the proposition.

On the other hand the proof of the proposition above together with the precise growth rate of $td(l_p^n)$ as $n \to \infty$ shows what balance must be maintained between the Banach-Mazur distance and the projection norm in order to satisfy Grothendieck's conjecture. The proposition above also provides us with properties that any counterexample to the conjecture must satisfy. For example, the spherical sections theorem of Dvoretzky shows that any infinite dimensional Banach space contains l_2^n and by the result of F. John we can suppose that a projection P_n onto l_2^n has norm $\leq \sqrt{n}$. By the Hardy-Littlewood inequality we have $td(l_2^n) \sim n$. Hence if E is a counterexample to Grothendieck's conjecture then

$$\infty > td(E) \geq \limsup_n \frac{td(l_2^n)}{||P_n||^2} = \limsup_n \frac{n}{||P_n||^2}.$$

Hence there exists $c > 0$ such that $||P_n|| \geq c\sqrt{n}$. Thus l_2^n has asymptotically the worst possible projection norm and may be said to be badly located. G. Pisier, [3], showed that his counterexample $\tilde{E}$ has the following stronger property:
there exists $c > 0$ such that for any finite dimensional subspace F of $\tilde{E}$ and any projection P of $\tilde{E}$ on F

$$||P|| \geq c \cdot \sqrt{\dim(F)}.$$

References

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