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IRISH MATHEMATICAL SOCIETY
BULLETIN

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The aim of the Bulletin is to inform Society members about the activities of the Society and about items of general mathematical interest. It appears twice each year, at Easter and at Christmas. The Bulletin is supplied free of charge to members; it is sent abroad by surface mail. Libraries may subscribe to the Bulletin for IR£20.00 per annum.

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Minutes of the Meeting of the Irish Mathematical Society

Ordinary Meeting
11th April 1995

The Irish Mathematical Society held an ordinary meeting at 12.15pm on Tuesday 11th April 1995 in the Dublin Institute for Advanced Studies, 10 Burlington Road. 11 members were present. The President, D. Hurley, was in the chair. Apologies were received from Pauline Mellon.

1. The minutes of the 21st December 1994 meeting were approved and signed.

2. **Matters arising**

Gordon Lessells reported that the current (Easter 1995) issue of the I.M.S. bulletin is now available for circulation. It was noted that the bulletin is now completely up–to–date and on time. Members were urged to submit articles for the next issue. It was suggested that the talks at the September meeting could be a source of articles for the bulletin.

3. **Treasurer's business**

The Treasurer circulated an interim report on the state of the society's finances. Additional postage costs of £400 for the bulletin were added. It was agreed that the I.M.S. is *pro tem* no longer in a position to offer conference support. When the financial situation of the I.M.S. improves, such support may be resumed. The projected deficit for 1995 is £550. The issue of the societies' subscription to the European Mathematical Society was raised in this regard. There are presently six I.M.S. members also in the E.M.S. It was reported that individuals have an alternative route to membership of the E.M.S. through the London Mathematical Society. A motion was proposed by Diarmuid Ó Mathúna and seconded by Séan Tobin that the I.M.S. explain its financial difficulties to the E.M.S. and offer 120 ECU as our annual subscription. The motion was passed.

4. Membership fees

The committee proposed to the meeting an increase in the annual membership fees as follows:

Ordinary membership £15 (or $25 U.S.).
Student membership £6.
Institutional membership £50 (unchanged).
Reciprocity membership unchanged.

A motion was proposed by Diarmuid Ó Mathúna and seconded by Timothy Murphy that a new category of membership, *viz.* life membership, be introduced concurrently with these changes and that this be set at £200. The meeting voted to approve this motion and the changes above, to be implemented as from 1st January 1996. It was noted that members would require good advance notice of these changes to facilitate changes in standing orders *etc.*

5. September meeting 1995

Eugene Gath reported that the planning of the 1995 September Meeting at University of Limerick on 7th and 8th September is well under way. Most of the principal speaker have been invited. The conference dinner will take place at Gooser's Eating House in Killaloe, Co. Clare and this will be preceded by a cruise on Lough Derg. It was suggested that Belfast be the venue for the 1996 September Meeting. Séan Dineen undertook to discuss this with members at Queen's University, Belfast when he visits there in June.

6. Mathematics education

It was noted that Elizabeth Oldham gave an invited address to the I.M.S. at the Dublin Institute of Advanced Studies on the morning of 10th April 1995. The talk, and the discussion which followed, were agreed to have been stimulating and successful. The meeting agreed that such talks should become a regular feature of the I.M.S. calendar. A discussion took place as to the best time for these talks. It was agreed that the morning before the first day of the D.I.A.S. Christmas Symposium or after the sherry reception that evening would be best. The President, on behalf of all members of the I.M.S., recorded his gratitude to the Dublin Institute for

Advanced Studies for providing the facilities for these talks as well as for the committee meetings and ordinary meetings of the I.M.S.

The meeting closed at 1.15pm.

Eugene Gath
University of Limerick.

TENSOR PRODUCTS AND PROJECTIONS

Seán Dineen and Mamoru Yoshida

Summary In this article, using a conjecture of Grothendieck as focal point, we give a display of the interaction between various concepts from the geometry of Banach spaces. These concepts include tensor norms, the Banach-Mazur distance and uniformly complemented subspaces. The interaction is achieved with the aid of three powerful results:

(a) an inequality on bilinear forms due to Hardy and Littlewood,
(b) F. John's upper bound for projection norms,
and
(c) Dvoretzky's spherical sections theorem.

1. Tensor Products

For a vector space E, the tensor product $E \otimes E$ of E with itself consists of all finite sums of the form $\sum_i x_i \otimes y_i$. A. Grothendieck was mainly responsible for the development of a theory of tensor products in Banach spaces. He investigated norms on $E \otimes E$ satisfying

$$||x \otimes y|| = ||x|| \cdot ||y||. \qquad (*)$$

He observed that there is a largest norm π (and a smallest norm ε) which satisfies $(*)$, where, taking $z = \sum_i x_i \otimes y_i$,

$$||z||_\pi = \inf\{\sum_i ||x_i|| \cdot ||y_i||\}$$

and

$$||z||_\varepsilon = \sup\{|\sum_i \varphi(x_i)\psi(y_i)| : \varphi, \psi \in E', ||\varphi|| \leq 1, ||\psi|| \leq 1\}.$$

Since

$$|\sum_i \varphi(x_i)\psi(y_i)| \leq \sum_i ||\varphi|| \cdot ||x_i|| \cdot ||\psi|| \cdot ||y_i|| \leq \sum_i ||x_i|| \cdot ||y_i||,$$

we have

$$|| \cdot ||_\varepsilon \leq || \cdot ||_\pi. \qquad (**)$$

Grothendieck conjectured that $|| \cdot ||_\varepsilon$ and $|| \cdot ||_\pi$ were equivalent norms on E if and only if $\dim E < \infty$. Over the years it has been shown that Grothendieck's conjecture is true for large (and important) collections of Banach spaces. However, in 1983, G. Pisier, [2], showed that the conjecture is false in general. We show that the conjecture is true for Banach spaces which contain uniformly complemented l_p^n's and that any counterexample must contain a large number of badly located almost Euclidean subspaces.

A comprehensive study of Grothendieck's conjecture is given in G. Pisier, [3], and our results are special cases of results given there. The monograph [3] is extremely well written but technically demanding. Our aim in this article is to provide some insight for the non-expert.

2. An inequality of Hardy and Littlewood

Let j, k and n denote positive integers and let $\alpha_{j,k} = e^{2\pi ijk/n}$, where $i = \sqrt{-1}$. Let A denote the $n \times n$ matrix $(\alpha_{j,k})_{1 \leq j,k \leq n}$. With A we can associate, in a canonical fashion, a bilinear form $\tilde{A}$ as follows:

$$\tilde{A}((x_j)_{j=1}^n, (y_j)_{j=1}^n) = \sum_{j,k} \alpha_{jk} x_j y_k.$$

Let

$$||\tilde{A}||_p = \sup\{|\sum_{j,k=1}^n \alpha_{jk} x_j y_k| : \sum_{j=1}^n |x_j|^p \leq 1, \sum_{k=1}^n |y_k|^p \leq 1\}.$$

Hardy and Littlewood, [1], proved that if

$$a(p) = \begin{cases} 3/2 - 2/p & \text{for } p \geq 2 \\ 1 - 1/p & \text{for } 1 \leq p \leq 2, \end{cases}$$

then there exists $c > 0$ such that

$$c \cdot n^{a(p)} \leq ||\tilde{A}||_p \leq n^{a(p)}$$

for all n and p.

If $z \in E \otimes E$, with $z = \sum_i x_i \otimes y_i$, we associate with z the bilinear form $\tilde{z}$ on $E' \otimes E'$ where E' denotes the space of all continuous real valued linear forms on E, by the formula

$$\tilde{z}(\varphi, \psi) = \sum_i < \varphi, x_i >< \psi, y_i >$$

and with this identification we have

$$||z||_\varepsilon = \sup\{|\tilde{z}(\varphi, \psi)| : \ ||\varphi|| \leq 1, ||\psi|| \leq 1\}.$$

For a positive integer n and $1 \leq p < \infty$, we let l_p^n denote $\mathbf{C}^n$ endowed with the norm

$$||(z_i)_{i=1}^n||_p = \left(\sum_{j=1}^n |z_j|^p \right)^{1/p}$$

and for $p = \infty$, we let l_∞^n denote $\mathbf{C}^n$ endowed with the supremum norm

$$||(z_i)_{i=1}^n||_\infty = \sup_i |z_i|.$$

For $1 \leq p < \infty$, let $\dfrac{1}{p'} = 1 - \dfrac{1}{p}$ and let $p' = 1$ when $p = \infty$. Translated into the language of tensor products the Hardy-Littlewood inequality says that

$$|| \sum_{j,k=1}^n \alpha_{j,k} e_j \otimes e_k ||_{l_{p'}^n \otimes_\varepsilon l_{p'}^n} \sim n^{a(p)},$$

where $e_j = (0, \cdots, 0, 1, 0, \cdots)$, the entry 1 occurring in the j-th position.

For all finite dimensional Banach spaces (and many infinite dimensional spaces) we have

$$(E \otimes_\pi E)' = E' \hat{\otimes}_\varepsilon E'$$

where $E' \hat{\otimes}_\varepsilon E'$ is the completion of $E' \otimes_\varepsilon E'$.

If $E = l_p^n$, $E' = l_{p'}^n$, and $1/p + 1/p' = 1$, then this duality is given by

$$< \sum_{j,k=1}^n a_{j,k} e_j \otimes e_k, \sum_{j,k=1}^n b_{j,k} e_j' \otimes e_k' >= \sum_{j,k=1}^n a_{j,k} b_{j,k},$$

where $(e_j')_{j=1}^n$ is the standard dual basis to $(e_j)_{j=1}^n$, that is,

$$< e_j', e_k >= \delta_{jk} \quad \text{(the Kronecker } \delta \text{ function)}.$$

Hence

$$n^2 = | < \sum_{j,k=1}^n \alpha_{j,k} e_j \otimes e_k, \sum_{j,k=1}^n \bar{\alpha}_{j,k} e_j' \otimes e_k' > |$$
$$\leq n^{a(p)} || \sum_{j,k=1}^n \bar{\alpha}_{j,k} e_j' \otimes e_k' ||_{l_{p'}^n \otimes_\pi l_{p'}^n}$$

and

$$|| \sum_{j,k=1}^n \alpha_{j,k} e_j \otimes e_k ||_{l_p^n \otimes_\pi l_p^n} \geq n^{2-a(p')}. \qquad (***)$$

To simplify our notation, we introduce the concept of tensorial diameter (td). For a Banach space E, the tensorial diameter of E, $td(E)$, is defined by

$$td(E) = \sup \frac{||z||_\pi}{||z||_\varepsilon}, \text{ where } z \in E \otimes E, \ z \neq 0.$$

By $(**)$, $td(E) \geq 1$ and an infinite dimensional Banach space E is a counterexample to Grothendieck's conjecture if and only if $td(E) < \infty$. By $(***)$,

$$td(l_p^n) \geq \frac{|| \sum_{j,k} \alpha_{j,k} e_j \otimes e_k ||_{(l_p^n \otimes l_{p'}^n, \pi)}}{|| \sum_{j,k} \alpha_{j,k} e_j \otimes e_k ||_{(l_p^n \otimes l_{p'}^n, \varepsilon)}}$$
$$\geq n^{2-a(p)-a(p')} = n^{b(p)},$$

where

$$b(p) = \begin{cases} 3/2 - 1/p & \text{for } 1 \le p \le 2 \\ 1/2 + 1/p & \text{for } p \ge 2. \end{cases}$$

3. The Banach-Mazur distance

When two Banach spaces are isomorphic, the Banach-Mazur distance d measures how close they are isometrically. For isomorphic Banach spaces E and F

$$d(E,F) = \inf\{||T||\cdot||T^{-1}|| : T : E \to F \text{ is a linear isomorphism}\}.$$

The function $\log d$ is symmetric and obeys the triangle inequality. If E (and hence F) is finite dimensional then $d(E,F) = 1$ if and only if E and F are isometrically isomorphic.

Lemma 1. *If E and F are isomorphic Banach spaces then*

$$td(E) \le (d(E,F))^2 \cdot td(F).$$

Proof: Every linear mapping $T : E \to F$ gives rise to a canonical linear mapping $T \otimes T : E \otimes E \to F \otimes F$, where

$$(T \otimes T)(x \otimes y) = Tx \otimes Ty$$

and moreover,

$$||T \otimes T||_\pi = ||T \otimes T||_\varepsilon = ||T||^2.$$

In addition, if T is a linear isomorphism then so also is $T \otimes T$ for both ε and π and

$$||(T \otimes T)^{-1}||_\pi = ||(T \otimes T)^{-1}||_\varepsilon = ||T^{-1}||^2.$$

Now suppose $T : E \to F$ is a linear isomorphism. For $z \in E \otimes E$

$$\begin{aligned} ||z||_\pi &= ||(T \otimes T)^{-1}(T \otimes T)(z)||_\pi \\ &\le ||(T \otimes T)^{-1}||_\pi \cdot ||(T \otimes T)(z)||_\pi \\ &= ||T^{-1}||^2 \cdot ||(T \otimes T)(z)||_\pi. \end{aligned}$$

Moreover,

$$||(T \otimes T)(z)||_\varepsilon \le ||T||^2 \cdot ||z||_\varepsilon$$

and hence, if $z \neq 0$,

$$\frac{1}{||z||_\varepsilon} \le \frac{||T||^2}{||(T \otimes T)(z)||_\varepsilon}.$$

This implies

$$\frac{||z||_\pi}{||z||_\varepsilon} \le \frac{||T||^2}{||(T \otimes T)(z)||_\varepsilon} \cdot ||T^{-1}||^2 \cdot ||(T \otimes T)(z)||_\pi.$$

By first taking the supremum with respect to z and then the infimum with respect to T we get the required result.

Lemma 2. *If E and F are Banach spaces and P is a projection of E onto F then*

$$td(F) \le ||P||^2 \cdot td(E).$$

Proof: This is similar to the the proof of Lemma 1, using the easily verifiable fact that $||P \otimes P||_\pi \le ||P||^2$.

We also require the following result of F. John: if F is a finite dimensional subspace of a Banach space E then there exists a projection P of E onto F such that

$$||P|| \le \sqrt{\dim(F)}.$$

4. Local theory of Banach spaces

The study of the properties of the finite dimensional subspaces of a Banach space is known as the local theory of Banach spaces. This often leads to global results. For instance, if all the finite dimensional subspaces of a Banach space E are isometric to a Hilbert space, then E itself is a Hilbert space. The Dvoretzky spherical sections theorem says that for every $\varepsilon > 0$, every positive integer n and every infinite dimensional Banach space E, there exists an n-dimensional subspace F of E such that

$$d(F, l_2^n) \le 1 + \varepsilon.$$

We say that a Banach space E contains l_p^n's uniformly if for every $\varepsilon > 0$ there exists $F_n \subset E$ such that

$$d(F_n, l_p^n) \leq 1 + \varepsilon.$$

E is said to contain l_p^n's uniformly complemented if, in addition, for each n there exists a projection $P_n : E \to E$ such that $P_n(E) = F_n$ and $||P_n|| \leq 1 + \varepsilon$. The infinite dimensional Banach space l_p contains l_p^n's uniformly complemented.

Proposition 3. *If for some p the Banach space E contains uniformly complemented l_p^n's, then E satisfies Grothendieck's conjecture.*

Proof: We have

$$td(E) \geq \limsup_n \frac{td(F_n)}{||P_n||^2} \geq \limsup_n \frac{td(l_p^n)}{||P_n||^2 d^2(F_n, l_p^n)}$$
$$\geq \limsup_n n^{b(p)} = \infty,$$

the first inequality holding by Lemma 2, the second by Lemma 1. This proves the proposition.

On the other hand the proof of the proposition above together with the precise growth rate of $td(l_p^n)$ as $n \to \infty$ shows what balance must be maintained between the Banach-Mazur distance and the projection norm in order to satisfy Grothendieck's conjecture. The proposition above also provides us with properties that any counterexample to the conjecture must satisfy. For example, the spherical sections theorem of Dvoretzky shows that any infinite dimensional Banach space contains l_2^n and by the result of F. John we can suppose that a projection P_n onto l_2^n has norm $\leq \sqrt{n}$. By the Hardy-Littlewood inequality we have $td(l_2^n) \sim n$. Hence if E is a counterexample to Grothendieck's conjecture then

$$\infty > td(E) \geq \limsup_n \frac{td(l_2^n)}{||P_n||^2} = \limsup_n \frac{n}{||P_n||^2}.$$

Hence there exists $c > 0$ such that $||P_n|| \geq c\sqrt{n}$. Thus l_2^n has asymptotically the worst possible projection norm and may be said

to be badly located. G. Pisier, [3], showed that his counterexample $\tilde{E}$ has the following stronger property:
there exists $c > 0$ such that for any finite dimensional subspace F of $\tilde{E}$ and any projection P of $\tilde{E}$ on F

$$||P|| \geq c \cdot \sqrt{\dim(F)}.$$

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[3] G. Pisier, Factorization of Linear Operators and Geometry of Banach Spaces, Regional Conference Series in mathematics, **60**, A.M.S., 1986.

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ON THE CONTINUATION OF HOLOMORPHIC MAPPINGS

Mamoru Yoshida*

1. Introduction

In the theory of one complex variable, it is well known that holomorphic functions cannot in general be extended as holomorphic functions to larger domains. For example, if $\Omega(\neq \mathbf{C})$ is open and $a \in \partial\Omega$, then

$$f(z) = \frac{1}{z-a}$$

cannot be extended across the boundary point a. Another classical example is

$$g(z) = \Sigma_{n=1}^{\infty} z^{n!}$$

which cannot be extended across any point of the boundary of the unit disc. For any domain Ω of $\mathbf{C}$, there is a holomorphic function which cannot be extended across any point of the boundary $\partial\Omega$.

For functions of two variables, we have a new phenomenon. For example if we consider the annulus

$$\Omega_1 = \{(z_1, z_2) : \ 1 < |z_1|^2 + |z_2|^2 < 4\},$$

then all f holomorphic on Ω can be extended to the ball

$$\tilde{\Omega}_1 = \{(z_1, z_2) : \ |z_1|^2 + |z_2|^2 < 4\}.$$

A simple consequence of this result is that zeros of holomorphic functions of more than one variable cannot be isolated.

* The author thanks University College Dublin for support during the period this research was undertaken.

Another classical example concerns the domain

$$\Omega_2 = \{(z_1, z_2) : \ |z_1| < 1, |z_2| < 1, ||z_1| - |z_2|| < \frac{1}{3}\}.$$

All f holomorphic on Ω_2 can be extended to the polydisc

$$\tilde{\Omega}_2 = \{(z_1, z_2) : \ |z_1| < 1, |z_2| < 1\}.$$

When this phenomenon, which was discovered by Hartogs, occurs, it is natural to ask the following questions:

(A) Given f holomorphic on Ω, is there a natural domain to which all f holomorphic on Ω can be extended holomorphically ?

(B) Given Ω, is there a natural domain Ω' to which all holomorphic functions on Ω can be extended ?

If a natural domain for f exists, we call such a domain (in (A)) a *domain of existence* of f and denote it by Ω_f. For example the unit disc in $\mathbf{C}$ is the natural domain of existence of $g(z) = \Sigma_{n=1}^{\infty} z^{n!}$.

Is Ω_f unique? How does one construct Ω_f?

Such a natural domain Ω' (in(B)) is called a *domain of holomorphy.*

It is natural to expect that, in some sense, Ω' is related to $\bigcap_{f\in\mathcal{H}(\Omega)} \Omega_f$ where $\mathcal{H}(\Omega)$ is the family of all holomorphic functions in Ω. Is it possible to characterize when $\Omega = \Omega'$ geometrically or algebraically?

Geometric characterization

The proper mathematical definitions are rather technical but we can state some results which are readily understandable. For example the following notion of *holomorphically convex domain* is related, by the Hahn-Banach Theorem, to the linear notion of convexity.

Let Ω be a domain of $\mathbf{C}^n$ and K any compact subset in Ω. We put

$$\hat{K}_\Omega = \{z \in \Omega : \ |f(z)| \le \|f\|_K, \ f \in \mathcal{H}(\Omega)\},$$

where $\|f\|_K = \sup\{|f(z)| : \ z \in K\}$.

If $\hat{K}_\Omega$ is also compact for any K, Ω is said to be *holomorphically convex.* It is easily seen that convex domains are holomorphically convex.

We have the following two equivalent properties for a domain Ω of $\mathbf{C}^n$:

(I) Ω is a domain of holomorphy,

(II) Ω is holomorphically convex.

Metric characterization

A function μ on a connected domain D ($\subset \mathbf{C}$) is called *subharmonic*, if the following conditions are satisfied:

(1) μ is upper semicontinuous,

(2) μ is not identically $-\infty$,

(3) $\mu(c) \leq \frac{1}{2\pi}\int_0^{2\pi} \mu(c + re^{i\theta})d\theta$, $(r > 0$: sufficiently small).

Condition (1) is a smoothness condition, while (2) is just to exclude a trivial case and (3) is the main property which says that the average over a disc dominates the value at the centre.

Let Ω be a connected domain of $\mathbf{C}^n$ and let z_0 be any point in Ω. We put

$$E_a(z_0) = \{z = z_0 + at : \ t \in \mathbf{C}\}, \ (a \in \mathbf{C}^n).$$

If μ is not identically $-\infty$ and μ satisfies (1) and (3) as a function of t in $\Omega \cap E_a(z_0)$ for all $z_0 \in \Omega$ and $a \in \mathbf{C}^n$, then μ is said to be *plurisubharmonic* on Ω.

If $-\log d(z, \mathrm{C}\Omega)$ is plurisubharmonic in Ω, then Ω is called *pseudoconvex.*

The following condition for a domain Ω of $\mathbf{C}^n$ is equivalent to conditions (I) and (II) above:

(III) Ω is pseudoconvex.

It is not difficult to show that domains of existence are pseudoconvex. The converse was conjectured by E. E. Levi in 1911 and became known as the *Levi problem* and was solved by K. Oka in 1942 for domains in $\mathbf{C}^2$ and for domains in $\mathbf{C}^n$ by F. Norguet and H. Bremermann in 1954. The term 'Levi problem' (see section 2) is still used for problems of this type in more abstract and general settings.

Algebraic characterizations

For a given family (data) of meromorphic functions which are defined locally on a domain Ω of $\mathbf{C}^n$, can we find a global meromorphic function which is locally identified, in a sense, by the data? More precisely given an open covering $\mathcal{U} = \{U_i : \ i \in I\}$ of Ω and a collection of meromorphic functions m_i, with m_i meromorphic (i.e. a quotient of holomorphic functions) on U_i. Supposing that $m_i - m_j$ is holomorphic on $U_i \cap U_j$, then we wish to find a global meromorphic function m on Ω such that $m - m_i$ is holomorphic on U_i for all i. Let $\mathcal{O}$ be the sheaf of germs of holomorphic functions over a domain Ω of $\mathbf{C}^n$. If a collection $Z^1(\Omega, \mathcal{O}) := \{z_{ij} : \ i, j \in I\}$ satisfies conditions (1), (2) and (3) below, we call $Z^1(\Omega, \mathcal{O})$ a *holomorphic cocycle* of degree one on $\mathcal{U}$:

(1) z_{ij} is holomorphic on $U_i \cap U_j$ ($\neq \emptyset$) for all i and j in I,

(2) $z_{ij} = -z_{ji}$ on $U_i \cap U_j$ whenever $U_i \cap U_j \neq \emptyset$,

(3) $z_{ij} + z_{jk} + z_{ki} = 0$ on $U_i \cap U_j \cap U_k$ whenever $U_i \cap U_j \cap U_k \neq \emptyset$.

If $C^1(\Omega, \mathcal{O}) := \{z_i : \ i \in I\}$ satisfies condition (4) below, we call $C^1(\Omega, \mathcal{O})$ a *holomorphic cochain* of degree one on $\mathcal{U}$:

(4) z_i is holomorphic on U_i for all $i \in I$.

We put

$$B^1(\Omega, \mathcal{O}) = \{z_i - z_j : \ z_i, \ z_j \in C^1(\Omega, \mathcal{O})\}$$

and call $B^1(\Omega, \mathcal{O})$ a *holomorphic coboundary* of degree one on $\mathcal{U}$. $Z^1(\Omega, \mathcal{O})$ and $B^1(\Omega, \mathcal{O})$ are additive groups and $B^1(\Omega, \mathcal{O})$ is a subgroup of $Z^1(\Omega, \mathcal{O})$.

We put $H^1(\Omega, \mathcal{O}) = Z^1(\Omega, \mathcal{O}) \ / \ B^1(\Omega, \mathcal{O})$ and call $H^1(\Omega, \mathcal{O})$ the first *holomorphic cohomology group* on $\mathcal{U}$.

Clearly $H^1(\Omega, \mathcal{O}) = 0$ if and only if $Z^1(\Omega, \mathcal{O}) = B^1(\Omega, \mathcal{O})$.

The following condition is equivalent to (I):

(IV) $H^1(\Omega, \mathcal{O}) = 0$.

A very different algebraic characterization has been investigated by other authors. If $\mathcal{H}(\Omega)$ is endowed with the compact open topology, then it becomes a complete metrizable topological algebra (i.e. the operations of addition and multiplication are continuous). We may then consider $\mathcal{M}(\mathcal{H}(\Omega))$, the set of all continuous $\mathbf{C}$-valued multiplicative linear functions on $\mathcal{H}(\Omega)$. Clearly point evaluations belong to $\mathcal{M}(\mathcal{H}(\Omega))$ and thus the the mapping

$$\varphi : \ x \in \Omega \to \delta_x \in \mathcal{M}(\mathcal{H}(\Omega)),$$

where δ_x is point evaluation at x, is well defined and easily seen to be injective for Ω open in $\mathbf{C}^n$. The space $\mathcal{M}(\mathcal{H}(\Omega))$ can be endowed with the structure of a complex manifold and with this structure the envelope of holomorphy of Ω can be identified with the connected component of $\mathcal{M}(\mathcal{H}(\Omega))$ which contains $\varphi(\Omega)$. This gives the following characterization equivalent to condition (I) for a domain Ω of $\mathbf{C}^n$.

(V) $\Omega = \mathcal{M}(\mathcal{H}(\Omega))$.

Conditions (II) and (III) are geometric characterizations of domains of holomorphy while (IV) and (V) are algebraic characterizations. In particular, (II) reduces to whether $-\log d(z, \mathbf{C}\Omega)$ is subharmonic on a domain of the complex plane $\mathbf{C}$. Condition (IV) is suitable for concrete calculations.

These are parts of the basic classical background to more recent results which we will now describe. The results we describe involve a more general setting.

2. Setting up of the problem

We describe the more general setting.

(1) First it is more natural to discuss such problems over Riemann domains than open sets of $\mathbf{C}^n$, since for example there exists an open subset of $\mathbf{C}^n$, Ω, say, such that holomorphic functions on Ω cannot be extended to any larger open subset of $\mathbf{C}^n$, but if Ω is considered as a complex manifold, then all holomorphic functions on Ω can be extended to a strictly larger complex manifold.

The definition of Riemann domain is as follows:

if there exists a local biholomorphic mapping ψ of a complex manifold Ω into the locally convex space E, (Ω, ψ) is called a *Riemann domain over* E.

(2) We discuss Riemann domains over infinite dimensional spaces rather than over $\mathbf{C}^n$. In this case a mapping is *holomorphic* if it is continuous and its restriction to each finite dimensional section is holomorphic as a mapping of several complex variables.

(3) We discuss mappings of a Riemann domain Ω into a complex manifold N.

(4) We consider subsets $\mathcal{F}$ of the space of all holomorphic

mappings. The two simplest examples are $\mathcal{F} = \mathcal{H}(\Omega)$ which leads to the concept of domain of holomorphy and $\mathcal{F} = \{f\}$ (i.e. $\mathcal{F}$ consists of a single function f) and this leads to the concept of domain of existence. We confine ourselves here to these two examples but mention a further natural and useful example, $\mathcal{H}^\infty(\Omega)$–the space of bounded holomorphic functions on Ω. Note that if $\Omega \subset \Omega'$ and each holomorphic function on Ω extends, as a holomorphic function, to Ω' then if $f \in \mathcal{H}(\Omega)$, $\lambda \in \mathbf{C}$ and $\lambda \notin f(\Omega)$ then

$$g := \frac{1}{f - \lambda} \in \mathcal{H}(\Omega)$$

and hence admits a holomorphic extension $\tilde{g}$ to Ω'. By uniqueness of holomorphic extensions it follows that $(f - \lambda) \cdot g = 1$ on Ω and $(\tilde{f} - \lambda) \cdot \tilde{g} = 1$ on Ω', where $\tilde{f}$ is the extension of f to Ω'. Hence $\lambda \notin \tilde{f}(\Omega')$ and it is now easy to see that whenever $f \in \mathcal{H}^\infty(\Omega)$ then $\tilde{f} \in \mathcal{H}^\infty(\Omega')$.

Next we provide some terminology for mappings between Riemann domains in order to set up our problem precisely. Let (Ω, ψ) and (Ω', ψ') be Riemann domains over a local convex space E. If a holomorphic mapping λ of Ω into Ω' satisfies $\psi = \psi' \circ \lambda$, the mapping λ is called a *morphism* of (Ω, ψ) into (Ω', ψ'). Let N be a complex manifold and let $\mathcal{F} \subset \mathcal{H}(\Omega, N)$, where

$$\mathcal{H}(\Omega, N) = \{f : f \text{ is a holomorphic mapping of } \Omega \text{ into } N\}.$$

A morphism λ of (Ω, ψ) into (Ω', ψ') is said to be an $\mathcal{F}N$-*extension* of Ω if for each $f \in \mathcal{F}$ there exists a unique $f' \in \mathcal{H}(\Omega', N)$ such that $f' \circ \lambda = f$. When $\mathcal{F} = \mathcal{H}(\Omega)$ and $N = \mathbf{C}$ we simply say a *holomorphic extension.*

Ω is said to be an $\mathcal{F}N$-*domain of holomorphy* if each $\mathcal{F}N$-extension of Ω is an isomorphism.

Ω is said to be a *domain of holomorphy* if each holomorphic extension is an isomorphism.

Ω is said to be a *domain of existence* if there exists $f \in \mathcal{H}(\Omega)$ such that Ω is an $\{f\}\mathbf{C}$-domain of holomorphy.

Let (Ω, ψ) be a Riemann domain over the space E and let $\mathcal{F} \subset \mathcal{H}(\Omega, N)$. A morphism $\lambda : \Omega \to \Omega'$ is called an *$\mathcal{F}N$-envelope of holomorphy* of Ω if:

(a) λ is an $\mathcal{F}N$-extension of Ω,

(b) if $\mu : \Omega \to \Omega''$ is an $\mathcal{F}N$-extension of Ω, then there exists a morphism $\nu : \Omega'' \to \Omega'$ such that $\nu \circ \mu = \lambda$.

Then $(\lambda, \Omega', \psi')$ or Ω' is also called *$\mathcal{F}N$-envelope of holomorphy* of (Ω, ψ) or Ω. Note that condition (b) says that an $\mathcal{F}N$-extension is maximal.

Malgrange, [10, pp.29–34] for the finite dimensional case and Mujica, [11, Theorem 56.4] for the infinite dimensional case, proved that there exists an $\mathcal{F}N$-envelope of holomorphy of Ω and that this envelope is unique up to complex analytic isomorphism.

Let (Ω, ψ) be a Riemann domain over the space E, $(\tilde{\lambda}, \tilde{\Omega}, \tilde{\psi})$ its envelope of holomorphy and N a complex manifold.

Problem $(*)$: Can every $f \in \mathcal{H}(\Omega, N)$ be extended holomorphically to an element $\tilde{f}$ of $\mathcal{H}(\tilde{\Omega}, N)$ such that $f = \tilde{f} \circ \tilde{\lambda}$?

3. Some results

We require certain restrictions on N. If $N = \mathbf{C}^n$, it is clear that there is a function $\tilde{f}$ which solves Problem $(*)$. (See, for example, [7].)

If the complex manifold G is a group and the mappings $G \times G \ni (x, y) \to x \cdot y \in G$ and $G \ni x \to x^{-1} \in G$ are holomorphic, we call G a *complex Lie group*. If $\mathcal{G}$ is the tangent space at the identity $e \in G$, we call $\mathcal{G}$ the *Lie algebra* of G. For example, the complex general linear group $GL(n, \mathbf{C})$ consisting of all invertible $n \times n$ complex matrices is a complex Lie group and the set of all matrices of degree n, $\mathbf{gl}(n, \mathbf{C})$, is the Lie algebra with respect to the commutator product $[A, B] = AB - BA$. Between a complex Lie group G and its Lie algebra $\mathcal{G}$, there exists an exponential mapping $\exp : \mathcal{G} \to G$ which is a local biholomorphism of a neighbourhood of 0 in $\mathcal{G}$ onto a neighbourhood of the identity e in G. In the example above,

$$\exp(A) = I_n + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots\cdots$$

for any $A \in \mathbf{gl}(n, \mathbf{C})$, where I_n is the identity matrix.

If $m = \dim G < \infty$, we may take $\mathcal{G}$ as $\mathbf{C}^m$ and we consider a complex Lie group G of positive dimension as a complex manifold.

Let (Ω, ψ) be a Riemann domain over a Stein manifold S, $(\tilde{\lambda}, \tilde{\Omega}, \tilde{\psi})$ its envelope of holomorphy, G a complex Lie group and f a holomorphic mapping of Ω into G.

Let $(\lambda, \Omega', \psi')$ be the $\{f\}G$-envelope of holomorphy. By pseudoconvexity of (Ω', ψ') and the positive solution of Docquier-Grauert, [3], to the Levi problem, (Ω', ψ') is a domain of existence of a holomorphic function on Ω. Since $(\tilde{\lambda}, \tilde{\Omega}, \tilde{\psi})$ is its envelope of holomorphy, there is a mapping μ of $(\tilde{\Omega}, \tilde{\psi})$ into (Ω', ψ') such that $\lambda = \mu \circ \tilde{\lambda}$. Let f' be the holomorphic continuation of f to $(\lambda, \Omega', \psi')$. Then $f' \circ \mu$ is the holomorphic continuation of f to $(\tilde{\lambda}, \tilde{\Omega}, \tilde{\psi})$. Thus we obtain the following theorem.

Theorem 1. *Let (Ω, ψ) be a Riemann domain over a Stein manifold S and $(\tilde{\lambda}, \tilde{\Omega}, \tilde{\psi})$ its envelope of holomorphy. Any holomorphic mapping of Ω into a complex Lie group G has a holomorphic continuation to $(\tilde{\lambda}, \tilde{\Omega}, \tilde{\psi})$.*

Let E be a complex linear space with the finite open topology T_0. S. Dineen, [4], proved the vanishing theorem $H^1(D, \mathcal{O}) = 0$ for D pseudoconvex and $\mathcal{O}$ the structural sheaf over the $\mathbf{C}$-linear space (E, T_0) and L. Gruman, [5], solved the Levi problem by proving that any finitely pseudoconvex domain D of the space (E, T_0) is the domain of existence of a holomorphic function on D.

A complex manifold N is called a *complex Banach manifold* if it is a complex manifold modelled on a complex Banach space.

A *Banach complex Lie group* is a group G which is a complex Banach manifold and a complex Lie group.

Let $\mathcal{U} = \{U_i : i \in I\}$ be an open covering of E. Assume that a set

$$\mathcal{Z} := \{z_{ij} : z_{ij} \in \mathcal{H}(U_i \cap U_j, G),\ i, j \in I\}$$

satisfies conditions similar to conditions (1), (2) and (3) for a holomorphic cocycle of degree one $Z^1(\Omega, \mathcal{O})$ which we gave in our algebraic characterization of domains of holomorphy. We call $\mathcal{Z}$ a *holomorphic cocycle on $\mathcal{U}$ with values in G* and call the pair

$F = \{\mathcal{U}, \mathcal{Z}\}$ a *holomorphic principal fibre bundle* with *base space* E and *structure group* G.

We consider the continuation of holomorphic mappings of a Riemann domain (Ω, ψ) over (E, T_0) into a complex Banach Lie group G. Let Λ be the set of finite dimensional $\mathbf{C}$-linear subspaces of E. For any $L \in \Lambda$, we denote $\Omega \cap \psi^{-1}(L)$ and $\psi|\Omega \cap \psi^{-1}(L)$, by Ω_L and ψ_L, respectively. Any holomorphic mapping f of (Ω_L, ψ_L) into G can be continued holomorphically to its envelope of holomorphy by the finite dimensional result. The authors, [6], proved the following theorem, using the method of Kajiwara-Shon, [9], and transfinite induction as in Dineen, [4].

Theorem 2. *Let E be a complex linear space with the finite open topology, (Ω, ψ) a Riemann domain over E and $(\tilde{\lambda}, \tilde{\Omega}, \tilde{\psi})$ the envelope of holomorphy of (Ω, ψ). Let G be a complex Banach Lie group. Then any holomorphic mapping of Ω into G can be holomorphically continued to the envelope of holomorphy $(\tilde{\lambda}, \tilde{\Omega}, \tilde{\psi})$ of (Ω, ψ).*

The main problem with the finite open topology is that it allows of too many holomorphic functions. In two cases in which the base space E is endowed with stronger topologies, the authors, [12], prove, using positive results of Schottenloher, [13], for the Levi problem and methods similar to those used in finite dimension, the following theorem.

Theorem 3. *Let E be a separable Fréchet space with the bounded approximation property or a DFN-space and (Ω, ψ) a Riemann domain over the space E. Let G be a complex Banach Lie group and F a holomorphic principal fibre bundle with structure group G over E. Then a holomorphic section s of F over Ω can be continued holomorphically to the envelope of holomorphy of the Riemann domain Ω.*

The proofs of the results above are rather technical and so we do not provide details but give one brief example of the technical aspects. Let

$$D = \{z = (z_1, z_2) \in \mathbf{C}^2 : \ |z_1| < 1 + \varepsilon, |z_2| < 1\} \cup$$

$$\{z = (z_1, z_2) \in \mathbf{C}^2 : \ 1 - \varepsilon < |z_1| < 1 + \varepsilon, |z_2| < 1 + \varepsilon\}$$

and

$$\hat{D} = \{z = (z_1, z_2) \in \mathbf{C}^2 : \ |z_j| < 1 + \varepsilon \ (j = 1, 2)\}$$

for a positive number ε (< 1). It is classical that $\hat{D}$ is the envelope of holomorphy of D.

Lemma. *Let G be a complex Lie group. For any holomorphic mapping f of D into G, there is a holomorphic mapping g of $\hat{D}$ into G such that $g = f$ in D.*

Proof: We may assume that G is connected. We introduce in $\mathcal{H}(D, G)$ the compact-open topology τ_0 and let $\mathcal{H} = (\mathcal{H}(D, G), \tau_0)$. As D is analytically contractible to a point, $\mathcal{H}$ is a connected topological group. Let

$$K(\delta) = \{z \in \mathbf{C}^2 : \ |z_1| \leq 1 + \varepsilon - \delta, |z_2| < 1 - \delta\} \cup$$
$$\{z \in \mathbf{C}^2 : \ 1 - \varepsilon + \delta \leq |z_1| \leq 1 + \varepsilon - \delta, |z_2| < 1 + \varepsilon - \delta\}$$

and

$$K'(\delta) = \{z \in \mathbf{C}^2 : \ |z_j| < 1 + \varepsilon - \delta \ (j = 1, 2)\}$$

for any positive number δ with $\delta < \varepsilon$. Let m be the complex dimension of G and exp the exponential mapping of $\mathbf{C}^m$ into G. The mapping exp maps an open neighbourhood

$$U = \{w \in \mathbf{C}^m : \ |w_j| < a \ (j = 1, 2, \ldots, m)\}$$

of the origin in $\mathbf{C}^m$ biholomorphically onto an open neighbourhood W of the identity element e of G. Hence $\log := (\exp|U)^{-1}$ is a biholomorphic mapping of W onto U. We let

$$\mathcal{V}(1) = \{h \in \mathcal{H} \ ; \ h(K(\delta)) \subset W\}.$$

Then $\mathcal{V}(1)$ is a neighbourhood of the identity element 1 of the topological group $\mathcal{H}$. Since $\mathcal{H}$ is connected, $\mathcal{H}$ is generated by $\mathcal{V}(1)$. There is a finite number s of elements $f_1, f_2, \cdots, f_{s-1}$ and f_s in $\mathcal{V}(1)$ such that

$$f = f_1 f_2 \cdots f_s$$

in D. Each $\log f_i$ is a holomorphic mapping of $K(\delta)$ into the polydisc U.

There is a holomorphic mapping G_i of $K'(\delta)$ into U such that

$$G_i = \log f_i \text{ in } K(\delta) \cap K'(\delta)$$

for $j = 1, 2, \cdots, s$. We let

$$g = \exp G_1 \exp G_2 \cdots \exp G_s$$

in $K'(\delta)$. Then g is a holomorphic mapping of $K'(\delta)$ in G such that $g = f$ in $K(\delta) \cap K'(\delta)$. Since δ is arbitrary, we obtain the Lemma by the identity theorem for holomorphic mappings.

Further details and precise definitions are given in [1], [6] and [12].

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MUSICAL SCALES

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Introduction

Our aim is to provide precise definitions of some musical concepts, mainly *tuning system* and *mode*, in order to begin a rigorous search of the theorems uncovered by the main scales currently used in Western art music.

We shall consider musical concepts that only depend on the tones of sounds, disregarding any other characteristic of sounds (timbre, volume, duration, ...). We shall ground our study on the structure of pitches, assuming that each musical pitch is fully determined by the frequency of vibration of the sound wave that produces it. Given a pitch t and a positive real number λ, we have a pitch λt whose frequency is the product of λ by the frequency of t and so we obtain a free and transitive action of the group of all positive real numbers on the pitches. This is the only structure of sounds that we shall consider and our first goal is to show how the concepts of musical scale and mode may be reduced to this simple structure.

Given a pitch t, the most consonant pitch is t itself, then $2t$, $3t$ and so on. At the basis of the whole theory is the natural identification between t and $2t$ that most men make unconsciously. Hence we consider families of pitches $\mathcal{S}$ such that $t \in \mathcal{S}$ implies:

(i) $2^n t$ belongs to $\mathcal{S}$ for any integer n.
(ii) only a finite number of elements $s \in \mathcal{S}$ satisfy $t \leq s < 2t$ (this number does not depend on t and it is said to be the number of notes of $\mathcal{S}$).

Two families $\mathcal{S}$, $\mathcal{S}'$ are said to be equivalent when $\mathcal{S}' = \lambda\mathcal{S}$ for some positive number λ and *tuning systems* or *scales* are defined to be equivalence classes.

Any finite family of pitches F, such as a melody or the keys of a piano, defines a scale $\mathcal{F} = \{2^n t : n \in \mathbf{Z},\ t \in F\}$. For example, the classical diatonic scale is defined by any geometric progression $t, 3t, \ldots, 3^6 t$, and it is a good scale from the melodic point of view. From the point of view of modulation good scales are tempered scales (scales defined by geometric progressions t, rt, $r^2 t$, ..., $r^n t = 2t$) and the scale defined by the keys of a piano is the tempered scale of 12 notes. From the point of view of harmony, one should like to have $3t$ and $5t$ in the scale whenever t is. Therefore (neglecting the temperament for the moment) we should look for a scale $\mathcal{S}$ such that $3t, \ldots, 3^6 t$ and $5t$ belong to $\mathcal{S}$ whenever $t \in \mathcal{S}$; but any one of these conditions contradicts the finiteness of the number of notes, so that no scale may fulfil them. However, men cannot distinguish two pitches when their frequencies are very close, so that a scale fulfilling these conditions up to a small error would be a perfect one for human hearing. We shall prove that any scale improving the error of the tempered scale of 12 notes must have 16 or more notes. Even if one disregards modulation, the usual tempered scale is the best scale (from the point of view of melody and harmony) with less than 16 notes.

1. Tuning Systems

Given a pitch t and a positive real number λ, we shall denote by λt the pitch whose frequency is the product of λ by the frequency of t; so that $1t = t$ and $\lambda(\mu t) = (\lambda\mu)t$. Moreover, given two pitches s and t, there exists a unique positive real number λ such that $t = \lambda s$ and this number λ is said to be the *interval* from s to t. This is the only structure of sounds that we shall consider, so that our starting point is the following definition of the structure of tones of sounds:

Definition: Any set $\mathcal{P}$ endowed with a free transitive action of the (multiplicative) group $\mathcal{I}$ of all positive real numbers is said to be a *system of pitches*. (By a free transitive action, we mean any

map $\mathcal{I} \times \mathcal{P} \to \mathcal{P}, (\lambda, t) \mapsto \lambda t$, such that $1t = t$, $\lambda(\mu t) = (\lambda\mu)t$ and such that, for any pair $t, s \in \mathcal{P}$, we have $s = \lambda t$ for a unique $\lambda \in \mathcal{I}$.) The elements of $\mathcal{P}$ are said to be *pitches* and the elements of $\mathcal{I}$ are said to be *intervals*.

We shall always consider the usual order on $\mathcal{I} = \mathbf{R}_+$, so that pitches inherit an order: $s \leq t$ when $t = \lambda s$, $\lambda \geq 1$.

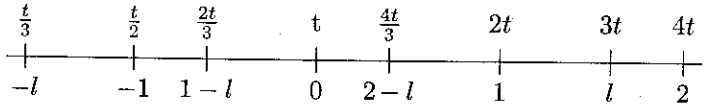
Some important intervals have a proper name: 2 is the *octave* (so that between the pitches t and $2^\alpha t$ there are α octaves), 3/2 is the *perfect fifth* and 5/4 is the *major third*. The basis of any tuning system is the identification between sounds forming octaves, so that we consider the subgroup $2^{\mathbf{Z}} = \{\lambda \in \mathcal{I} : \lambda = 2^n, n \in \mathbf{Z}\}$.

Definition: The quotient set $\mathcal{O} = \mathcal{P}/2^{\mathbf{Z}}$ is said to be the *Octave*, so that $\mathcal{I}$ acts transitively on the Octave and the quotient group $\mathcal{I}/2^{\mathbf{Z}}$ acts transitively and freely on $\mathcal{O}$.

Geometric representation:

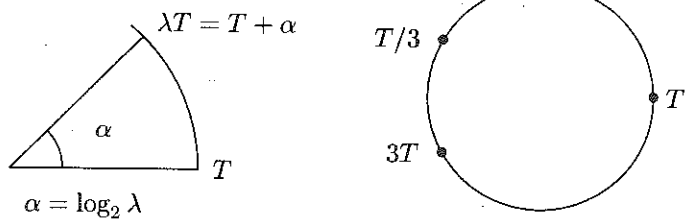
We denote pitches by Latin letters and their projection on the Octave by the corresponding capital letter.

If a pitch t is fixed, then pitches correspond with positive real numbers, but this representation takes octaves into segments of different lengths. To avoid this problem it is convenient to use an additive notation; hence, we represent the pitch λt by the real number $\alpha = \log_2 \lambda$, so that the interval from t to λt is represented by a segment of length α. We put $t + \alpha$ instead of λt when this additive notation is used ($+\alpha$ is translation by α octaves). For example, if $l = \log_2 3$, we have:



When the group $\mathcal{I}$ is identified with $\mathbf{R}$ via $\log_2$, the subgroup $2^{\mathbf{Z}}$ is identified with $\mathbf{Z}$, so that $\mathcal{I}/2^{\mathbf{Z}}$ is isomorphic to $\mathbf{R}/\mathbf{Z}$. Therefore, we may represent the Octave by the points of a circle and it is quite natural to fix the length of this circle as the unit of length

and to measure angles by octaves (i.e., complete turns, so that the angle α has $2\pi\alpha$ radians):



This geometric representation of the Octave allows us to define the *distance* between two elements of $\mathcal{O}$ as the distance of their corresponding points in the circle.

Note that the order of $\mathcal{P}$ defines an order on the complement $\mathcal{O} - T$ of any element $T \in \mathcal{O}$, so that any finite subset of the Octave inherits a "circular order" (we always represent it in the counter–clockwise sense)

Definition: Two finite subsets S and S' of the Octave $\mathcal{O}$ are said to be *equivalent* if $S' = \lambda S$ for some interval λ (if there exists a rotation of the Octave transforming S into S'). Equivalence classes of finite subsets of $\mathcal{O}$ are said to be *tuning systems* or *scales*. The *number of notes* of a scale is the common cardinal number of all finite subsets of $\mathcal{O}$ representing it.

By definition, a scale $\mathbf{S}$ may be represented by a finite subset S of the Octave (whose elements are said to be *notes*) or by a family of pitches $\mathcal{S}$ with the following property: if $t \in \mathcal{S}$, then $2^n t \in \mathcal{S}$ for all $n \in \mathbf{Z}$ and there is a finite number of elements of $\mathcal{S}$ between t and $2t$. Two such families $\mathcal{S}$ and $\mathcal{S}'$ define the same scale when $\mathcal{S}' = \lambda\mathcal{S}$ for some interval λ. Moreover, any finite family of pitches, such as the keys of a piano or a melody, define a scale when we project it on the Octave.

Take a finite subset S of the Octave representing a given scale $\mathbf{S}$ of n notes and let us consider the circular order $(T_1, \ldots, T_n)$ of its notes. Then we get an n-cycle $(\alpha_1, \ldots, \alpha_n)$ of intervals (in fact of

elements of $\mathcal{I}/2^{\mathbf{Z}}$), where $T_{i+1} = T_i + \alpha_i$. This n-cycle $(\alpha_1, \ldots, \alpha_n)$ only depends on the scale $\mathbf{S}$ and it is called the *symbol* of $\mathbf{S}$ because it is clear that any scale is determined by its symbol.

The symbol of a scale is not an arbitrary cycle of intervals because we have $\alpha_1 + \ldots + \alpha_i < 1$ when $i < n$ and $\alpha_1 + \ldots + \alpha_n = 1$ (identifying $\mathcal{I}/2^{\mathbf{Z}}$ with $(0, 1]$).

Modes: A finite subset M of the group $\mathcal{I}/2^{\mathbf{Z}}$ is said to be a *mode* if it contains the neutral element 1.

Each mode M defines, once you fix a pitch t, a finite set $Mt = \{\lambda T : \lambda \in M\}$ in the Octave; hence M defines a scale, since $M\lambda t = \lambda(Mt)$. Conversely, given a scale represented by a finite subset S of the Octave, each note $T \in S$ defines a mode $M = \{\lambda \in \mathcal{I}/2^{\mathbf{Z}} : \lambda T \in S\}$ but this mode depends on the note T. Each scale of n notes defines, in general, n different modes.

Since $\mathcal{I}/2^{\mathbf{Z}} \cong \mathbf{R}/\mathbf{Z} \cong [0, 1)$, every mode M is a sequence $0 = m_1 < m_2 < \ldots < m_n < 1$, so that M is determined by the sequence $\alpha_1, \ldots, \alpha_n$ where $\alpha_i = m_{i+1} - m_i$ and $\alpha_n = 1 - m_n$. The symbol of the scale defined by M is just $(\alpha_1, \ldots, \alpha_n)$. Conversely, the modes defined by the scale of symbol $(\alpha_1, \ldots, \alpha_n)$ are just the modes corresponding to the n sequences:

$$\alpha_1, \ldots, \alpha_{n-1}, \alpha_n$$
$$\alpha_2, \alpha_3, \ldots, \alpha_n, \alpha_1$$
$$\ldots\ldots\ldots\ldots$$
$$\alpha_n, \alpha_1, \ldots, \alpha_{n-1}$$

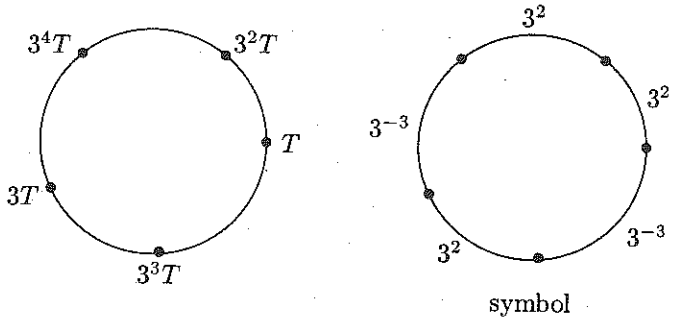
Tempered scales: A scale is said to be *tempered* if it divides the Octave in equal parts; that is to say, the symbol of the tempered scale of n notes is $(1/n, \ldots, 1/n)$.

The scale defined by the sounds of a piano is the tempered scale of 12 notes. The reader may obtain the symbol of the scale of 7 notes defined by the white keys and the corresponding 7 modes.

Scales of fifths: The *scale of fifths* of n notes is the scale defined by any geometric progression of ratio 3 and n terms, $n \geq 2$. It is the scale represented by $\{T, 3T, \ldots, 3^{n-1}T\}$. In this scale every

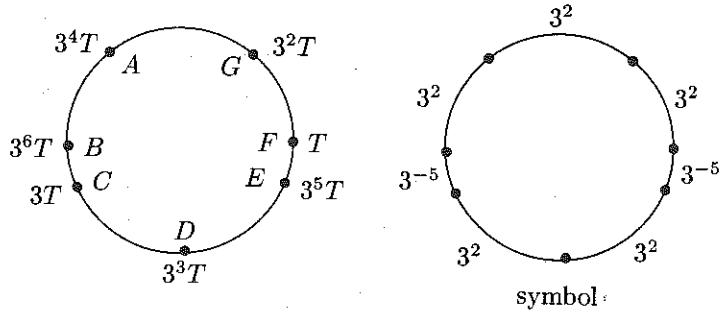
note S, except for $3^{n-1}T$, has its perfect fifth $3S$, but no one has its major third $5S$. Scales of fifths cannot be tempered because 3^n is not a power of 2.

When $n = 5$, one finds the *pentatonic* scale, frequently used in folk music (according to the *Britannica*, the pentatonic scale is used more widely than any other scale and Western art music is one of the few traditions in which pentatonic scales do not predominate):



symbol

When $n = 7$, one obtains the classical *diatonic* scale (the traditional name of each note figures inside the circle):



symbol

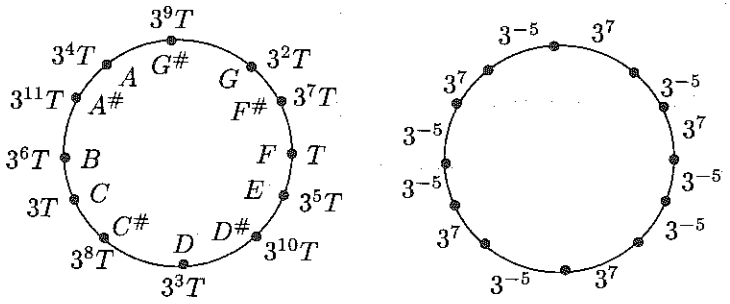
$$Do = C, Re = D, Mi = E, Fa = F, Sol = G, La = A, Si = B$$

and each note of this scale defines one of the seven classical modes:

Major	C	1	3^2	3^4	3^{-1}	3	3^3	3^5	Ionian
	D	1	3^2	3^{-3}	3^{-1}	3	3^3	3^{-2}	Dorian
	E	1	3^{-5}	3^{-3}	3^{-1}	3	3^{-4}	3^{-2}	Phrygian
	F	1	3^2	3^4	3^6	3	3^3	3^5	Lydian
	G	1	3^2	3^4	3^{-1}	3	3^3	3^{-2}	Mixolydian
Minor	A	1	3^2	3^{-3}	3^{-1}	3	3^{-4}	3^{-2}	Aeolian
	B	1	3^{-5}	3^{-3}	3^{-1}	3^{-6}	3^{-4}	3^{-2}	Hypophrygian

The last one is rarely used because it does not contain the perfect fifth (= 3). On the other hand, no mode contains the major third (= 5), but the major mode contains $3^4 = 81/64 = 1.265625$ which is much closer to $5/4 = 1.25$ than the interval $3^{-3} = 32/27 = 1.185$ of the minor mode.

When $n = 12$, one obtains the *chromatic* scale (a sharped note $S^{\#}$ denotes $3^7 S$ and a flatted note S^b denotes $3^{-7}S$):



symbol

In this scale $A^{\#}$ is the unique note without perfect fifth in the scale. In fact, the distance from F to $3A^{\#}$ is about 0.02. No note has its major third in this scale. The distance from $5S$ to the closest note in the scale is about 0.018 when S=A, B, C, D, E, F, G, $F^{\#}$ and it is about 0.002 when S=$A^{\#}$, $C^{\#}$, $D^{\#}$ and $G^{\#}$. The temperament of this scale is quite good, since its symbol only has two different intervals (3^7 and 3^{-5}) of similar length: the length of 3^7 is about 0.095 and the length of 3^{-5} is about 0.075.

The figure above shows that scales of fifths of 5 and 7 notes also have symbols with only two different intervals, while scales of fifths of 4, 6, 8, 9, 10 and 11 notes do not have this property. A scale of fifths is said to be *pythagorean* when its symbol has only two different intervals.

Theorem 1. *The numbers of notes of the pythagorean scales form the following sequence* (a_i)

$$2,\underbrace{3,5}_{2},\underbrace{7,12}_{2},\underbrace{17,29,41}_{3},53,\underbrace{94,\ldots,306}_{5},\underbrace{359,665}_{2},\underbrace{971,\ldots,15601}_{23},\ldots$$

where $a_{i+1} = a_i + b_i$ *and* b_i *is the term preceding the group of* a_i. *Moreover, the lengths of these groups are the terms (or partial quotients) of the continued fraction*

$$\frac{\log_2(3/2)}{\log_2(4/3)} = 1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{\ddots}}}}}}}$$

Furthermore, if $\alpha_1 > \alpha_2$ *are the lengths of the two intervals of the symbol of the pythagorean scale of* a_i *notes, then the lengths of the two intervals of the next one (the scale of* a_{i+1} *notes) are* α_2 *and* $\alpha_1 - \alpha_2$, *which is the distance of* $3^{\alpha_1}T$ *to* T.

Note 1.1: The first 36 terms of the continued fraction above are:
1, 2, 2, 3, 1, 5, 2, 23, 2, 2, 1, 1, 55, 1, 4, 3, 1, 1, 15, 1, 9, 2, 5, 7, 1, 1, 4, 8, 11, 1, 20, 2, 1, 10, 1, 4, ...

It is has been well-known for a long time that continued fractions are closely related to musical scales (see [1], [2], [6], [7] and [8]) and the theorem above shows a new relation.

If l_1, l_2 are two positive real numbers, then the terms r_1, r_2, r_3, ..., of the continued fraction l_1/l_2 have the following geometric

interpretation: if I_1, I_2 are two segments of lengths l_1, l_2 in a straight line, then I_2 fits r_1 times in I_1 and there remains a smaller segment I_3, then I_3 fits r_2 times in I_2 and there remains a smaller segment I_4, then I_4 fits r_3 times in I_3, and so on.

Since $\log_2(3/2)$ and $\log_2(4/3)$ are the lengths of the two intervals of the pythagorean scale of 2 notes, we conclude that

(1.2): *The last terms of the groups of the sequence (a_i) above are the natural numbers n such that 3^nT it is closer to T than 3^mT for any number $1 \leq m < n$ and they correspond to the pythagorean scales such that the smaller interval does not fit twice in the bigger one.*

The symbol of a pythagorean scale only has two different intervals $\alpha_1 > \alpha_2$, but it may be that this scale is highly non-tempered because α_1 may be much bigger than α_2. A pythagorean scale is said to be *quasi–tempered* when α_1 is smaller than $2\alpha_2$ and, by 1.2, these scales correspond to the last terms of the groups of the sequence (a_i) above. According to the geometric interpretation of the terms of a continued fraction, in such case the difference of the two intervals fits $r(n)$ times in α_2, where $r(n)$ denotes the length of the group following n; so that the scale has "better temperament" when $r(n)$ is bigger. From 1 and 1.1, one directly obtains the numbers of notes of the first 36 quasi-tempered pythagorean scales:

(1.3) :

n	= 2	5	12	41	53	306	665	15601	31867	79335	111202	190537
$r(n)$	= 2	2	3	1	5	2	23	2	2	1	1	55

We see that the first one improving the temperament of the chromatic scale has 53 notes and that the scale of 665 notes has a very good temperament. Moreover, (1.1) shows that the scale of 190537 notes has the best temperament among the first 36 terms. Since the 36th term is easily estimated to be greater than 10^{18}, any pythagorean scale improving the temperament of the scale of 190537 notes must have more than 10^{18} notes. However, if one looks for a scale improving the temperament more than it increases the number of notes, that is to say a number $n \geq 13$ such that

$$\frac{r(12)}{12} < \frac{r(n)}{n} \qquad \text{or} \qquad \frac{n}{4} < r(n)$$

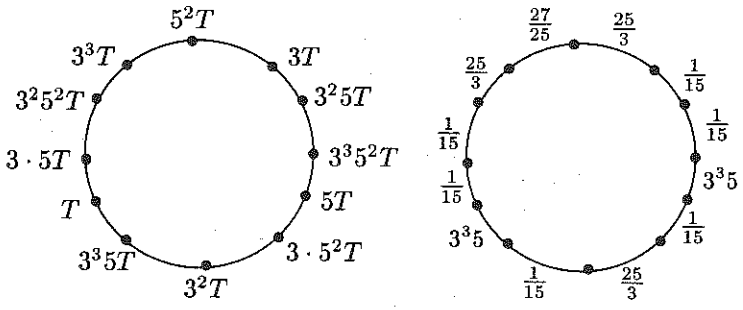
then (1.1) shows that the first 36 terms of this sequence do not fulfil this condition. We should wonder at the existence of a quasi–tempered pythagorean scale of more than 12 notes such that $r(n)/n$ is bigger than $r(12)/12$; but, unfortunately, we have no evidence to conjecture that no pythagorean scale with more than 12 notes improves the temperament more than it increases the number of notes (remark that $r(n)/n$ is not a decreasing sequence).

Euler's construction: The divisors of a given natural number form a mode, hence they define a scale. Euler considered the scale defined by the divisors of $d = 3^a5^b$. It is a scale of $(a+1)(b+1)$ notes such that any note, except 3^a5^iT, has its perfect fifth and any note, except 3^j5^bT, has its major third. These scales cannot be tempered, but Euler remarked that the scale of 12 notes corresponding to $d = 3^35^2 = 675$ is quite close to the chromatic scale and to the tempered scale of 12 notes:

Eulerian scale for 675											
1	3^35	3^2	35^2	5	3^35^2	3^25	3	5^2	3^3	3^25^2	$3\cdot5$
1	1.055	1.125	1.17	1.25	1.318	1.406	1.5	1.562	1.69	1.76	1.875
Pythagorean scale of 12 notes											
1	3^7	3^2	3^9	3^4	3^{-1}	3^6	3	3^8	3^3	3^{10}	3^5
1	1.068	1.125	1.2	1.27	1.333	1.424	1.5	1.602	1.69	1.8	1.898
Tempered scale of 12 notes											
1	r	r^2	r^3	r^4	r^5	r^6	r^7	r^8	r^9	r^{10}	r^{11}
1	1.059	1.122	1.19	1.26	1.335	1.414	1.498	1.588	1.68	1.78	1.888

$r = 2^{1/12}$

(Instead of the scales, this table shows one mode of each scale. For the chromatic scale, it is the mode corresponding to *C=Do.*) The symbol of the eulerian scale is



symbol

In this scale, the distance from $3 \cdot 3^3T$ to the closest note (in fact $5T$) is about 0.018 and the distance from $5 \cdot 5^2T$ to the closest note (in fact T) is about 0.034.

2. Approximation Indices

We should like to have "ideal scales" containing the diatonic scale T, $3T$, ..., 3^6T based on any note T of the scale, as well as its major third $5T$. Clearly no scale may satisfy these conditions (since no power of 2 is a power of 3 or 5); but we may look for scales fulfilling them approximately.

Definition: Let us consider a scale $\mathbf{S}$ and let S be a finite subset of the Octave representing $\mathbf{S}$. We define $\alpha_3(\mathbf{S})$ to be the infimum of all real numbers α such that the distance of $3T$ to S is $\leq \alpha$ for any $T \in S$ (hence the perfect fifth of any note in S is in S up to an error bounded by $\alpha_3(\mathbf{S})$). We define $\alpha_5(\mathbf{S})$ to be the infimum of all real numbers α such that the distance of $5T$ to S is $\leq \alpha$ for any $T \in S$. We define the *approximation index* of the scale $\mathbf{S}$ to be $\alpha(\mathbf{S}) = \max\{6\alpha_3(\mathbf{S}), \alpha_5(\mathbf{S})\}$.

By definition, if $T \in S$, then $3T, \ldots, 3^6T$ and $5T$ may be replaced by notes of S at a distance $\leq \alpha(\mathbf{S})$. Therefore, if $\alpha(\mathbf{S})$ were smaller than the human perception of acoustic pitch differences, then $\mathbf{S}$ could be considered as a human realization of the

"impossible ideal scale". In fact it seems that most men cannot differentiate two pitches when the distance is smaller than 0.003 (3 thousandths of the octave).

Let us consider the tempered scale $\mathbf{T}_n$ of n notes. This scale is represented by all rational numbers with denominator n. Hence, if a/n is the best approximation of $\log_2 3$ with denominator n, then we have $\alpha_3(\mathbf{T}_n) = |\log_2 3 - a/n|$. Therefore:

$$n \cdot \alpha_3(\mathbf{T}_n) = \{n \cdot \log_2 3\}$$
$$n \cdot \alpha_5(\mathbf{T}_n) = \{n \cdot \log_2 5\}$$

where $\{x\}$ denotes the distance of x to the closest integer number. For the 13 first tempered scales we obtain

n	2	3	4	5	6	7	8	9	10	11	12	13
α_3	85	82	85	15	82	13.5	40	29.4	15	39.5	1.63	30.4
α_5	178	14	72	78	11.4	36	53	11.4	22	42	11.4	14
α	510	490	510	90	490	81.2	240	176	90	237	11.4	182
$n\alpha_3$	170	245	340	75	490	94.5	320	265	150	435	19.6	395

where all the values are given in thousandths of octave, so that $\alpha(\mathbf{T}_{12}) \cong 0.0114$. We may see that the approximation index of $\mathbf{T}_{12}$ is much better than the indices of the previous tempered scales. The first tempered scales improving the index α_3 of $\mathbf{T}_{12}$ are $\mathbf{T}_{29}$ and $\mathbf{T}_{41}$, and the first tempered scale improving the index α of $\mathbf{T}_{12}$ is $\mathbf{T}_{41}$. In fact we have:

$$\alpha_3(\mathbf{T}_{41}) \cong 0.0004, \quad \alpha(\mathbf{T}_{41}) = \alpha_5(\mathbf{T}_{41}) \cong 0.0049$$

so that $\alpha(\mathbf{T}_{41})$ is quite close to the human perception of tonal differences.

Moreover, the tempered scale of 12 notes has a very good approximation index even if we consider arbitrary tuning systems. Any scale improving the index of $\mathbf{T}_{12}$ has more than 15 notes:

Theorem 2. *If the approximation index of a musical scale* $\mathbf{S}$ *is smaller than the approximation index of the tempered scale of 12 notes, then the number of notes of* $\mathbf{S}$ *is greater than 15.*

Proof of Theorem 1

If x is a real number, we shall denote by $[x]$ the decimal part of x; that is to say, $0 \leq [x] < 1$ and $x = n + [x]$ for some integer n, so that $[x]$ may be considered as the image of x in $\mathbf{R}/\mathbf{Z}$.

We shall always identify the Octave with the interval $[0, 1]$, where the end points 0 and 1 are identified. Hence, the scale of fifths of n notes is represented by $0 = [0l]$, $[l]$, $[2l]$, $\ldots$, $[(n-1)l]$, where $l = \log_2 3$. In fact, we shall only use that l is an irrational number.

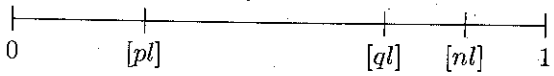
Let l be a fixed irrational number and let n be a natural number, $n \geq 2$.

If we consider the arithmetic progression $0l$, l, $\ldots$, $(n-1)l$ of n terms and we consider the increasing order of $[0l]$, $[l]$, $\ldots$, $[(n-1)l]$, we obtain a partition of the interval $[0, 1]$:

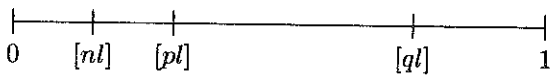
with an initial interval of length $[pl]$ and a final interval of length $[-ql] = 1 - [ql]$. This arithmetic progression of n terms is said to be *pythagorean* when the distance between two consecutive points of the partition is $[pl]$ or $[-ql]$. A pythagorean arithmetic progression is said to be *quasi-tempered* when the smaller interval is greater than a half of the bigger one.

Lemma. *If the arithmetic progression of n terms is pythagorean, then the corresponding partition of $[0, 1]$ has q intervals of length $[pl]$ and p intervals of length $[-ql]$, so that $p + q = n$. Moreover*

1) If the initial interval is smaller than the final one, $[pl] < [-ql]$, then the next pythagorean progression has $n + p$ terms and the initial and final intervals of the corresponding partition are



2) If the final interval is smaller than the initial one, $[-ql] < [pl]$, then the next pythagorean progression has $n + q$ terms and the initial and final intervals of the corresponding partition are



Proof: First we prove 1 and 2 assuming that the corresponding partition has q intervals of length $[pl]$ and p intervals of length $[-ql]$, $n = p + q$; so that the p intervals of length $[-ql]$ are just the intervals from $[(q+i)l]$ to $[il]$, $0 \leq i < p$, and the q intervals of length $[pl]$ are just the intervals from $[jl]$ to $[(j+p)l]$, $0 \leq j < q$ (remark that the two ends of any interval of length $[pl]$ or $[-ql]$ are consecutive points of the partition because no integer multiple of $[pl]$ may coincide with $[-ql]$ and no integer multiple of $[-ql]$ may coincide with $[pl]$, since l is assumed to be irrational).

We shall only consider the first case, the other being quite similar. In this case we have $[ql] + [pl] < 1$, so that $[nl] = [pl] + [ql]$ lies between $[ql]$ and 1; hence $[nl]$ divides the final interval in two intervals of lengths $[pl] = [nl] - [ql]$ and $[-nl]$. For the same reason, $[(n+i)l]$ divides the interval from $[(q+i)l]$ to $[il]$, $0 \leq i \leq p-1$, in two intervals of lengths $[pl]$ and $[-nl]$. Therefore, the next pythagorean progression has $n + p$ terms, the length of the initial interval is again $[pl]$ and the length of the final interval is just $[-nl]$. Hence assertion 1 is proved. Moreover, we obtain that the pythagorean progression of $n + p$ terms has p intervals of length $[-nl]$ and n intervals of length $[pl]$.

We conclude the proof of this lemma by induction on n, since it is obvious when $n = 2$.

Corollary. *If $\alpha_1 > \alpha_2$ are the lengths of the intervals of the partition of $[0, 1]$ defined by a pythagorean progression of n terms, then we have:*

1) the lengths of the intervals of the next pythagorean progression are α_2 and $\alpha_1 - \alpha_2$;

2) the distance from $[nl]$ to $0 = 1$ is just $\alpha_1 - \alpha_2$;

3) If $r\alpha_2 < \alpha_1 < (r+1)\alpha_2$, then the consecutive pythagorean progressions have

$$n+p,\ldots,n+rp \text{ terms when } [pl] < [-ql]$$

$$n+q,\ldots,n+rq \text{ terms when } [-ql] < [pl].$$

Corollary. *If $[-l] < [l]$ and $r_0, r_1, r_2, r_3, \ldots$ are the partial quotients of the continued fraction of $[l]/[-l]$, then the numbers of terms of the pythagorean arithmetic progressions defined by l are*

$$\underbrace{2,\ldots,r_0+1}_{r_0},\underbrace{a_1=r_0+2,\ldots,a_{r_1}}_{r_1},\underbrace{a_{r_1+1},\ldots,a_{r_1+r_2}}_{r_2},\ldots$$

where $a_{i+1} = a_i + b_i$ and b_i is the term preceding the group of a_i.

Proof: According to the geometric interpretation of the partial quotients of a continued fraction, this result follows directly from the corollary above, since the first pythagorean progression has 2 terms and the lengths of the corresponding intervals are just $[l]$ and $[-l]$.

Corollary. *The last terms of the groups of the sequence above are the natural numbers n such that $[nl]$ is closer to 0 than $[ml]$ for any $1 \le m < n$ and they correspond to the numbers of terms of the quasi-tempered pythagorean progressions.*

Proof of Theorem 2

Let $\alpha = \alpha(\mathbf{T}_{12}) = \frac{4}{12} - \log_2(\frac{5}{4})$ and $\beta = \log_2(\frac{3^4}{2^4 5})$ be the approximation of the major third in $\mathbf{T}_{12}$ and the chromatic scale respectively (the interval 81/80 from $5T$ to 3^4T is usually named the *comma*).

(1) *The distance between any two notes of the chromatic scale is larger than $5\alpha+\beta$ and 6α.*

In fact, the smallest interval between two notes of the chromatic scale is $2^8/3^5$. Since it is easy to check that $\alpha < \beta$, we must show that $2^8/3^5$ is greater than

$$\left(\frac{2^{4/12}4}{5}\right)^5 \cdot \frac{3^4}{2^4 5},$$

or, equivalently, $3^{27} < 2 \cdot 5^{18}$ and this inequality may be tested directly.

Now, let $\mathbf{S}$ be a scale such that $\alpha(\mathbf{S}) < \alpha$ and let S be a subset of the Octave representing $\mathbf{S}$. If $T \in S$, then there are notes $T_1,\ldots,T_6$ in S approaching $3T,\ldots,3^6T$ more than α. There are notes $T_7,\ldots,T_{11}$ in S approaching $3T_6,\ldots,3^6T_6$ more than α; hence approaching $3^7T,\ldots,3^{11}T$ more than 2α. Therefore, the notes in $F = \{T{=}T_0, T_1,\ldots,T_{11}\}$ approach the notes of the chromatic scale $\{T, 3T,\ldots,3^{11}T\}$ with an error $< 2\alpha$.

(2) *The distance between T_i and T_j is larger than 2α when $i \neq j$. In particular F has 12 different notes.*

Otherwise, the distance from 3^iT to 3^jT would be bounded by 6α, contradicting (1).

(3) *The distance from T_j to $5T_i$ is larger than α when $j \neq i+4$.*

Otherwise, the distance from 3^jT to $3^{4+i}T$ would be bounded by $d(3^jT,T_j) + d(T_j,5T_i) + d(5T_i,3^4T_i) + d(3^4T_i,3^43^iT) \le 2\alpha + \alpha + \beta + 2\alpha$, contradicting (1).

Finally, we compare the 12 intervals of F with the equal intervals of $\mathbf{T}_{12}$. Let $a_1,\ldots,a_{12}$ be the differences with 1/12 of the lengths of the 12 consecutive intervals that F defines in the Octave, so that $a_1 + \ldots + a_{12} = 0$.

If $a_j, a_{j+1}, a_{j+2}, a_{j+3}$ are the differences corresponding to the four intervals following a note $T_i \in F$, then the end of the fourth interval is just T_{i+4} (the index must be considered modulo 12, see the symbol of the chromatic scale). Hence:

$$T_{i+4} = T_i + a_j + \ldots + a_{j+3} + 4/12$$

and we obtain that the distance from T_{i+4} to $5T$ is exactly $\alpha + a_j + \ldots + a_{j+3}$, because the distance from $5T$ to $T_i + 4/12$ is

$\alpha(\mathbf{T}_{12}) = \alpha$. If $a_j + \ldots + a_{j+3} \geq 0$, by (3) we conclude that no note in F approaches $5T_i$ more than α, so that the note $S_i \in S$ approaching $5T_i$ more than α does not belong to F. Now, each one of the following additions $(i = 1, 2, 3, 4)$

$$(*)\ \ (a_i + \ldots + a_{i+3}) + (a_{i+4} + \ldots + a_{i+7}) + (a_{i+8} + \ldots + a_{i+11}) = 0$$

has at least one non-negative addend. So we obtain four different notes $T^1, T^2, T^3, T^4 \in F$ such that the notes $S^i \in S$ approaching $5T^i$ more than α do not belong to F. We conclude that S contains at least 16 different notes $T_0, T_1, \ldots, T_{11}, S^1, \ldots, S^4$ because we have $S^i \neq S^j$ when $i \neq j$. Otherwise, the distance from $5T^i$ to $5T^j$ would be smaller than 2α, contradicting (2).

Note: Let S represent a scale $\mathbf{S}$ such that $\alpha(\mathbf{S}) = \alpha(\mathbf{T}_{12})$. The argument above also shows that $F \subseteq S$ has 12 different notes. Hence, if $\mathbf{S}$ only has 12 notes, then $F = S$ and $\alpha_5(\mathbf{S}) > \alpha$ when some addend $a_j + \ldots + a_{j+3}$ is positive. By (*), it follows that $a_j + \ldots + a_{j+3} = 0$ for any index j and $\alpha_5(\mathbf{S}) = \alpha = \alpha_5(\mathbf{T}_{12})$. Now, it is easy to prove that the closest note to $3T_i$ in $F = S$ is just T_{i+1}, the end of the seventh interval following T_i; so that $\alpha_3(\mathbf{S}) \geq \alpha_3(\mathbf{T}_{12}) - a_{j+4} - a_{j+5} - a_{j+6} = \alpha_3(\mathbf{T}_{12}) + a_{j+7}$. Therefore, if $\alpha_3(\mathbf{S}) \leq \alpha_3(\mathbf{T}_{12})$, then $a_j \leq 0$ and we conclude that $a_j = 0$ for any index j; that is to say, $\mathbf{S} = \mathbf{T}_{12}$. Resuming, the tempered scale of 12 notes is the best tuning system with no more than 12 notes in a very precise sense:

Suppose that $n \leq 12$. If $\mathbf{S}$ is a scale of n notes such that $\alpha(\mathbf{S}) \leq \alpha(\mathbf{T}_{12})$, then $n = 12$, $\alpha(\mathbf{S}) = \alpha(\mathbf{T}_{12})$, $\alpha_5(\mathbf{S}) = \alpha_5(\mathbf{T}_{12})$, $\alpha_3(\mathbf{S}) \geq \alpha_3(\mathbf{T}_{12})$ and the last inequality is strict when $\mathbf{S} \neq \mathbf{T}_{12}$.

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SOME GALWAY PROFESSORS OF MATHEMATICS AND OF NATURAL PHILOSOPHY

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The first professor of mathematics at Queen's College, Galway was John Mulcahy, LL.D., (1810?-1853), who held the chair from 1849 until his death. He was a graduate of Trinity College, Dublin, and received a gold medal as best answerer in science in the B. A. degree examinations of 1829. According to [5], he was one of only two Roman Catholics appointed to the Faculty of Arts when the Galway college opened. He wrote a textbook, entitled *Principles of Modern Geometry*, published in Dublin in 1852. A second revised edition appeared in 1862. (*Principles of modern geometry, with numerous applications to plane and spherical figures; and an appendix, containing questions for exercise. Intended chiefly for the use of junior students.*) This book was recommended by Boole for purchase by the library of Queen's College, Cork in 1852 (see [7, pp.102-103]).

Mulcahy was succeeded by George Johnston Allman (1824-1904), who retained the professorship until 1893, when he retired. He was the son of William Allman, Professor of Botany at Trinity College, Dublin. His best known work is the book *Greek Geometry from Thales to Euclid*, published in 1889 in the Dublin University Press Series. This book was highly esteemed by contemporary historians of mathematics. It is based on a paper written in six parts in *Hermathena* between 1877 and 1887. Allman also wrote articles on Ptolemy, Pythagoras and Thales for the 9th edition of the *Encyclopaedia Britannica*. For obituaries of Allman, see Proc. Roy. Soc. London **78 A** (1907), p. xii, and Nature **LXX** (1904), 83.

Allman's successor was Alfred Cardew Dixon (1865-1936), who held the chair until 1901, when he was appointed to the chair of mathematics in Queen's College, Belfast. In his article, [13], James Ward has given some biographical details about Dixon. Dixon wrote one textbook, *The Elementary Properties of Elliptic Functions*, published in 1894 by Macmillan & Co.

Dixon was succeeded by Thomas John I'Anson Bromwich (1875-1929) in 1902 and Bromwich held the chair until 1907, when he took up a permanent lectureship at St John's College, Cambridge. Bromwich's best known contribution to mathematics is the book *An Introduction to the Theory of Infinite Series*, published in 1908 by Macmillan & Co. This was based on lectures on elementary analysis given at Galway. A revised second edition was published in 1926 and this has been reprinted several times. He also published another book, *Quadratic Forms and their Classification by Means of Invariant Factors*, in 1906. This book is an early example in English of the more abstract methods introduced into algebra by researchers such as Kronecker and Weierstrass. It is particularly concerned with the simultaneous reduction of two quadratic forms, a problem which, in its modern presentation, requires almost the full repertoire of the theory of a single linear transformation. In his obituary of Bromwich, Proc. Roy. Soc. London, **129 A** (1930), i-x, G. H. Hardy expressed the opinion that Bromwich's best work had been completed by 1908. The obituary makes interesting reading, as it gives a critical assessment of Bromwich's work and is certainly no mere eulogy. Bromwich seems to have worked both as a pure and applied mathematician, although not at the highest levels, according to Hardy. A slightly different form of the obituary was published in the Journal of the London Math. Soc. **5** (1930), 209-220. In this obituary, unlike the first, Hardy points out that Bromwich died by suicide. (See also *Collected Papers of G. H. Hardy*, vol. 7, 732-743.)

The next Professor of Mathematics was William A. Houston, who held the chair from 1908 until 1912.

After Houston, the next Professor of Mathematics is Michael Power who held the chair for over 40 years, from 1912 until his retirement in 1955. He obtained his B.A. in 1907, M.A. in 1908,

and M.Sc. in 1909.

There have been several distinguished holders of the chair of natural philosophy in Galway, some of whom did considerable work in mathematics, and we would like to discuss the first five of them briefly.

The first Professor of Natural Philosophy was Morgan William Crofton (1826-1915), who held the chair between 1849 and 1852. Crofton's career is interesting and we shall give some details about it, largely based on the obituary in the Proceedings of the London Math. Soc. (second series) **14** (1915), pp. xxix-xxx, and on [4].

Morgan Crofton was born in Dublin, and was the eldest son of Rev. W. Crofton, Rector of Skreen, in Co. Sligo. This is remarkable, as Crofton senior must have been the successor of George Gabriel Stokes's father, who was rector of Skreen until his death in 1834. Morgan Crofton obtained his degree from Trinity College, Dublin, topping the list of Senior Moderators in Mathematics ahead of G. J. Stoney (about whom, more below) in 1847. According to the obituary in the Proc. LMS, Crofton was denied the chance to stand for a fellowship at Trinity, as he had become a Roman Catholic. (This information is somewhat at variance with that contained in [4, p.96], which states that Crofton resigned the Galway professorship in 1853, about which time he entered the Catholic Church. According to [4, p.97], Crofton's son, Father William Crofton, S.J., wrote: "my father was not with Newman in Dublin, but he was instructed and received into the Church by Newman himself in the early 'fifties at Birmingham".) In any case, Crofton received a prize in the Fellowship Examination at Trinity in 1848.

After leaving Galway, Crofton worked in various Jesuit educational establishments in France. He seems to have come to England and got to know J. J. Sylvester, who was Professor of Mathematics at the Royal Military Academy in Woolwich. On Sylvester's recommendation, Crofton was appointed an instructor in mathematics at the Academy and he succeeded Sylvester as professor in 1870, holding the professorship until 1884. J. D. North's article on Sylvester in Vol. XIII of the *Dictionary of Scientific*

Biography mentions that one of Sylvester's last published papers, in the 1890's, on Buffon's needle problem, had been motivated by conversations with Crofton in the 1860's but that Crofton had already published identical material in 1868 (*On the theory of local probability*). The Royal Society's Catalogue of Scientific Papers lists 18 papers by Crofton, almost entirely on pure mathematics, especially probability theory, geometry and the calculus of operations. Some of his ideas on probability theory are discussed by John Venn in [12]. His teaching at Woolwich was directed towards mechanics and engineering mathematics, in keeping with the Army's needs, and he wrote one textbook for the Academy on applied mechanics, [2], as well as contributing to another one, [3].

Crofton retired from Woolwich in 1884 and became a member of the Mathematical Staff of the newly formed University College, Dublin. He cannot have done much teaching, as he continued to reside in England, and came over to Dublin mainly as an examiner in mathematics. He clearly collaborated with John Casey, also a member of the Mathematical Staff at University College, Dublin, on geometrical questions, as several of the exercises in Casey's books are attributed to Crofton. (There are 13 exercises in [1] that bear Crofton's name and Casey acknowledges his debt to Crofton in the preface to [1].) Crofton retired in 1895 and died in Brighton in 1915. He was awarded an honorary Doctorate in Science by Trinity College, Dublin in 1898.

Crofton wrote a substantial article on Probability for the 9th edition of the *Encyclopaedia Britannica*, which is still worth looking at. Interestingly enough, the article on Probability for the 11th edition of the *Encyclopaedia Britannica* was also written by a person with Trinity College, Dublin connections, Francis Ysidro Edgeworth (1845-1926). Edgeworth obtained a scholarship in classics from Trinity in 1862, but took his degree at Oxford in 1869. He wrote much initially in moral science and was later a pioneer worker in mathematical statistics, probability theory and economics.

Crofton was succeeded as Professor of Natural Philosophy in Galway in 1853 by George Johnstone Stoney (1826-1911), whom we have seen was second behind Crofton in the list of Senior Mod-

erators at Trinity College in 1847. He was the uncle of the physicist George Francis Fitzgerald, noted in connection with the Lorentz-Fitzgerald contraction. Stoney held the professorship until 1857, when he resigned to become Secretary of the Queen's University in Ireland. Although he died in London, Stoney is buried in the graveyard of St Nahi's Church, Dundrum, Dublin, where his tomb bears the inscription *Felix qui potuit rerum cognoscere causas*. Several other members of the Stoney family are also buried there.

Stoney wrote numerous scientific papers, and was especially interested in properties of spectral lines and also the measurement of fundamental physical units. He is best known for having coined the name *electron*, which arose in connection with the unit of electrical charge on an atom. Articles about Stoney may be found in [9], [10] and Vol. XIII of [6]. [10] in particular gives several references to Stoney's life and work.

Stoney's successor in the professorship was Arthur Hill Curtis, (1827?-1886?), who held the position from 1857 until 1879. He was also Registrar of Queen's College, Galway from 1877 until 1879. In 1880, he became Assistant Commissioner of Intermediate Education. He may possibly have died in 1886, as there are no further references to him in Thom's Directory after this date. There are some parallels in the early careers of Crofton and Curtis, as Curtis also topped the list of Senior Moderators in Mathematics at Trinity College, Dublin, this time in 1849, and both were Lloyd Exhibitioners (1846, 1848) and Bishop Law's Prizemen (1848, 1850) at Trinity. Curtis was also the first recipient of the MacCullagh Prize in Mathematics in 1855, for an essay on the subject of physical optics. The prize had been funded by subscriptions raised in memory of James MacCullagh, a former Professor of Mathematics and of Natural Philosophy at Trinity College, who had committed suicide in 1847. (A sum of £977 10s. 4d. was raised to fund the prize.) MacCullagh's research work was concerned mainly with mathematical models of the aether and the geometry of surfaces of the second order. See, for example, [11]. It is interesting to observe that four of the first scientific papers of Curtis were devoted to the geometry of surfaces, one in particular relating to MacCullagh's work (*A geometrical proof of Professor MacCullagh's theorem on the polar plane*, Quart. J. Math 1 (1857), 134-141, this paper having been written in 1855). It seems likely that MacCullagh's work influenced Curtis's early research. Papers written by Curtis after 1860 have titles reflecting an interest in physical questions, in keeping with his position in Galway. He published a book, *A Mathematical Deduction of the Principal Properties of the Gyroscope*, in Dublin in 1862.

Curtis was followed in the professorship by Joseph Larmor (1857-1942), who held the chair from 1880 until 1885. Larmor is a major figure in late 19th century physics, having contributed important ideas in electromagnetic theory and early relativity theory. He was especially interested in the motion of matter through the aether and wrote a related book, *Aether and Matter*, in 1900. He was an important administrator in scientific bodies and edited various collected editions of scientific papers (those of Stokes and Fitzgerald, for example). He succeeded G. G. Stokes as Lucasian professor of mathematics in 1903 and held this position until his retirement in 1932. He also served as M.P. for Cambridge University from 1911 until 1922. An article about Larmor, with bibliography, may be found in Vol VIII of [6].

The final Professor of Natural Philosophy in Galway whom we shall describe is Alexander Anderson (1858-1936), who succeeded Larmor in 1885 and retired in 1934. He was also President of Queen's College, Galway from 1899 until his retirement. He wrote many papers on a variety of physical topics. The present writer owns several of Anderson's books, on such subjects as electricity, optics, geomagnetism, elasticity and the electron. For further information about Anderson, we refer to the article [8].

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GEORGE GABRIEL STOKES 1819-1903 AN IRISH MATHEMATICAL PHYSICIST

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The name of Stokes, a contemporary of Kelvin and Maxwell, has become well known to generations of international scientists, mathematicians and engineers, through its association with various physical laws and mathematical formulae. In standard textbooks of mathematics, physics and engineering we find Stokes' Law, Stokes' Theorem, Stokes' Phenomenon, Stokes' conjecture and the Navier-Stokes equations. George Gabriel Stokes has long been associated with the University of Cambridge, where he spent all of his working life, occupying the Lucasian Chair of Mathematics from 1849 until his death in 1903. This prestigious chair was once held by Isaac Newton, and is currently occupied by Stephen Hawking, who has reached a wide audience outside mathematics with his "Brief History of Time". What is not well known is that Stokes was born in Skreen, County Sligo, where his father was Rector of the Church of Ireland, and received his early education there and in Dublin. Like William Thomson, later Lord Kelvin, who is often associated with Scotland (he occupied the chair of Natural Philosophy in Glasgow University) rather than with Belfast, where he was born, the contribution of Stokes has not been fully recognized in Ireland. Kelvin at least had a statue outside Queen's University, but Stokes lacked any memorial in the land of his birth. Perhaps this is a commentary on the importance which Irish society attaches to scientific vis-a-vis literary achievement. The situation was rectified, however, with the unveiling by former EU Commissioner for Agriculture, Mr Ray MacSharry, of a memorial at Stokes' birthplace in Skreen on Saturday 10th June 1995 as part of a meeting organized at Sligo RTC by the Institutes of Physics and of Mathematics and its Applications,

under the auspices of the Royal Irish Academy, as part of the Sligo 750 celebrations.

The first of the Stokes family to be recorded in history was Gabriel Stokes, born in 1682, a mathematical instrument maker residing in Essex Street, Dublin, who became Deputy Surveyor General of Ireland. Among his concerns was the use of "hydrostatic balance" to ensure a piped water supply to Dublin. His great grandson, George Gabriel, returned to this problem in one of his earliest papers "The internal friction of fluids in motion" where he discussed an application to the design of an aqueduct to supply a given quantity of water to a given place. Gabriel's elder son, John, was Regius Professor of Greek and his younger son, another Gabriel, was Professor of Mathematics, both in Dublin University.

The descendants of this professor of mathematics became an important medical family in Ireland and internationally (see the article by J. B. Lyons). The first of the medical Stokes was Whitley (1763-1845), a medical Fellow of Trinity College, Dublin, whose career was temporarily interrupted from 1798 to 1800 when he was suspended for his association with the United Irishmen. Besides holding, at various times, medical chairs in Dublin University and the College of Surgeons, Whitley was Donegall Professor of Mathematics for one year (1795) and published in 1821 "Observations on the population and resources of Ireland", charging Malthus with errors. The name of his son, William Stokes (1804-1878), is preserved in medicine through Cheyne-Stokes respiration and the Stokes-Adams syndrome in cardiology. His son, Sir William Stokes (1839-1900), was Professor of Surgery at the College of Surgeons and many of his descendants are working with distinction in medicine and academia today. It is interesting to note that George Gabriel, while primarily a mathematical physicist, did, like his great-uncle Whitley, cross the boundary between mathematics and medicine by discovering the respiratory function of haemoglobin.

It is from the first Gabriel's elder son, John Stokes, that George Gabriel Stokes is descended. Much less is known about his branch of the family. Almost all of G. G. Stokes's published papers appear in the five volume Mathematical and Physical

Papers (Cambridge, 1880-1905), together with obituaries, mainly assessing the value of his contributions to science, by Lord Kelvin and Lord Rayleigh. The latter contains some personal detail of Stokes, including a much quoted anecdote, which seems to have originated with his mathematics teacher in Bristol College, "His habit, often remarked in later life, of answering with a plain yes or no, when something more elaborate was expected, is supposed to date from his transference from an Irish to an English school, when his brothers chaffed him and warned him that if he gave long Irish answers he would be laughed at by his school fellows". The additional information presented here has been obtained from manuscripts in the Cambridge University Library. These include his correspondence with, and a memoir of his life produced by, the Rev. H. P. Stokes (no relation), Vicar of St Paul's Church, Cambridge. Pembroke College, of which George Gabriel was a Fellow, lies in this Parish, and he was Churchwarden during the incumbency of H. P. Stokes. Information has also been obtained from the Notes and Recollections of his daughter, Mrs Laurence Humphry, which appear in the book edited by Larmor, [2].

In 1798, Gabriel Stokes, son of John Stokes and Rector of Skreen, married Elizabeth, the daughter of John Haughton, the Rector of Kilrea. Their first child, Sarah, died in infancy, but they produced seven further children, of whom George Gabriel was the youngest. All of his four brothers became clergymen, the oldest, John Whitley, who was already 20 when George Gabriel was born, becoming Archdeacon of Armagh. In later life Stokes talked fondly of the scenery of his boyhood and his rambles within sound of the Atlantic breakers. Even in his paper "On the theory of oscillatory waves" he writes, in the midst of mathematical equations, of "the surf which breaks upon the western coasts as the result of storms out in the Atlantic". This paper also records a visit to the Giant's Causeway to observe wave phenomena. This very private and reserved Victorian scientist had the occasional habit of breaking into poetical descriptions in the middle of mathematical proofs. In his 1902 paper on asymptotics, he describes what is now known as Stokes' phenomenon as "the inferior term enters as it were into a mist, is hidden for a little from view, and comes out with its

coefficient changed". Perhaps as a boy he had watched the mists skim the surface of flat-topped Ben Bulben across the bay, an area which was later to influence the poet W. B. Yeats. There can be no doubt that George Gabriel was greatly inspired by his upbringing in the West of Ireland, and he returned regularly for the summer vacation, a non-trivial exercise in the pre-railway era, while a student in England. Even after the death of his parents he continued to visit his brother John Whitley, then a clergyman in Tyrone, and his sister, Elizabeth Mary, to whom he was greatly attached, in Malahide almost annually until his death.

His first mathematics teacher was the Clerk of Skreen Parish, who recorded George Gabriel as "working out for himself new ways of doing sums, better than the book". He read classics with his father, who by this time was getting old; he had been 52 when George Gabriel was born. In 1832 he sent the young George Gabriel to live with his oldest brother John Whitley in Dublin so that he could attend, as a day boarder, a Dr Wall's School in Hume Street where he attracted attention by his elegant solution of geometrical problems. Gabriel Stokes died in 1834, and his widow and two daughters had to leave Skreen Rectory, but money was found to send George Gabriel to school in England. His second brother, William Haughton, had been 16th Wrangler in the Cambridge Mathematical Tripos of 1828, and obtained a Fellowship at Caius College. It was he who recommended Bristol College, whose Headmaster was Joseph Henry Jerrard, an honorary Fellow of Caius. Most of Stokes' family connections had been with Trinity College. A link with University College, Dublin, now existed in the person of his mathematics teacher in Bristol, Francis Newman, brother of Cardinal Newman. Francis Newman wrote that Stokes "did many of the propositions of Euclid as problems, without looking at the book". Stokes appears to have had a great affection for Newman, whom he records as having "a very pleasing countenance and kindly manners".

George Gabriel Stokes entered Pembroke College, the third oldest in Cambridge, as an undergraduate in 1837. H. P. Stokes points out that Queen Victoria, who had been born in the same year, 1819, as Stokes, ascended the throne in the same year as he

entered university, although he outlived her by two years. Distinguished graduates from Pembroke included the martyr, Bishop Ridley, the poets Spenser and Gray, and the statesman William Pitt. Although a mathematical prodigy at school, Stokes was beaten into second place in his first year at Pembroke by one John Sykes. From second year onwards he studied, as was the custom at that time, for the highly competitive Mathematical Tripos with a private tutor, William Hopkins. So effective were these studies that Stokes was Senior Wrangler (that is, placed first in mathematics in the whole university) in 1841 and elected to a Fellowship at Pembroke. His early research was in the area of hydrodynamics, both experimental and theoretical, during which he put forward the concept of "internal friction" of an incompressible fluid. This work was independent of the work of Navier, Poisson and Saint-Venant which was appearing in the French literature at the same time, but Stokes' methods could also be applied to other continuous media such as elastic solids. He then turned his attention to oscillatory waves in water, producing the subsequently verified conjecture on the wave of greatest height, which now bears his name.

Such was Stokes' reputation as a promising young man, familiar with the latest Continental literature, that in 1849 he was appointed to the Lucasian Chair of Mathematics. At the same time, to augment his income from this poorly endowed chair, he taught at the School of Mines in London throughout the 1850's. Although appointed to the Lucasian Chair for his outstanding research, Stokes showed a concern in advance of his time for the welfare of his students, stating that he was "prepared privately to be consulted by and to assist any of the mathematical students of the university". It is recorded that Babbage, an earlier incumbent, never once addressed classes. Stokes immediately advertised that "the present professor intends to commence a lecture course in Hydrostatics", which he was still delivering 53 years later, in the last year of his life. Stokes' manuscript notes still exist in the University Library in Cambridge, although his writing was so bad that he eventually became one of the first people in Britain to make regular use of a typewriter.

The pure mathematical results of Stokes arose mainly from the needs of the physical problems which he and others studied. He was a mathematician very much driven by the needs of industrial applications in his own time. Besides his links with the School of Mines, he acted, over a period of many years, as consultant to the lensmaker Howard Grubb who ran a successful and internationally-known optical works in Rathmines. He also acted as advisor on lighthouse illuminants to Trinity House. Stokes' collected works include a paper on a differential equation relating to the breaking of railway bridges and, following the Tay Bridge disaster, he served on a Board of Trade committee to report on wind pressure on railway structures. His paper on periodic series concerned conditions for the expansion of a given function in what we now know as a Fourier series. He is also credited with having had the idea of uniform convergence of a series. His major work on the asymptotic expansion of integrals and solutions of differential equations arose from the optical research of G. B. Airy. The well-known theorem in vector calculus which bears his name is sadly not due to Stokes, but was communicated to him in a letter by Lord Kelvin. The confusion appears to have arisen because Stokes set the proof of this theorem as Question 8 in the Smith's Prize Examination Paper for 1854! There is justice in this, however, as Stokes was undoubtedly generous in sharing his unpublished ideas with others, notably with Kelvin over spectral analysis. In its leader of 3rd February 1903, following his death two days earlier, The Times wrote that "Sir G. Stokes was remarkable ... for his freedom from all personal ambitions and petty jealousies".

Stokes continued his researches in the principles of geodesy (another link with his surveyor great-grandfather) and in the theory of sound, which he treated as a branch of hydrodynamics. But perhaps his major advance was in the wave theory of light, by then well established at Cambridge, examining mathematically the properties of the ether which he treated as a sensibly incompressible elastic medium. This enabled him to obtain major results on the mathematical theory of diffraction, which he confirmed by experiment, and on fluorescence, which led him into the field of spectrum analysis. His last major paper on light was his study

of the dynamical theory of double refraction, presented in 1862. After this his time was increasingly taken up with scientific and academic administration.

A major reason for this change was that in 1851 he had been elected a Fellow of the Royal Society and shortly afterwards, in 1854, became Secretary of the Society, where he performed an important role in advising authors of research papers of possible improvements and related work. A fellow member of the Council of the Society wrote "One of the distinguishing characteristic qualities of Sir George was the generous way in which he was always ready to lay aside at once, for the moment, his own scientific work, and give his whole attention and full sympathy to any point of scientific theory or experiment about which his correspondent had sought his counsel". He acted as a sounding board for many famous scientists, including Lord Kelvin, with whom he carried on an extensive correspondence, recently edited by David B. Wilson and published by Cambridge University Press (1990), [7]. He was also extremely active in the British Association for the Advancement of Science. Many of his colleagues, including Kelvin, regretted his taking on these administrative duties and P. G. Tait even went so far as to write a letter to Nature protesting at "the spectacle of a genius like that of Stokes' wasted on drudgery [and] exhausting labour".

In 1859 Stokes vacated his Fellowship at Pembroke, as he was compelled to do by the regulations at that time, on his marriage to Mary Susannah, daughter of Dr Thomas Romney Robinson, FRS, Astronomer at Armagh. Following a change in regulations, he was subsequently able to resume his Fellowship and for the last year of his life served as Master of Pembroke. After a short stay in a house adjacent to Addenbrookes Hospital, the couple moved to Lensfield Cottage, which lay in a large garden opposite the south side of Downing College. This was by all accounts a happy and charming home, in which Stokes had a "simple study" and conducted experiments "in a narrow passage behind the pantry, with simple and homely apparatus". Do not forget that his great-grandfather had started out as an instrument maker! Unfortunately, the family life of George Gabriel and Mary was marked by tragedy: their first

two daughters died in infancy, and Stokes himself was seriously ill with scarlet fever; their second son, William George, survived to qualify as a medical doctor, but died in 1893 of an accidental overdose of morphine while a trainee general practitioner in Durham. But their elder son, Arthur Romney, a graduate of King's College, became a master at Shrewsbury School, and their youngest daughter, Isabella Lucy, married Dr Lawrence Humphry in 1889. The couple lived with Stokes at Lensfield Cottage and cared for him after the death of his wife in 1899.

Prior to their marriage Stokes, who was a tireless writer of letters, had carried on an extensive (one letter ran to 55 pages) and frank correspondence with his fiancée. In one letter, the theme of which will be familiar to all spouses of research mathematicians, he states that he has been up until 3 a.m. wrestling with a mathematical problem and fears that she will not permit this after their marriage! Based on remarks on loneliness, brooding and lack of domestic affection in other letters in this highly personal correspondence, David Wilson, [6], has suggested that "Stokes himself may have welcomed what others regretted - his abandonment of the lonely rigours of mathematical physics for domestic life and the collegiality of scientific administration".

At the General Election of 1887, Stokes offered himself as Member of Parliament for Cambridge University. As was the custom, his nomination was unopposed, but he issued a single election address, the main plank of which was opposition to the disestablishment and disendowment of the Church of England, a not surprising position for the son of an Anglican clergyman. His election caused dissension among the Fellows of the Royal Society, of which he was then President (1885-1890). Some Fellows, to judge from correspondence in Nature at the time, felt it improper that both positions should be held simultaneously and saw a possible conflict of interest. It was pointed out, however, that his distinguished predecessor, Isaac Newton, had successfully combined the holding of these academic and political offices. In Westminster, Stokes sat with the Conservatives and supported them on the Irish Question (that is, against Home Rule). He is recorded as having spoken only three times in Parliament: on 13th August 1888, in favour

of University representation on the Town Councils of Oxford and Cambridge; and on 15th August 1889 in support of two officials of the British Museum (of which he was a Trustee) who had been permitted, on behalf of The Times, to do some work for the special Irish Commission. He assured the House that the work had been done entirely out of hours! His third contribution, on 1st July 1891, was to support an amendment to the Free Education Act to enable ten shillings to be paid to every child attending school during forty weeks of the year. The amendment was defeated, and Stokes did not speak again. He found the hours of Parliament most uncongenial and he did not stand for re-election in 1892.

A deeply religious man, Stokes had always been interested in the relationship between science and religion. From 1886 to 1903 he was President of the Victoria Institute, whose aims were "To examine, from the point of view of science, such questions as may have arisen from an apparent conflict between scientific results and religious truths; to enquire whether the scientific results are or are not well founded". He delivered the Burnett lectures (on light) in the University of Aberdeen from 1883-85 and the Gifford lectures (on natural theology) in the University of Edinburgh in 1891 and 1893. Many honours were bestowed on him in later life. He was made a baronet (Sir George Gabriel Stokes) by Queen Victoria in 1889, was awarded the Copley Medal of the Royal Society in 1893, and in 1899 given a Professorial Jubilee (50 years as Lucasian Professor) by the University of Cambridge. Stokes died at Lensfield Cottage at 1 a.m. on Sunday, 1st February 1903. As a mathematician I can do no better than quote to you the leading article of The Times, which appeared two days after his death:-

"It is sometimes supposed-and instances in point may sometimes be adduced-that minds conversant with the higher mathematics are unfit to deal with the ordinary affairs of life. Sir George Stokes was a living proof that if the mathematician is only big enough, his intellect will handle practical questions so easily and as well as mathematical formulas".

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SIR GEORGE GABRIEL STOKES: THE MALAHIDE CONNECTION

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While the Irish origin and family connections of Sir George Gabriel Stokes are well documented, [1], [2], the fact that he maintained continuous contact with his family in Ireland is probably not as well known. This connection is recorded on a brass tablet which is mounted on the wall of the east (right) transept of Saint Andrew's Church, Malahide. The inscription reads

To the Glory of God and in memory of
Sir George Gabriel Stokes Bart
Master of Pembroke College
and for 53 years Professor of Mathematics
in the University of Cambridge
which he represented in Parliament 1887–92.
President of the British Association 1869,
and of the Royal Society 1885–90.
He was a Member of the French Academy of Science,
was decorated with the Prussian Order of Merit,
& received many other honours for his discoveries
in previously unexplored regions of Science
Endowed with rare intellectual gifts
yet simple hearted as a child
and seeking truth above all things
He was a devout believer in Him
Whom he often worshipped in this church
and in the Knowledge of Whom is everlasting life
Born in County Sligo, 1819. Died at Cambridge, 1903.
"The Lord is my Light"

The name of the tablet maker is inscribed on the lower margin, namely, T. R. Scott & Co., Dublin. This was a firm of wholesale cabinet makers, shop fitters and straw case makers which supplied school and church furnishings from their joinery works at Upper Abbey Street, [3].

In relation to the plaque itself a "form of Certificates of consent to Alterations proposed to be made in a Church vested in the representative Church Body" is held in the Archives of the Representative Body of the Church of Ireland. This dates the certification of the permission by the Bishop or Ordinary to "introduce" the brass tablet as 10th July 1903. There is no record of the date on which the tablet was dedicated but it is reasonable to surmise that this Church ceremony took place on or near the birth date of Stokes, 13th August.

A gravestone in the adjacent churchyard provides further evidence of the Malahide connection. The inscription on the stone is

> *Sacred to the memory of Elizabeth Stokes*
> *relict of the late Revd. Gabriel Stokes*
> *rector of Skreen and Vicar General of the*
> *Diocese of Killala who departed this life*
> *November 30th AD 1866 in the 86th year of*
> *her age.*
> *Revelation XIV Ch. 13V.*
> *"Her children arise up and call her blessed"*
> *Prov XXXI Ch. 28 V.*
> *Also of her daughter Elizabeth Mary Stokes*
> *who died June 18 1904 aged 93*
> *Continuing instant in prayer*

The firstnamed Elizabeth Stokes was the mother of Sir George Gabriel Stokes. She was the daughter of John Haughton, rector of Kilrea, County Derry, while the secondnamed Elizabeth Mary Stokes was her unmarried daughter and older sister of Sir George Gabriel. His mother lived at 4 Windsor Terrace, Malahide, during the last years of her life. An entry in the 1865 edition of Thom's Directory, [4], which lists the householders of Malahide, shows a

Mrs Stokes at that address for the first time. The house, a semi-detached two storey over basement type, still stands as a private residence to the north of (downhill from) Saint Andrew's Church on the opposite side of Church Road. Mrs Stokes continued to be recorded as the householder in Thom's Directory until the 1867 edition following which there was a break of two years, the listing being "Vacant" for that period. In the 1869 edition the entry was changed to Miss Stokes, which continued unchanged until the 1905 edition.

There is no documentation relating to the Malahide connection in the collection of his academic papers and writings which is held at Pembroke College Library, Cambridge, [5]. However, the following extract from *Alumni Cantabrigienses*, [6], provides evidence of the continued link with his family in Ireland.

In 1837, the year of Queen Victoria's accession, he commenced residence at Cambridge, where he was to find his home, almost without intermission, for sixty-five years. In those days sport was not the fashion of reading men, but he was a good walker, and astonished his contemporaries by the strength of his swimming. Even at a much later date he enjoyed encounters with wind and waves in his summer holidays on the north coast of Ireland.

With the development of the railway system and sea ferries between Ireland and Great Britain the Victorian traveller had frequent and rapid transport between London and Dublin. At the end of the 19th century three express trains departed from Euston Station, London for Westland Row Station, Dublin via Kingstown from Monday to Saturday with one express train on Sundays. The total rail and sea journey time was 9 hours, the City of Dublin Steam Packet Company providing the sea ferry service from Holyhead to Kingstown. A frequent rail service was provided by the Great Northern Railway (Ireland) from the Amiens Street terminus to Malahide.

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Book Review

Introduction to MAPLE

Andre Heck
Springer-Verlag 1993, 497 pp
ISBN 0-387-97662-0 (New York)
ISBN 3-540-97662-0 (Berlin)

Reviewed by Pat O'Leary

This book is the first real introduction to Maple and, as such, is very welcome. The author is managing director of the CAN (Computer Algebra in the Netherlands) which stimulates and coordinates the use of computer algebra in education and research. The book is an introduction and has to be viewed as such. It begins by discussing computer algebra and as well as discussing the advantages, some limitations of computer algebra are mentioned. The version of Maple used is release 2 of Maple V, which has been superseded by the launch of version 3 in April 1994 (a common problem with books on software) but given the introductory nature of the book, and the nature of changes in the new release, this does not cause major problems.

After the introduction, the basic syntax of Maple is introduced at a very reasonable pace and there are many good exercises at the end of each chapter. There is a very clear exposition of the structure of the language and of data types (a subject that often causes problems for students). The author also illustrates some difficulties that arise with examples, particularly with plotting. In the chapter on solving equations there is a nice demonstration of the use of Gröbner basis for solving non-linear differential equations. The last chapter looks at applications using the Linear Algebra package. The book has an extensive list of references on the material of the book. Given the large number of examples of code in the book, it would have been greatly enhanced if a diskette with code had been included with it, or was even available as a

companion to it. Also there is a very sparse amount of material on procedures in Maple. Overall this book is a very welcome addition to the literature on Maple.

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Book Review

Theory of Singular Boundary Value Problems

D. O'Regan
World Scientific, Singapore, 1994, xi+154pp
ISBN 981-02-1760-9
Price $ 38.00, hardback.

Reviewed by Johnny Henderson

The last decade has given rise to much activity in the area of boundary value problems (BVP's) for singular ordinary differential equations (ODE's), with this book's author contributing significantly to that activity. The book under review presents some topics of current interest in the theory of regular and singular BVP's (singular in both independent and dependent variables), with the two objectives to serve as a graduate text on the existence theory for these problems, as well as acquainting researchers new to the field with results and methods. The author states that no attempt has been made to deal in greatest generalities, and yet while the book is restricted to second order ODE's, a very general theory is developed for singular two-point BVP's in this context. While the book is self-contained, a reasonable background in real and functional analysis is assumed on the part of the reader.

There are ten clearly written chapters. While there are no formally listed exercises, the work involved in verifying results for cases analogous to those the author presents in detail serves as an adequate set of exercises. References are included at the end of each chapter.

Chapter 1 is an introduction, which serves as motivation for the study of singular two-point BVP's for second order ODE's, via presentation of problems involving, for example, the study of steady-state oxygen diffusion in a cell with Michaelis-Menten kinetics, the determination of the electrical potential in an atom due

to Thomas and Fermi, and the study of the Emden-Fowler equation for the non-linear phenomena in non-Newtonian fluid theory. Chapter 2 is devoted to Fixed Point Theory which the author primarily will use in establishing solutions of regular and singular BVP's including the problem mentioned in the first chapter. More precisely, the author develops in detail a non-linear alternative theory known as the Leray-Schauder Alternative, based on essential mappings and homotopy equivalence within the framework of topological transversality which A. Granas introduced in 1976. Many of the book's existence results rely on the application of the following result.

Theorem (Non-linear Alternative). *Let C be a convex subset of a normed linear space E and let U be an open subset of C with $p^* \in U$. Let $F : \bar{U} \to C$ be a compact continuous map. Then at least one of the following holds:*

(i) F has a fixed point.

(ii) There is an $x \in \partial U$ with $x = \lambda F(x) + (1-\lambda)p^*$, for some λ with $0 < \lambda < 1$.

Application of this theorem is first made in Chapter 3 in obtaining solutions in Section One of the equation

$$\frac{1}{p}(py')' = qf(t, y, py'), \quad 0 < t < 1, \tag{1}$$

and in Section Two of the equation

$$\frac{1}{p}(py')' + ry + \kappa py' = f(t, y, py'), \quad \text{a.e. on } [0,1], \tag{2}$$

satisfying boundary conditions of various types:

$$\text{(Sturm-Liouville)} \begin{cases} -\bar{\alpha}y(0) + \bar{\beta}\lim_{t\to 0+} p(t)y'(t) = c_0, \\ ay(1) + b\lim_{t\to 1-} p(t)y'(t) = c_1, \end{cases} \tag{3}$$

where $\bar{\alpha} > 0$, $\bar{\beta} \geq 0$ in the first equation and $a \geq 0$, $b \geq 0$, $a^2 + b^2 > 0$ in the second.

$$\text{(Mixed)} \begin{cases} \lim_{t\to 0+} p(t)y'(t) = c_0, \\ ay(1) + b\lim_{t\to 1-} p(t)y'(t) = c_1, \quad a > 0, b \geq 0, \end{cases} \tag{4}$$

$$\text{(Neumann)} \begin{cases} \lim_{t\to 0+} p(t)y'(t) = c_0, \\ \lim_{t\to 1-} p(t)y'(t) = c_1, \end{cases} \tag{5}$$

$$\text{(Periodic)} \begin{cases} y(0) = y(1), \\ \lim_{t\to 0+} p(t)y'(t) = \lim_{t\to 1-} p(t)y'(t), \end{cases} \tag{6}$$

$$\text{(Bohr)} \begin{cases} y(0) = c_0, \\ \int_0^1 \frac{ds}{p(s)} \lim_{t\to 1-} p(t)y'(t) - y(1) = c_1, \end{cases} \tag{7}$$

where in the case of (1) with (3) and (1) with (4), $f : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ is continuous, $q \in C(0,1)$, $p \in C[0,1] \cap C^{(1)}(0,1)$ and both $p > 0$ and $q > 0$ on $(0,1)$, while in the case of (2) with any of (3) to (7), assumptions include $pf : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ is L^1-Caratheodory, and $p\kappa \in L^1[0,1]$. To apply the Non-linear Alternative to obtain solutions of, say, (1) with (3), *a priori* bounds, independent of parameter λ, are exhibited on solutions of an associated family of problems,

$$\frac{1}{p}(py')' = \lambda qf(t, y, py'), \quad 0 < t < 1, \quad 0 < \lambda < 1, \tag{8}$$

satisfying condition (3). The first existence theorem, Theorem 3.3, states that if this *a priori* bound exists on solutions of (8) with condition (3), for all λ, and if certain integrability is assumed,

$$\int_0^1 \frac{ds}{p(s)} < \infty \text{ and } \int_0^1 p(s)q(s)ds < \infty,$$

then (1) with condition (3) has a solution. The *a priori* bound arguments employed by the author involve tremendous amounts of calculations, often tedious, yet these arguments give an excellent display of the work required to obtain bounds necessary to apply the Alternative. As such, the arguments are rather elegant. The statement of the first existence theorem is typical of most throughout the book and the methods set the tone for the arguments to be employed. For example, in dealing with the singular problems (2) with conditions (3)-(7), it is also necessary to assume that the corresponding homogeneous BVP has only the

trivial solution, in effect, giving rise to a Green's function, which is then used to define a compact mapping F to which the Non-linear Alternative is applied.

The purpose of Chapter 4 is to apply operator theory methods to obtain solutions of regular and singular eigenvalue problems

$$Ly = \lambda y, \quad 0 < t < 1, \tag{9}$$

satisfying any of the homogeneous boundary conditions corresponding to (3)-(7), where

$$Ly = \frac{1}{pq}(py')'.$$

In addition to the hypotheses above on p and q in the case of (1) and condition (3), it is assumed that the domain of L, $D(L)$, is given by

$$\begin{aligned} D(L) = \{v \in C[0,1] : v, pv' \in AC[0,1],\ (pv')' \in L^2[0,1], \\ \text{and } -\bar{\alpha}v(0) + \bar{\beta} \lim_{t\to 0^+} p(t)v'(t) \\ = av(1) + b \lim_{t\to 1^-} p(t)v'(t) = 0\}. \end{aligned}$$

The operator

$$L^{-1} : L^2_{pq}[0,1] \to D(L) \subseteq L^2_{pq}[0,1]$$

is a completely continuous, symmetric operator (making use of the Green's function), by which operator theory results yield an infinite number of real eigenvalues of L with corresponding eigenvectors in $D(L)$, as well as establishing some Rayleigh-Ritz integral inequalities, such as the Wirtinger inequality when $\beta = b = 0$ and $p = q = 1$. Similar results are obtained for the cases of mixed, Neumann, periodic and Bohr problems with each involving appropriate $D(L)$. Again the mechanics could be described as tedious, yet they are beautifully done. Within the chapter's development, a well-written and detailed exposition is given on the spectrum of a symmetric, completely continuous operator.

The first part of Chapter 5 is devoted to upper and lower solutions methods for obtaining solutions of (1) with condition (3) (where $a > 0$ and q is assumed positive on $(0,1)$ such that $pq \in L^1[0,1]$, with the same hypotheses on p and f as when (1) with condition (3) was treated in Chapter 3). An upper solution β is defined in the natural way to mean $\beta \in C[0,1] \cap C^{(2)}(0,1)$ such that $p\beta' \in C[0,1]$ and satisfies

$$qf(t,\beta,p\beta') \geq \frac{1}{p}(p\beta')', \quad 0 < t < 1,$$

$$\bar{\alpha}\beta(0) + \bar{\beta} \lim_{t\to 0^+} p(t)\beta'(t) \leq c_0,$$

$$a\beta(1) + b \lim_{t\to 1^-} p(t)\beta'(t) \geq c_1,$$

with a lower solution α defined by reversing the inequalities. The results are such that, if there exist upper and lower solutions β and α of (1) with $\alpha \leq \beta$ on $[0,1]$ in conjunction with a Nagumo type condition for the singular setting, along with additional technical hypotheses, then (1), (3) has a solution $y \in C[0,1] \cap C^{(2)}(0,1)$ with $\alpha \leq y \leq \beta$ and $py' \in C[0,1]$. The chapter is fairly complete, including a discussion of radial solutions of elliptic PDE's in spherical domains for which $\frac{1}{p} \notin L^1[0,1]$, since $p(t) = t^{n-1}$. Yet exchanging that integrability condition for

$$\int_0^1 \frac{1}{p(s)} \int_0^s p(x)q(x)dx\,ds < \infty,$$

along with some of the technical hypotheses, upper and lower solutions methods are successfully applied. The chapter then returns to applications of the Non-linear Alternative, when, motivated by the Thomas-Fermi equation, *a priori* bounds are established for solutions of the associated one-parameter family

$$\frac{1}{p}(py')' = \lambda qf(t,y), \quad 0 < \lambda < 1, \quad 0 < t < b,$$

satisfying

$$y(0) = a, \quad p(b)\int_0^b \frac{ds}{p(s)} y'(b) - y(b) = 0.$$

Chapter 6 first makes use of Rayleigh-Ritz minimization theorems with respect to the L^2_{pq}-norm to establish existence of solutions of (1) with (3), with $c_0 = c_1 = 0$, where f can be decomposed as

$$f(t,u,v) = g(t,u,v) + h(t,u,v),$$

with $g, h : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ both continuous,

$$|h(t,u,v)| \leq K\{|u|^\gamma + |v|^\tau + 1\}, \quad 0 \leq \gamma,\ \tau < 1,$$

and for $C \in \mathbf{R}$ and $d \leq 0$,

$$ug(t,u,v) \geq C|u|^2 + d|uv|,$$

and

$$|g(t,u,v)| \leq A(t,u)|v|^2 + B(t,u),$$

where A and B are bounded on bounded sets. Also $p(t)\sqrt{q(t)}$ is bounded on $[0,1]$. Then very nice applications of Hölder's inequality and the results of Chapter 4 yield *a priori* bounds on solutions of corresponding one-parameter family of problems, so that the Non-linear Alternative can be applied to yield solutions of (1) with (3) (with $c_0 = c_1 = 0$), provided

$$C^+ - dN_0\sqrt{\mu} < \mu,$$

where μ is the first eigenvalue of $Ly = \lambda y$ satisfying condition (3) (with $c_0 = c_1 = 0$ and L is as in Chapter 4), $C^+ = \max\{0, -C\}$, $N_0 = \sup_{[0,1]} p(t)\sqrt{q(t)}$. Similar treatment is given to the case when $p\sqrt{q}$ is singular at $t = 0$ and/or $t = 1$, even including

$$\int_0^1 \frac{ds}{p(s)} = +\infty.$$

The chapter concludes with a discussion of the non-existence and existence of solutions of (1) satisfying, for example, $y(0) = y(1) = 0$. With the equations

$$y'' = (y')^2 + \pi \text{ and } y'' = (y')^2 - \pi^2$$

as models, the author points out that what is important is not the growth of solutions, as $|y'| \to \infty$, but is rather the zero set of f. Once again for the case of existence, *a priori* bounds on solutions of an associated one-parameter family are exploited, leading to an application of the Non-linear Alternative.

In Chapter 7, the author considers singular BVP's for

$$\frac{1}{p}(py')' + \mu qy = f(t,y,y') \text{ a.e. on } [0,1], \tag{10}$$

for the non-resonant case $\lambda_{m-1} < \mu < \lambda_m$, and for the resonant case $\mu = \lambda_m$, $m = 1, 2, \ldots$, where $\lambda_0 = -\infty$ and the λ_i are assumed to be eigenvalues of the appropriate homogeneous problem associated with Sturm-Liouville, Neumann or periodic boundary conditions, and where f decomposes as

$$f(t,u,v) = \eta v + g(t,u,v),$$

with $\eta p \in L^1[0,1]$, $pq \in L^1$-Caratheodory, and

$$|g(t,u,v)| \leq \varphi_1(t) + \varphi_2(t)|u|^\gamma + \varphi_3(t)|v|^\theta,$$

$p\varphi_i \in L^1[0,1]$, and

$$\sup_{[0,1]} \int_0^1 |p(t)G(t,s)\eta(s)|ds < 1$$

($G(t,s)$ is the Green's function for the respective BVP). In the case of non-resonance, *a priori* bounds are established again for solutions of an associated one-parameter family of BVP's, so that the Non-linear Alternative can be applied. For the case of resonance, two types of existence results are presented: the first is for singular problems on the "left" of the eigenvalue and the second is for singular problems on the "right" of the eigenvalue. The arguments give much insight of the work required to obtain the necessary *a priori* bounds on solutions, and in these cases exhibit nice applications of the Hölder inequality.

The author's search in Chapter 8 for non-negative solutions of (1) on $(0,\infty)$ satisfying (in one case),

$$\begin{aligned} &\lim_{t\to 0+} p(t)y'(t) = 0, \\ &y(t) \to 0, \quad \text{as} \quad t \to \infty, \end{aligned} \tag{11}$$

is motivated by the classical problem of finding positive solutions of Poisson's equation in $\mathbf{R}^n$ reduced to finding radial solutions to

$$u'' + \frac{n-1}{r}u' + h(u) = 0, \quad 0 < r < \infty,$$

and satisfying $u'(0) = 0$, and $u(r) \to 0$, as $r \to \infty$. It is assumed here that $f : [0,\infty) \times \mathbf{R}^2 \to \mathbf{R}$ is continuous, $q \in C(0,\infty)$, $p \in C[0,\infty) \cap C^{(1)}(0,\infty)$, both $p > 0$ and $q > 0$ on $(0,\infty)$, and

$$\int_0^{a_0} p(x)q(x)dx < \infty \text{ and } \int_0^{a_0} \frac{1}{p(s)} \int_0^s p(x)q(x)dx\, ds < \infty,$$

for each $a_0 > 0$. In addition, also assumed are $f(t,0,0) \le 0$, for all $t \ge 0$, there exists $r_0 > 0$ such that $f(t,r_0,0) \ge 0$, for all $t \ge 0$, and a Nagumo type condition. Consideration of a corresponding one-parameter family in the spirit of previous arguments leads to, for each $N \in \mathbf{N}$, a non-negative solution $y_N(t)$ of (1) on $(0,N)$ satisfying

$$\begin{aligned} \lim_{t\to 0+} p(t)y'(t) &= 0, \\ y(N) &= 0, \end{aligned}$$

and such that $0 \le y_N(t) \le r_0$ on $[0,N]$. To discuss the boundary condition in (11) at infinity (that is, to pass to the limit with the sequence $\{y_N(t)\}$), cases are considered. The first case deals with $p(t) = t^\gamma$, $\gamma > 1$. The arguments, while tedious, are provided in entirety, and an illustrative example is given with the singular BVP,

$$y'' + \frac{\gamma}{t}y' = (y^n - e^{-t})t^{-1/2}, \quad 0 < t < \infty,$$

satisfying (11), where $n \ge 0$, $\gamma > 1$.

Chapter 9 introduces singular BVP's in which the singularity occurs in the dependent variable as is the case in pseudoplastic fluids and some boundary layer theory. Positive solutions are sought for

$$y'' + f(t,y) = 0, \quad 0 < t < 1, \tag{12}$$

satisfying

$$\begin{aligned} -\alpha y(0) + \beta y'(0) = 0, \quad \alpha^2 + \beta^2 > 0, \\ ay(1) + by'(1) = 0, a^2 + b^2 > 0, \end{aligned} \tag{13}$$

with $\alpha, \beta, a, b \ge 0$ and $\alpha + a > 0$, where $f : [0,1] \times (0,\infty) \to (0,\infty)$ is continuous. For the case when

$$0 < f(t,y) \le Ay + h(y) + By^{-\gamma}$$

on $(0,1) \times (0,\infty)$, A, B, $\gamma \ge 0$ and $h \ge 0$ on $(0,\infty)$, and $y^\gamma h(y) \le Cy^{\gamma+\tau} + D$, for $y > 0$ and some C, $D \ge 0$, $0 \le \tau \le 1$, *a priori* bounds are obtained for solutions of an associated one-parameter family of equations and satisfying

$$\begin{aligned} -\alpha y(0) + \beta y'(0) &= \frac{\alpha}{n}, \\ ay(1) + by'(1) &= \frac{a}{n}, \quad n \in \mathbf{N}, \end{aligned}$$

so that the Non-linear Alternative can be applied to obtain a positive solution $y_n(t)$, for each $n \in \mathbf{N}$. It is then assumed that there exist $M > 0$ and $\psi \in C[0,1]$ which is positive on $(0,1)$ such that $f(t,x) \ge \psi(t)$ on $(0,1) \times (0,M]$. An application of Arzela-Ascoli yields a solution of (13) and (14). This method also extends to obtain solutions of (13) satisfying many of the other boundary conditions of the book.

The last chapter, Chapter 10, is devoted to the existence of positive solutions of

$$\frac{1}{p}(py')' = \varphi^2 q f(t,y,y'), \quad 0 < t < 1, \tag{14}$$

$$y(0) = a > 0, \quad y(1) = b \ge 0, \tag{15}$$

where p may have singularities at $t = 0$ and/or 1, and f may be singular at $y = 0$, in that

$$f : [0,1] \times (0,\infty) \times (-\infty,\infty) \to (0,\infty)$$

is continuous, $\lim_{y\to 0+} f(x,y,v) = +\infty$ uniformly on compact subsets of $[0,1] \times (-\infty,\infty)$, and

$$f(t,y,v) \leq [g(y) + h(y)]k(v),$$

where g is continuous, positive and non-increasing, and $h \geq 0$, $k > 0$ are both continuous on $[0,\infty)$. The most interesting case for (14), (15) is when $b = 0$. This is addressed by considering the appropriate one-parameter homotopy family for (14) satisfying

$$y(0) = a > 0, \quad y(1) = \frac{a}{n}, \quad n \in \mathbf{N}, \tag{16}$$

making sufficient assumptions so that *a priori* bounds are obtained for this associated family of BVP's, again obtaining positive solutions $y_n(t)$, and then passing to the limit, with Arzela-Ascoli providing a positive solution of (14), (15), with $b = 0$.

This is a very readable and attractive book, containing much basic information and with a contemporary outlook on singular BVP's for second order ODE's. The references give an adequate sample of the relevant literature on this topic.

A PROBLEM OF BOURBAKI ON FIELD THEORY

Rod Gow

The following problem appeared in one of Bourbaki's early chapters on algebra, [1, p.146]. Let K be a commutative field of characteristic different from 2 and let f be a mapping of K into itself such that

$$f(x+y) = f(x) + f(y)$$

for all x and y in K and

$$f(x)f(x^{-1}) = 1$$

for all non-zero x. Show that f is an isomorphism of K onto a subfield of K (or alternatively, a monomorphism of K). In other words, we must show that

$$f(xy) = f(x)f(y)$$

for all x and y.

In fact, Bourbaki's result is not strictly true as it stands. For it follows from the relation $f(x)f(x^{-1}) = 1$ that $f(1)^2 = 1$ and thus $f(1) = \pm 1$. Now if $f(1) = -1$, f is not a monomorphism, but it can be proved that $-f$ defined by $(-f)(x) = -f(x)$ is a monomorphism. We will assume throughout this discussion that $f(1) = 1$. We note that Bourbaki's exercise was still being presented in the incorrect form in later editions such as [2, p.175].

A hint is given in Bourbaki's exercise: show that $f(x^2) = f(x)^2$ for all x (there is a misprint of this in [1]). It took us some time to prove the equality above and, to allow people to try to prove this for themselves, if they so wish, we will not present our

proof (which is totally elementary). The proof does not require any restriction on the characteristic of K. Now, assuming that $f(x^2) = f(x)^2$, we can easily prove Bourbaki's result by considering the expansion of $f((x+y)^2)$.

We would like to raise two other questions. The first is: is the same result true when K has characteristic 2? We have checked that it is true for finite fields of characteristic 2. The other question we would like to mention is: what can be said if K is not necessarily commutative (that is, when K is a skew-field)? It is straightforward to see that the relation $f(x^2) = f(x)^2$ still holds in case K is a skew-field but the only general relation connecting x and y that we have been able to obtain is

$$f(xy(x+y)^{-1}) = f(x)f(y)(f(x)+f(y))^{-1},$$

which holds for all non-zero x and y, with $x \neq -y$, provided that x and y commute and $f(x)$ and $f(y)$ commute.

In conclusion, we suspect that it is quite likely that this question has already been discussed in the literature, although we have not seen anything ourselves.

References

[1] N. Bourbaki, Algèbre, Chapitre I-Structures algébriques (second edition). Hermann: Paris, 1951.

[2] N. Bourbaki, Algebra, Chapters 1-3. Springer-Verlag: Berlin-New York-Heidelberg, 1989.

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