MUSICAL SCALES

María José Garmendia Rodríguez
Juan Antonio Navarro González

Introduction

Our aim is to provide precise definitions of some musical concepts, mainly *tuning system* and *mode*, in order to begin a rigorous search of the theorems uncovered by the main scales currently used in Western art music.

We shall consider musical concepts that only depend on the tones of sounds, disregarding any other characteristic of sounds (timbre, volume, duration, ...). We shall ground our study on the structure of pitches, assuming that each musical pitch is fully determined by the frequency of vibration of the sound wave that produces it. Given a pitch t and a positive real number λ, we have a pitch λt whose frequency is the product of λ by the frequency of t and so we obtain a free and transitive action of the group of all positive real numbers on the pitches. This is the only structure of sounds that we shall consider and our first goal is to show how the concepts of musical scale and mode may be reduced to this simple structure.

Given a pitch t, the most consonant pitch is t itself, then $2t$, $3t$ and so on. At the basis of the whole theory is the natural identification between t and $2t$ that most men make unconsciously. Hence we consider families of pitches $\mathcal{S}$ such that $t \in \mathcal{S}$ implies:

(i) $2^n t$ belongs to $\mathcal{S}$ for any integer n.
(ii) only a finite number of elements $s \in \mathcal{S}$ satisfy $t \leq s < 2t$ (this number does not depend on t and it is said to be the number of notes of $\mathcal{S}$).

Two families $\mathcal{S}$, $\mathcal{S}'$ are said to be equivalent when $\mathcal{S}' = \lambda\mathcal{S}$ for some positive number λ and *tuning systems* or *scales* are defined to be equivalence classes.

Any finite family of pitches F, such as a melody or the keys of a piano, defines a scale $\mathcal{F} = \{2^n t : n \in \mathbf{Z},\, t \in F\}$. For example, the classical diatonic scale is defined by any geometric progression $t, 3t, \ldots, 3^6 t$, and it is a good scale from the melodic point of view. From the point of view of modulation good scales are tempered scales (scales defined by geometric progressions t, rt, $r^2 t$, ..., $r^n t = 2t$) and the scale defined by the keys of a piano is the tempered scale of 12 notes. From the point of view of harmony, one should like to have $3t$ and $5t$ in the scale whenever t is. Therefore (neglecting the temperament for the moment) we should look for a scale $\mathcal{S}$ such that $3t, \ldots, 3^6 t$ and $5t$ belong to $\mathcal{S}$ whenever $t \in \mathcal{S}$; but any one of these conditions contradicts the finiteness of the number of notes, so that no scale may fulfil them. However, men cannot distinguish two pitches when their frequencies are very close, so that a scale fulfilling these conditions up to a small error would be a perfect one for human hearing. We shall prove that any scale improving the error of the tempered scale of 12 notes must have 16 or more notes. Even if one disregards modulation, the usual tempered scale is the best scale (from the point of view of melody and harmony) with less than 16 notes.

1. Tuning Systems

Given a pitch t and a positive real number λ, we shall denote by λt the pitch whose frequency is the product of λ by the frequency of t; so that $1t = t$ and $\lambda(\mu t) = (\lambda\mu)t$. Moreover, given two pitches s and t, there exists a unique positive real number λ such that $t = \lambda s$ and this number λ is said to be the *interval* from s to t. This is the only structure of sounds that we shall consider, so that our starting point is the following definition of the structure of tones of sounds:

Definition: Any set $\mathcal{P}$ endowed with a free transitive action of the (multiplicative) group $\mathcal{I}$ of all positive real numbers is said to be a *system of pitches*. (By a free transitive action, we mean any map $\mathcal{I} \times \mathcal{P} \to \mathcal{P}, (\lambda, t) \mapsto \lambda t$, such that $1t = t$, $\lambda(\mu t) = (\lambda\mu)t$ and such that, for any pair $t, s \in \mathcal{P}$, we have $s = \lambda t$ for a unique $\lambda \in \mathcal{I}$.) The elements of $\mathcal{P}$ are said to be *pitches* and the elements of $\mathcal{I}$ are said to be *intervals*.

We shall always consider the usual order on $\mathcal{I} = \mathbf{R}_+$, so that pitches inherit an order: $s \leq t$ when $t = \lambda s$, $\lambda \geq 1$.

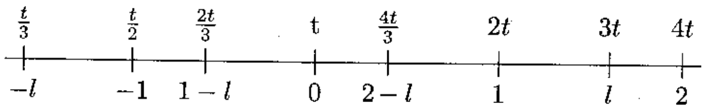
Some important intervals have a proper name: 2 is the *octave* (so that between the pitches t and $2^\alpha t$ there are α octaves), $3/2$ is the *perfect fifth* and $5/4$ is the *major third*. The basis of any tuning system is the identification between sounds forming octaves, so that we consider the subgroup $2^{\mathbf{Z}} = \{\lambda \in \mathcal{I} : \lambda = 2^n, n \in \mathbf{Z}\}$.

Definition: The quotient set $\mathcal{O} = \mathcal{P}/2^{\mathbf{Z}}$ is said to be the *Octave*, so that $\mathcal{I}$ acts transitively on the Octave and the quotient group $\mathcal{I}/2^{\mathbf{Z}}$ acts transitively and freely on $\mathcal{O}$.

Geometric representation:

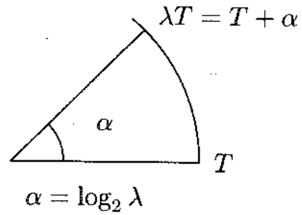
We denote pitches by Latin letters and their projection on the Octave by the corresponding capital letter.

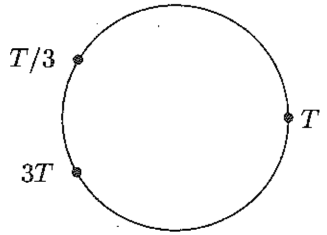
If a pitch t is fixed, then pitches correspond with positive real numbers, but this representation takes octaves into segments of different lengths. To avoid this problem it is convenient to use an additive notation; hence, we represent the pitch λt by the real number $\alpha = \log_2 \lambda$, so that the interval from t to λt is represented by a segment of length α. We put $t + \alpha$ instead of λt when this additive notation is used ($+\alpha$ is translation by α octaves). For example, if $l = \log_2 3$, we have:



When the group $\mathcal{I}$ is identified with $\mathbf{R}$ via $\log_2$, the subgroup $2^{\mathbf{Z}}$ is identified with $\mathbf{Z}$, so that $\mathcal{I}/2^{\mathbf{Z}}$ is isomorphic to $\mathbf{R}/\mathbf{Z}$. Therefore, we may represent the Octave by the points of a circle and it is quite natural to fix the length of this circle as the unit of length

and to measure angles by octaves (i.e., complete turns, so that the angle α has $2\pi\alpha$ radians):





This geometric representation of the Octave allows us to define the *distance* between two elements of $\mathcal{O}$ as the distance of their corresponding points in the circle.

Note that the order of $\mathcal{P}$ defines an order on the complement $\mathcal{O} - T$ of any element $T \in \mathcal{O}$, so that any finite subset of the Octave inherits a "circular order" (we always represent it in the counter-clockwise sense)

Definition: Two finite subsets S and S' of the Octave $\mathcal{O}$ are said to be *equivalent* if $S' = \lambda S$ for some interval λ (if there exists a rotation of the Octave transforming S into S'). Equivalence classes of finite subsets of $\mathcal{O}$ are said to be *tuning systems* or *scales*. The *number of notes* of a scale is the common cardinal number of all finite subsets of $\mathcal{O}$ representing it.

By definition, a scale $\mathbf{S}$ may be represented by a finite subset S of the Octave (whose elements are said to be *notes*) or by a family of pitches $\mathcal{S}$ with the following property: if $t \in \mathcal{S}$, then $2^n t \in \mathcal{S}$ for all $n \in \mathbf{Z}$ and there is a finite number of elements of $\mathcal{S}$ between t and $2t$. Two such families $\mathcal{S}$ and $\mathcal{S}'$ define the same scale when $\mathcal{S}' = \lambda\mathcal{S}$ for some interval λ. Moreover, any finite family of pitches, such as the keys of a piano or a melody, define a scale when we project it on the Octave.

Take a finite subset S of the Octave representing a given scale $\mathbf{S}$ of n notes and let us consider the circular order $(T_1, \ldots, T_n)$ of its notes. Then we get an n-cycle $(\alpha_1, \ldots, \alpha_n)$ of intervals (in fact of elements of $\mathcal{I}/2^{\mathbf{Z}}$), where $T_{i+1} = T_i + \alpha_i$. This n-cycle $(\alpha_1, \ldots, \alpha_n)$ only depends on the scale $\mathbf{S}$ and it is called the *symbol* of $\mathbf{S}$ because it is clear that any scale is determined by its symbol.

The symbol of a scale is not an arbitrary cycle of intervals because we have $\alpha_1 + \ldots + \alpha_i < 1$ when $i < n$ and $\alpha_1 + \ldots + \alpha_n = 1$ (identifying $\mathcal{I}/2^{\mathbf{Z}}$ with $(0, 1]$).

Modes: A finite subset M of the group $\mathcal{I}/2^{\mathbf{Z}}$ is said to be a *mode* if it contains the neutral element 1.

Each mode M defines, once you fix a pitch t, a finite set $Mt = \{\lambda T : \lambda \in M\}$ in the Octave; hence M defines a scale, since $M\lambda t = \lambda(Mt)$. Conversely, given a scale represented by a finite subset S of the Octave, each note $T \in S$ defines a mode $M = \{\lambda \in \mathcal{I}/2^{\mathbf{Z}} : \lambda T \in S\}$ but this mode depends on the note T. Each scale of n notes defines, in general, n different modes.

Since $\mathcal{I}/2^{\mathbf{Z}} \cong \mathbf{R}/\mathbf{Z} \cong [0, 1)$, every mode M is a sequence $0 = m_1 < m_2 < \ldots < m_n < 1$, so that M is determined by the sequence $\alpha_1, \ldots, \alpha_n$ where $\alpha_i = m_{i+1} - m_i$ and $\alpha_n = 1 - m_n$. The symbol of the scale defined by M is just $(\alpha_1, \ldots, \alpha_n)$. Conversely, the modes defined by the scale of symbol $(\alpha_1, \ldots, \alpha_n)$ are just the modes corresponding to the n sequences:

$$\alpha_1, \ldots, \alpha_{n-1}, \alpha_n$$
$$\alpha_2, \alpha_3, \ldots, \alpha_n, \alpha_1$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$\alpha_n, \alpha_1, \ldots, \alpha_{n-1}$$

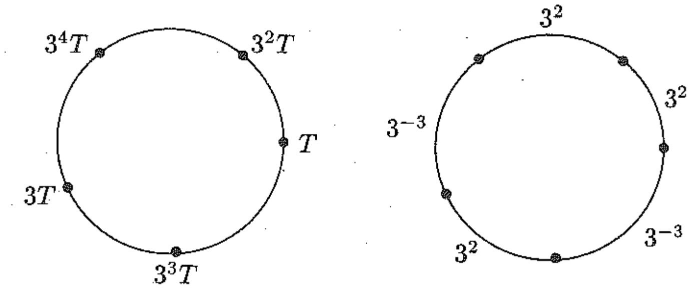
Tempered scales: A scale is said to be *tempered* if it divides the Octave in equal parts; that is to say, the symbol of the tempered scale of n notes is $(1/n, \ldots, 1/n)$.

The scale defined by the sounds of a piano is the tempered scale of 12 notes. The reader may obtain the symbol of the scale of 7 notes defined by the white keys and the corresponding 7 modes.

Scales of fifths: The *scale of fifths* of n notes is the scale defined by any geometric progression of ratio 3 and n terms, $n \geq 2$. It is the scale represented by $\{T, 3T, \ldots, 3^{n-1}T\}$. In this scale every

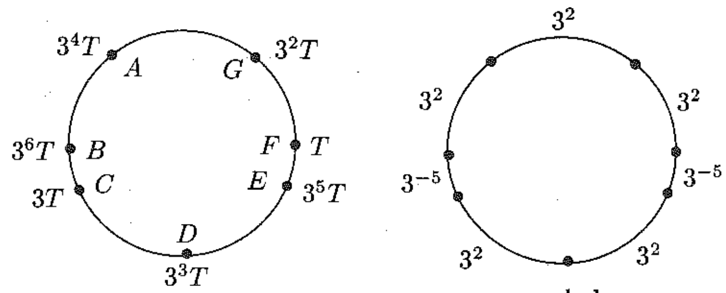
note S, except for $3^{n-1}T$, has its perfect fifth $3S$, but no one has its major third $5S$. Scales of fifths cannot be tempered because 3^n is not a power of 2.

When $n = 5$, one finds the *pentatonic* scale, frequently used in folk music (according to the *Britannica*, the pentatonic scale is used more widely than any other scale and Western art music is one of the few traditions in which pentatonic scales do not predominate):



symbol

When $n = 7$, one obtains the classical *diatonic* scale (the traditional name of each note figures inside the circle):



symbol

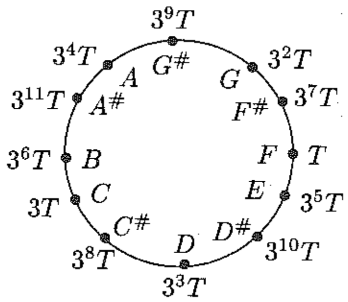
$$Do = C, Re = D, Mi = E, Fa = F, Sol = G, La = A, Si = B$$

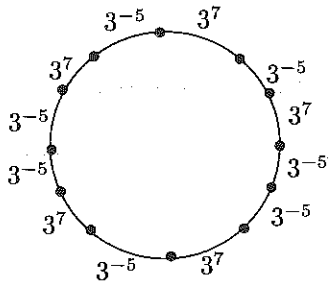
and each note of this scale defines one of the seven classical modes:

Major C	1	3^2	3^4	3^{-1}	3	3^3	3^5	Ionian
D	1	3^2	3^{-3}	3^{-1}	3	3^3	3^{-2}	Dorian
E	1	3^{-5}	3^{-3}	3^{-1}	3	3^{-4}	3^{-2}	Phrygian
F	1	3^2	3^4	3^6	3	3^3	3^5	Lydian
G	1	3^2	3^4	3^{-1}	3	3^3	3^{-2}	Mixolydian
Minor A	1	3^2	3^{-3}	3^{-1}	3	3^{-4}	3^{-2}	Aeolian
B	1	3^{-5}	3^{-3}	3^{-1}	3^{-6}	3^{-4}	3^{-2}	Hypophrygian

The last one is rarely used because it does not contain the perfect fifth (= 3). On the other hand, no mode contains the major third (= 5), but the major mode contains $3^4 = 81/64 = 1.265625$ which is much closer to $5/4 = 1.25$ than the interval $3^{-3} = 32/27 = 1.185$ of the minor mode.

When $n = 12$, one obtains the *chromatic* scale (a sharped note $S^\#$ denotes 3^7S and a flatted note S^b denotes $3^{-7}S$):





symbol

In this scale $A^\#$ is the unique note without perfect fifth in the scale. In fact, the distance from F to $3A^\#$ is about 0.02. No note has its major third in this scale. The distance from $5S$ to the closest note in the scale is about 0.018 when S=A, B, C, D, E, F, G, $F^\#$ and it is about 0.002 when S=$A^\#$, $C^\#$, $D^\#$ and $G^\#$. The temperament of this scale is quite good, since its symbol only has two different intervals (3^7 and 3^{-5}) of similar length: the length of 3^7 is about 0.095 and the length of 3^{-5} is about 0.075.

The figure above shows that scales of fifths of 5 and 7 notes also have symbols with only two different intervals, while scales of fifths of 4, 6, 8, 9, 10 and 11 notes do not have this property. A scale of fifths is said to be *pythagorean* when its symbol has only two different intervals.

Theorem 1. *The numbers of notes of the pythagorean scales form the following sequence* (a_i)

$$2, \underbrace{3,5}_{2}, \underbrace{7,12}_{2}, \underbrace{17,29,41}_{3}, 53, \underbrace{94,\ldots,306}_{5}, \underbrace{359,665}_{2}, \underbrace{971,\ldots,15601}_{23}, \ldots$$

where $a_{i+1} = a_i + b_i$ *and* b_i *is the term preceding the group of* a_i. *Moreover, the lengths of these groups are the terms (or partial quotients) of the continued fraction*

$$\frac{\log_2(3/2)}{\log_2(4/3)} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{\cdots}}}}}}$$

Furthermore, if $\alpha_1 > \alpha_2$ *are the lengths of the two intervals of the symbol of the pythagorean scale of* a_i *notes, then the lengths of the two intervals of the next one (the scale of* a_{i+1} *notes) are* α_2 *and* $\alpha_1 - \alpha_2$, *which is the distance of* $3^{a_i}T$ *to* T.

Note 1.1: The first 36 terms of the continued fraction above are: $1, 2, 2, 3, 1, 5, 2, 23, 2, 2, 1, 1, 55, 1, 4, 3, 1, 1, 15, 1, 9, 2, 5, 7, 1, 1, 4, 8, 11, 1, 20, 2, 1, 10, 1, 4, \ldots$

It is has been well-known for a long time that continued fractions are closely related to musical scales (see [1], [2], [6], [7] and [8]) and the theorem above shows a new relation.

If l_1, l_2 are two positive real numbers, then the terms r_1, r_2, r_3, …, of the continued fraction l_1/l_2 have the following geometric interpretation: if I_1, I_2 are two segments of lengths l_1, l_2 in a straight line, then I_2 fits r_1 times in I_1 and there remains a smaller segment I_3, then I_3 fits r_2 times in I_2 and there remains a smaller segment I_4, then I_4 fits r_3 times in I_3, and so on.

Since $\log_2(3/2)$ and $\log_2(4/3)$ are the lengths of the two intervals of the pythagorean scale of 2 notes, we conclude that

(1.2): *The last terms of the groups of the sequence (a_i) above are the natural numbers n such that 3^nT it is closer to T than 3^mT for any number $1 \leq m < n$ and they correspond to the pythagorean scales such that the smaller interval does not fit twice in the bigger one.*

The symbol of a pythagorean scale only has two different intervals $\alpha_1 > \alpha_2$, but it may be that this scale is highly non-tempered because α_1 may be much bigger than α_2. A pythagorean scale is said to be *quasi–tempered* when α_1 is smaller than $2\alpha_2$ and, by 1.2, these scales correspond to the last terms of the groups of the sequence (a_i) above. According to the geometric interpretation of the terms of a continued fraction, in such case the difference of the two intervals fits $r(n)$ times in α_2, where $r(n)$ denotes the length of the group following n; so that the scale has "better temperament" when $r(n)$ is bigger. From 1 and 1.1, one directly obtains the numbers of notes of the first 36 quasi-tempered pythagorean scales:

(1.3) :

$$\begin{array}{rcllllllllllll} n &=& 2 & 5 & 12 & 41 & 53 & 306 & 665 & 15601 & 31867 & 79335 & 111202 & 190537 \\ r(n) &=& 2 & 2 & 3 & 1 & 5 & 2 & 23 & 2 & 2 & 1 & 1 & 55 \end{array}$$

We see that the first one improving the temperament of the chromatic scale has 53 notes and that the scale of 665 notes has a very good temperament. Moreover, (1.1) shows that the scale of 190537 notes has the best temperament among the first 36 terms. Since the 36th term is easily estimated to be greater than 10^{18}, any pythagorean scale improving the temperament of the scale of 190537 notes must have more than 10^{18} notes. However, if one looks for a scale improving the temperament more than it increases the number of notes, that is to say a number $n \geq 13$ such that

$$\frac{r(12)}{12} < \frac{r(n)}{n} \qquad \text{or} \qquad \frac{n}{4} < r(n)$$

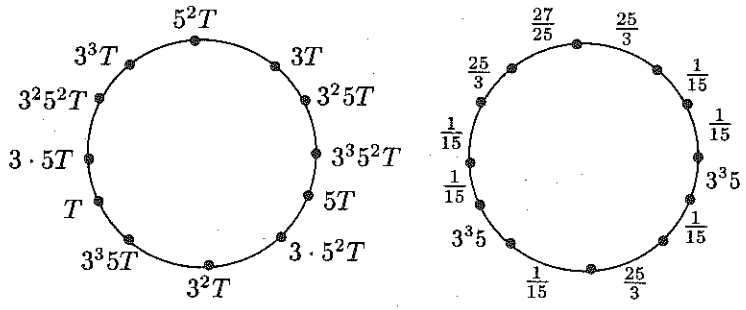
then (1.1) shows that the first 36 terms of this sequence do not fulfil this condition. We should wonder at the existence of a quasi–tempered pythagorean scale of more than 12 notes such that $r(n)/n$ is bigger than $r(12)/12$; but, unfortunately, we have no evidence to conjecture that no pythagorean scale with more than 12 notes improves the temperament more than it increases the number of notes (remark that $r(n)/n$ is not a decreasing sequence).

Euler's construction: The divisors of a given natural number form a mode, hence they define a scale. Euler considered the scale defined by the divisors of $d = 3^a5^b$. It is a scale of $(a+1)(b+1)$ notes such that any note, except 3^a5^iT, has its perfect fifth and any note, except 3^j5^bT, has its major third. These scales cannot be tempered, but Euler remarked that the scale of 12 notes corresponding to $d = 3^35^2 = 675$ is quite close to the chromatic scale and to the tempered scale of 12 notes:

Eulerian scale for 675											
1	3^35	3^2	35^2	5	3^35^2	3^25	3	5^2	3^3	3^25^2	$3 \cdot 5$
1	1.055	1.125	1.17	1.25	1.318	1.406	1.5	1.562	1.69	1.76	1.875
Pythagorean scale of 12 notes											
1	3^7	3^2	3^9	3^4	3^{-1}	3^6	3	3^8	3^3	3^{10}	3^5
1	1.068	1.125	1.2	1.27	1.333	1.424	1.5	1.602	1.69	1.8	1.898
Tempered scale of 12 notes											
1	r	r^2	r^3	r^4	r^5	r^6	r^7	r^8	r^9	r^{10}	r^{11}
1	1.059	1.122	1.19	1.26	1.335	1.414	1.498	1.588	1.68	1.78	1.888

$r = 2^{1/12}$

(Instead of the scales, this table shows one mode of each scale. For the chromatic scale, it is the mode corresponding to *C=Do*.) The symbol of the eulerian scale is



symbol

In this scale, the distance from $3\cdot3^3T$ to the closest note (in fact $5T$) is about 0.018 and the distance from $5\cdot5^2T$ to the closest note (in fact T) is about 0.034.

2. Approximation Indices

We should like to have "ideal scales" containing the diatonic scale T, $3T$, $\ldots$, 3^6T based on any note T of the scale, as well as its major third $5T$. Clearly no scale may satisfy these conditions (since no power of 2 is a power of 3 or 5); but we may look for scales fulfilling them approximately.

Definition: Let us consider a scale $\mathbf{S}$ and let S be a finite subset of the Octave representing $\mathbf{S}$. We define $\alpha_3(\mathbf{S})$ to be the infimum of all real numbers α such that the distance of $3T$ to S is $\leq\alpha$ for any $T\in S$ (hence the perfect fifth of any note in S is in S up to an error bounded by $\alpha_3(\mathbf{S})$). We define $\alpha_5(\mathbf{S})$ to be the infimum of all real numbers α such that the distance of $5T$ to S is $\leq\alpha$ for any $T\in S$. We define the *approximation index* of the scale $\mathbf{S}$ to be $\alpha(\mathbf{S})=\max\{6\alpha_3(\mathbf{S}),\alpha_5(\mathbf{S})\}$.

By definition, if $T\in S$, then $3T,\ldots,3^6T$ and $5T$ may be replaced by notes of S at a distance $\leq\alpha(\mathbf{S})$. Therefore, if $\alpha(\mathbf{S})$ were smaller than the human perception of acoustic pitch differences, then $\mathbf{S}$ could be considered as a human realization of the "impossible ideal scale". In fact it seems that most men cannot differentiate two pitches when the distance is smaller than 0.003 (3 thousandths of the octave).

Let us consider the tempered scale $\mathbf{T}_n$ of n notes. This scale is represented by all rational numbers with denominator n. Hence, if a/n is the best approximation of $\log_2 3$ with denominator n, then we have $\alpha_3(\mathbf{T}_n)=|\log_2 3-a/n|$. Therefore:

$$n\cdot\alpha_3(\mathbf{T}_n)=\{n\cdot\log_2 3\}$$
$$n\cdot\alpha_5(\mathbf{T}_n)=\{n\cdot\log_2 5\}$$

where $\{x\}$ denotes the distance of x to the closest integer number. For the 13 first tempered scales we obtain

n	2	3	4	5	6	7	8	9	10	11	12	13
α_3	85	82	85	15	82	13.5	40	29.4	15	39.5	1.63	30.4
α_5	178	14	72	78	11.4	36	53	11.4	22	42	11.4	14
α	510	490	510	90	490	81.2	240	176	90	237	11.4	182
$n\alpha_3$	170	245	340	75	490	94.5	320	265	150	435	19.6	395

where all the values are given in thousandths of octave, so that $\alpha(\mathbf{T}_{12})\cong 0.0114$. We may see that the approximation index of $\mathbf{T}_{12}$ is much better than the indices of the previous tempered scales. The first tempered scales improving the index α_3 of $\mathbf{T}_{12}$ are $\mathbf{T}_{29}$ and $\mathbf{T}_{41}$, and the first tempered scale improving the index α of $\mathbf{T}_{12}$ is $\mathbf{T}_{41}$. In fact we have:

$$\alpha_3(\mathbf{T}_{41})\cong 0.0004,\quad \alpha(\mathbf{T}_{41})=\alpha_5(\mathbf{T}_{41})\cong 0.0049$$

so that $\alpha(\mathbf{T}_{41})$ is quite close to the human perception of tonal differences.

Moreover, the tempered scale of 12 notes has a very good approximation index even if we consider arbitrary tuning systems. Any scale improving the index of $\mathbf{T}_{12}$ has more than 15 notes:

Theorem 2. *If the approximation index of a musical scale* $\mathbf{S}$ *is smaller than the approximation index of the tempered scale of 12 notes, then the number of notes of* $\mathbf{S}$ *is greater than 15.*

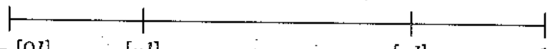
Proof of Theorem 1

If x is a real number, we shall denote by $[x]$ the decimal part of x; that is to say, $0 \leq [x] < 1$ and $x = n + [x]$ for some integer n, so that $[x]$ may be considered as the image of x in $\mathbf{R}/\mathbf{Z}$.

We shall always identify the Octave with the interval $[0, 1]$, where the end points 0 and 1 are identified. Hence, the scale of fifths of n notes is represented by $0 = [0l], [l], [2l], \ldots, [(n-1)l]$, where $l = \log_2 3$. In fact, we shall only use that l is an irrational number.

Let l be a fixed irrational number and let n be a natural number, $n \geq 2$.

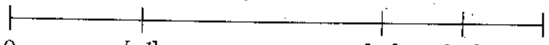
If we consider the arithmetic progression $0l, l, \ldots, (n-1)l$ of n terms and we consider the increasing order of $[0l], [l], \ldots, [(n-1)l]$, we obtain a partition of the interval $[0, 1]$:



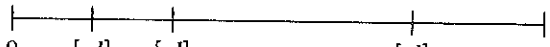
with an initial interval of length $[pl]$ and a final interval of length $[-ql] = 1 - [ql]$. This arithmetic progression of n terms is said to be *pythagorean* when the distance between two consecutive points of the partition is $[pl]$ or $[-ql]$. A pythagorean arithmetic progression is said to be *quasi-tempered* when the smaller interval is greater than a half of the bigger one.

Lemma. *If the arithmetic progression of n terms is pythagorean, then the corresponding partition of $[0, 1]$ has q intervals of length $[pl]$ and p intervals of length $[-ql]$, so that $p + q = n$. Moreover*

1) *If the initial interval is smaller than the final one, $[pl] < [-ql]$, then the next pythagorean progression has $n + p$ terms and the initial and final intervals of the corresponding partition are*



2) *If the final interval is smaller than the initial one, $[-ql] < [pl]$, then the next pythagorean progression has $n + q$ terms and the initial and final intervals of the corresponding partition are*



Proof: First we prove 1 and 2 assuming that the corresponding partition has q intervals of length $[pl]$ and p intervals of length $[-ql], n = p + q$; so that the p intervals of length $[-ql]$ are just the intervals from $[(q+i)l]$ to $[il], 0 \leq i < p$, and the q intervals of length $[pl]$ are just the intervals from $[jl]$ to $[(j+p)l], 0 \leq j < q$ (remark that the two ends of any interval of length $[pl]$ or $[-ql]$ are consecutive points of the partition because no integer multiple of $[pl]$ may coincide with $[-ql]$ and no integer multiple of $[-ql]$ may coincide with $[pl]$, since l is assumed to be irrational).

We shall only consider the first case, the other being quite similar. In this case we have $[ql] + [pl] < 1$, so that $[nl] = [pl] + [ql]$ lies between $[ql]$ and 1; hence $[nl]$ divides the final interval in two intervals of lengths $[pl] = [nl] - [ql]$ and $[-nl]$. For the same reason, $[(n+i)l]$ divides the interval from $[(q+i)l]$ to $[il], 0 \leq i \leq p-1$, in two intervals of lengths $[pl]$ and $[-nl]$. Therefore, the next pythagorean progression has $n + p$ terms, the length of the initial interval is again $[pl]$ and the length of the final interval is just $[-nl]$. Hence assertion 1 is proved. Moreover, we obtain that the pythagorean progression of $n + p$ terms has p intervals of length $[-nl]$ and n intervals of length $[pl]$.

We conclude the proof of this lemma by induction on n, since it is obvious when $n = 2$.

Corollary. *If $\alpha_1 > \alpha_2$ are the lengths of the intervals of the partition of $[0, 1]$ defined by a pythagorean progression of n terms, then we have:*

1) *the lengths of the intervals of the next pythagorean progression are α_2 and $\alpha_1 - \alpha_2$;*

2) *the distance from $[nl]$ to $0 = 1$ is just $\alpha_1 - \alpha_2$;*

3) If $r\alpha_2 < \alpha_1 < (r+1)\alpha_2$, then the consecutive pythagorean progressions have

$$n+p,\dots,n+rp \text{ terms when } [pl] < [-ql]$$

$$n+q,\dots,n+rq \text{ terms when } [-ql] < [pl].$$

Corollary. *If $[-l] < [l]$ and $r_0, r_1, r_2, r_3, \dots$ are the partial quotients of the continued fraction of $[l]/[-l]$, then the numbers of terms of the pythagorean arithmetic progressions defined by l are*

$$\underbrace{2,\dots,r_0+1}_{r_0}, \underbrace{a_1=r_0+2,\dots,a_{r_1}}_{r_1}, \underbrace{a_{r_1+1},\dots,a_{r_1+r_2}}_{r_2},\dots$$

where $a_{i+1}=a_i+b_i$ and b_i is the term preceding the group of a_i.

Proof: According to the geometric interpretation of the partial quotients of a continued fraction, this result follows directly from the corollary above, since the first pythagorean progression has 2 terms and the lengths of the corresponding intervals are just $[l]$ and $[-l]$.

Corollary. *The last terms of the groups of the sequence above are the natural numbers n such that $[nl]$ is closer to 0 than $[ml]$ for any $1 \le m < n$ and they correspond to the numbers of terms of the quasi-tempered pythagorean progressions.*

Proof of Theorem 2

Let $\alpha = \alpha(\mathbf{T}_{12}) = \frac{4}{12} - \log_2(\frac{5}{4})$ and $\beta = \log_2(\frac{3^4}{2^4 5})$ be the approximation of the major third in $\mathbf{T}_{12}$ and the chromatic scale respectively (the interval 81/80 from $5T$ to 3^4T is usually named the *comma*).

(1) *The distance between any two notes of the chromatic scale is larger than $5\alpha+\beta$ and 6α.*

In fact, the smallest interval between two notes of the chromatic scale is $2^8/3^5$. Since it is easy to check that $\alpha < \beta$, we must show that $2^8/3^5$ is greater than

$$\left(\frac{2^{4/12}4}{5}\right)^5 \cdot \frac{3^4}{2^4 5},$$

or, equivalently, $3^{27} < 2\cdot 5^{18}$ and this inequality may be tested directly.

Now, let $\mathbf{S}$ be a scale such that $\alpha(\mathbf{S}) < \alpha$ and let S be a subset of the Octave representing $\mathbf{S}$. If $T \in S$, then there are notes $T_1,\dots,T_6$ in S approaching $3T,\dots,3^6T$ more than α. There are notes $T_7,\dots,T_{11}$ in S approaching $3T_6,\dots,3^6T_6$ more than α; hence approaching $3^7T,\dots,3^{11}T$ more than 2α. Therefore, the notes in $F=\{T=T_0,T_1,\dots,T_{11}\}$ approach the notes of the chromatic scale $\{T,3T,\dots,3^{11}T\}$ with an error $< 2\alpha$.

(2) *The distance between T_i and T_j is larger than 2α when $i \neq j$. In particular F has 12 different notes.*

Otherwise, the distance from 3^iT to 3^jT would be bounded by 6α, contradicting (1).

(3) *The distance from T_j to $5T_i$ is larger than α when $j \neq i+4$.*

Otherwise, the distance from 3^jT to $3^{4+i}T$ would be bounded by $d(3^jT,T_j)+d(T_j,5T_i)+d(5T_i,3^4T_i)+d(3^4T_i,3^43^iT) \le 2\alpha+\alpha+\beta+2\alpha$, contradicting (1).

Finally, we compare the 12 intervals of F with the equal intervals of $\mathbf{T}_{12}$. Let $a_1,\dots,a_{12}$ be the differences with 1/12 of the lengths of the 12 consecutive intervals that F defines in the Octave, so that $a_1+\dots+a_{12}=0$.

If $a_j,a_{j+1},a_{j+2},a_{j+3}$ are the differences corresponding to the four intervals following a note $T_i \in F$, then the end of the fourth interval is just T_{i+4} (the index must be considered modulo 12, see the symbol of the chromatic scale). Hence:

$$T_{i+4} = T_i + a_j + \dots + a_{j+3} + 4/12$$

and we obtain that the distance from T_{i+4} to $5T$ is exactly $\alpha + a_j + \dots + a_{j+3}$, because the distance from $5T$ to $T_i + 4/12$ is

$\alpha(\mathbf{T}_{12}) = \alpha$. If $a_j + \ldots + a_{j+3} \geq 0$, by (3) we conclude that no note in F approaches $5T_i$ more than α, so that the note $S_i \in S$ approaching $5T_i$ more than α does not belong to F. Now, each one of the following additions ($i = 1, 2, 3, 4$)

$$(*) \quad (a_i + \ldots + a_{i+3}) + (a_{i+4} + \ldots + a_{i+7}) + (a_{i+8} + \ldots + a_{i+11}) = 0$$

has at least one non-negative addend. So we obtain four different notes $T^1, T^2, T^3, T^4 \in F$ such that the notes $S^i \in S$ approaching $5T^i$ more than α do not belong to F. We conclude that S contains at least 16 different notes $T_0, T_1, \ldots, T_{11}, S^1, \ldots, S^4$ because we have $S^i \neq S^j$ when $i \neq j$. Otherwise, the distance from $5T^i$ to $5T^j$ would be smaller than 2α, contradicting (2).

Note: Let S represent a scale $\mathbf{S}$ such that $\alpha(\mathbf{S}) = \alpha(\mathbf{T}_{12})$. The argument above also shows that $F \subseteq S$ has 12 different notes. Hence, if $\mathbf{S}$ only has 12 notes, then $F = S$ and $\alpha_5(\mathbf{S}) > \alpha$ when some addend $a_j + \ldots + a_{j+3}$ is positive. By (*), it follows that $a_j + \ldots + a_{j+3} = 0$ for any index j and $\alpha_5(\mathbf{S}) = \alpha = \alpha_5(\mathbf{T}_{12})$. Now, it is easy to prove that the closest note to $3T_i$ in $F = S$ is just T_{i+1}, the end of the seventh interval following T_i; so that $\alpha_3(\mathbf{S}) \geq \alpha_3(\mathbf{T}_{12}) - a_{j+4} - a_{j+5} - a_{j+6} = \alpha_3(\mathbf{T}_{12}) + a_{j+7}$. Therefore, if $\alpha_3(\mathbf{S}) \leq \alpha_3(\mathbf{T}_{12})$, then $a_j \leq 0$ and we conclude that $a_j = 0$ for any index j; that is to say, $\mathbf{S} = \mathbf{T}_{12}$. Resuming, the tempered scale of 12 notes is the best tuning system with no more than 12 notes in a very precise sense:

Suppose that $n \leq 12$. If $\mathbf{S}$ is a scale of n notes such that $\alpha(\mathbf{S}) \leq \alpha(\mathbf{T}_{12})$, then $n = 12$, $\alpha(\mathbf{S}) = \alpha(\mathbf{T}_{12})$, $\alpha_5(\mathbf{S}) = \alpha_5(\mathbf{T}_{12})$, $\alpha_3(\mathbf{S}) \geq \alpha_3(\mathbf{T}_{12})$ and the last inequality is strict when $\mathbf{S} \neq \mathbf{T}_{12}$.

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M. J. Garmendia Rodríguez,
I. N. B. A. D.

J. A. Navarro González,
Dpto. de Matemáticas,
Universidad de Extremadura,
06071–Badajoz,
Spain.