Book Review-Survey

Dynamical Systems VI

Singularity Theory I

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Dynamical Systems VIII

Singularity Theory II: Classification and Applications

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Reviewed by Charles Nash

§1. Introduction

I shall review both the above books together since they are parts I and II of a treatment of singularity theory. For brevity I shall also refer to them as part I and part II respectively.

First a few formal preliminaries about the origin of the books, their authors and the nature of their expository methods.

The books are translations from Russian and appeared, in that language, in 1988 and 1989 respectively. They are both *edited by* Vladimir Arnold but are *multi-authored*; however, any given chapter has, in the main, a single author. The same authors wrote parts I and II and are: *V. I. Arnold, V. V. Goryunov, O. V. Lyashko and V. A. Vasil'ev.* The preface of each book gives

the precise authorship details of each individual chapter and also informs us that *B. Z. Shapiro* wrote a little bit of part II. The translators are *A. Iacob* of Mathematical Reviews and *J. S. Joel* respectively. Finally we come to the matter of exposition.

As the phrase *Encyclopaedia of Mathematical Sciences* above indicates, they belong to a mathematical encyclopaedia, being volumes 6 and 39 thereof. This encyclopaedia, which is a translation from a Russian original, is under the general editorship of *R. V. Gamkrelidze.* Its style therefore is expository and the books are a survey of their subject matter. This means that theorems are almost always stated rather than proved; it also means that the books are about 250 pages long instead of being several times that length.

The authors are recognized experts in their fields and so are ideal choices to write such a survey. In addition Arnold, who is the senior author because of his prominent position in singularity theory, has already written many books and so has a good written style. Vasil'ev (Vassiliev) has recently made a big advance in applying singularity theory to knot theory, about which more below. The text of the book is liberally sprinkled with illustrative examples and so the style is not heavy going or turgid; nor is the significance, and relative importance, of the various theorems left totally to the reader to fathom. On the subject of indexes, each volume has an author and a subject index but in both cases the latter is far too short, especially so for reference books belonging to an encyclopaedia. The intended audience is a "student reader" who wishes to learn the subject, be he a mathematician, or a theoretical or mathematical physicist.

Let us place the present two books on singularity theory in context by first discussing dynamical systems themselves—that done we shall move on to singularity theory and the books under review.

§2. Historical background and origins

The founder of the modern theory of dynamical systems was Poincaré, cf. "Les méthodes nouvelles de la mécanique céleste", [1]. Poincaré was interested in answering questions about the *qualit-*

ative behaviour of the orbits of celestial bodies: for example one asks what happens to the planets if their orbits are perturbed slightly? Can the orbits remain stable, change wildly, fall into the Sun or rearrange in some new way? The difficulty of solving even the three body problem analytically meant that methods which could classify the qualitative behaviour were highly desirable.

What emerges is that, for n bodies, with $n \geq 3$, as the initial conditions vary, the orbits can be *chaotic* as well as *regular*: Chaotic motion can be exhibited by an asteroid close to what is known as a Kirkwood gap; for this initial data, its eccentricity can jump in a random manner and, in time, become larger and a fatal collision with a planet can occur. Regular motion is exhibited by a planet such as the the Earth; its initial data is such that its ecliptic plane oscillates a little around a fixed position. For more details cf. [2-4].

Poincaré's pioneering work then gave birth to the present day subject of dynamical systems. In this subject one studies an immense diversity of sophisticated mathematical problems usually no longer connected with celestial or Newtonian mechanics.

A very rough idea of what is involved goes as follows: Recall that the celestial mechanics of n bodies has a motion that is described by a set of differential equations together with their initial data. One then varies the initial data and asks how the motion changes.

§3. Dynamical systems in general

The modern mathematical setting is to view the orbits of the n bodies as integral curves for their associated differential equations. Then one regards the *qualitative study* of the orbits as being a study of the *global geometry* of the space of integral curves as their initial conditions vary smoothly. Integral curves $\gamma(t)$ are associated with vector fields $V(t)$ via the differential equation

$$\frac{d\gamma(t)}{dt} = V(\gamma(t)) \tag{3.1}$$

Hence one is now studying the vastly more general subject of the global geometry of the space of flows of a vector field V on a manifold M.

It turns out that two notions play a distinguished part in the theory of dynamical systems. One fundamental notion that emerges from the example treated below is the existence of a closed integral curve. A second notion, also fundamental, is that of a *singular point* which will be dealt with in the next section.

It is natural to regard two flows on M as *equivalent* if there is a homeomorphism of M which takes one flow into the other; one can also insist that this homeomorphism is smooth, i.e. a diffeomorphism. Finally an equivalence class of flows in the homeomorphic sense is a *topological dynamical system*, and one in the diffeomorphic sense is a *a smooth, or differentiable, dynamical system.*

A further key concept in dynamical systems is that of *structural stability* and to illustrate this we introduce the following example.

Example *The pendulum with friction*

Consider a simple pendulum subject to friction, [5]. One has to solve the second order differential equation

$$\ddot{x} = -x - \mu\dot{x} \tag{3.2}$$

where $\mu \geq 0$ is the coefficient of friction. This is equivalent to solving the pair of first order equations

$$\dot{x} = y, \qquad \dot{y} = -x - \mu y \tag{3.3}$$

A solution to this pair of equations is a curve in the (x, y)-plane and so is also a flow line of the vector field V on $\mathbf{R}^2$ whose components are just $(\dot{x}, \dot{y})$. Thus eq. (3.3) is now of the form (3.1) above with V as just given and $M = \mathbf{R}^2$.

Now it is easy to compute that for μ *strictly positive* the solutions are spirals winding round the origin; but when μ is *zero* the solutions are circles centered at the origin. In other words, a big qualitative change in the trajectories takes place if the pendulum is perturbed, μ increasing μ from zero to some positive value; however if μ is perturbed but stays positive then no qualitative change occurs.

One then says that the simple pendulum with $\mu > 0$ is *structurally stable* but the simple pendulum with $\mu = 0$ is *structurally unstable*.

Thus structurally stability of a dynamical system corresponds to its equivalence under a small perturbation of V.

We now turn to the second fundamental notion of dynamical systems, which is also the subject matter of the books under review, that of singular points.

§4. Singular points and dynamical systems: vector fields

For a vector field V, a singular point is just a point on M where V vanishes. We note that a closed integral curve cannot have a singular point. There are also topological restrictions on the nature and type of singular points of V: Suppose, for simplicity, that M is closed and compact. Then a celebrated and well known result is that the index* $i(V)$ of V is equal to the Euler characteristic $\chi(M)$ of M.

Singular points of V are also closely tied to structural stability, the key point is to study whether they are degenerate or not. The result (loosely) is that a structurally stable system only possesses non-degenerate singular points. The underlying intuition is not too difficult to explain: Consider a vector field V on $\mathbf{R}^2$, say with a non-degenerate zero at $z_0 \in \mathbf{R}^2$ so that, near z_0, V behaves like

$$(z - z_0) \tag{4.1}$$

If we perturb V slightly to a new vector field V_ϵ, then we can write

$$V \longmapsto V_\epsilon = V + \epsilon f(z), \qquad \epsilon \text{ small} \tag{4.2}$$

Clearly V_ϵ also has a non-degenerate zero at the nearby location $z_0 - \epsilon f(z_0)$ (if desired, the implicit function theorem can be used to

* $i(V)$ is the total number of singular points of V, it is an algebraic sum with signs and degeneracies taken into account and assumes that the zeroes are isolated.

create a rigorous version of this argument). Hence non-degenerate singular points perturb to new ones and do not change their total number. By contrast if the zero at z_0 is *degenerate* then, near z_0, V behaves like

$$(z - z_0)^n, \qquad n > 1 \tag{4.3}$$

So the perturbed vector field V_ϵ looks like

$$(z - z_0)^n + \epsilon f(z_0), \qquad \text{near } z_0 \tag{4.4}$$

But, in general,

$$(z - z_0)^n + \epsilon f(z_0) = (z - z_1)(z - z_2)\cdots(z - z_n) \tag{4.5}$$

Hence, on perturbation, the degenerate zero has *bifurcated* into n non-degenerate zeroes. Actually, more generally, degenerate zeroes, can even *disappear altogether* on perturbation because the bifurcation process may produce only complex zeroes which may not belong to the particular M under consideration.

In sum the perturbation of a system with one or more degenerate singular points is structurally unstable, and so we recover the fact that all the singular points of a structurally stable dynamical system are non-degenerate.

§5. Singular functions: the real case

As well as singular points of vector fields the study of dynamical systems requires us to consider singular points of functions. By a singular point of a function f we mean a *critical point*, or extremum, of f.

For example let M be a manifold and f a smooth real valued* function on M

$$f : M \longrightarrow \mathbf{R} \tag{5.1}$$

* As we shall see below both f and M can be generalized considerably: For f we can generalize to complex valued functions $f : M \to \mathbf{C}$ and even maps of the form $f : M \to N$, where N is another manifold. For M we should start with an M which is closed and then generalize to the case where M has a boundary; in fact cases where M is *infinite dimensional* arise naturally and are important, one of these latter is the original problem of Morse cf. § 8.

then, if p is a point in M with local coordinates $(x^1, x^2, \ldots, x^n)$, p is a *critical point* of f if

$$\left.\frac{\partial f}{\partial x^1}\right|_p = \left.\frac{\partial f}{\partial x^2}\right|_p = \cdots = \left.\frac{\partial f}{\partial x^n}\right|_p = 0 \tag{5.2}$$

or, in a more concise notation,

$$df = 0 \quad \text{at } p \tag{5.3}$$

Example *Gradient dynamical systems*

Using such a function $f : M \to \mathbf{R}$ we obtain an important class of dynamical systems known as *gradient dynamical systems*: We require M to have a (Riemannian) metric so that the grad operator is defined and then the flow equation is that of gradient flow

$$\frac{d\gamma(t)}{dt} = \operatorname{grad} f(\gamma(t)) \tag{5.4}$$

so that $V = \operatorname{grad} f$ and f is like a potential function. We see that the flow begins and ends at singular points of f.

We shall now discuss some of the theory of singularities of functions such as f from a qualitative topological viewpoint; for *real valued* functions this is known as Morse theory. The aim in Morse theory is to study the relation between critical points and topology. More specifically one extracts topological information from a study of the critical points of a smooth real valued function

$$f : M \longrightarrow \mathbf{R}, \tag{5.5}$$

where M is an n-dimensional compact manifold, without boundary. For a suitably behaved class of functions f, there exists quite a tight relationship between the number and type of critical points of f and topological invariants of M such as the Euler–Poincaré characteristic, the Betti numbers and other cohomological data. This relationship can then be used in two ways: one can take certain special functions whose critical points are easy to find and use this information to derive results about the topology of M; on the other hand, if the topology of M is well understood, one can use this topology to infer the existence of critical points of f in cases where f is too complicated, or too abstractly defined, to allow a direct calculation.

We begin with the smooth function $f : M \to \mathbf{R}$ and assume* that all the critical points p of f are distinct and non-degenerate; the non-degeneracy means that the Hessian matrix Hf of second derivatives is invertible at p, or

$$\det Hf(p) \neq 0 \quad \text{where} \quad Hf(p) = \left[\partial^2 f/\partial x^i \partial x^j \big|_p\right]_{n\times n} \tag{5.6}$$

Each critical point p has an index λ_p which is defined to be the number of *negative* eigenvalues of $Hf(p)$. In a neighbourhood of a non-degenerate critical point p of index λ_p we can represent f as

$$f(x) = f(p) - \overbrace{x_1^2 - x_2^2 - \cdots - x_{\lambda_p}^2}^{\lambda_p \text{ terms}} + \underbrace{x_{\lambda_p+1}^2 + \cdots + x_n^2}_{n-\lambda_p \text{ terms}} \tag{5.7}$$

for suitable coordinates $(x_1, \ldots, x_n)$.

We next associate to the function f and its critical points p the Morse series $M_t(f)$ defined by

$$M_t(f) = \sum_{\text{all } p} t^{\lambda_p} = \sum_i m_i t^i. \tag{5.8}$$

The sum will always converge since it only contains a finite number of terms; this is because the non-degeneracy makes the critical points all discrete and the compactness of M permits only a finite number of such discrete points. The topology of M now enters via

* Such functions are called Morse functions and it should be clear from what we have said earlier that when f is not a Morse function one can always perturb it slightly to obtain one.

$P_t(M)$: the Poincaré series of M. This is the following polynomial constructed out of the Betti numbers of M; we have

$$P_t(M) = \sum_{i=0}^{n} \dim H^i(M;\mathbf{R})t^i = \sum_{i=0}^{n} b_i t^i. \tag{5.9}$$

The fundamental result of Morse theory, known as the Morse inequalities, is the statement that

$$M_t(f) - P_t(M) \geq 0 \tag{5.10}$$

This can be refined further to say that

$$M_t(f) - P_t(M) = (1+t)R(t), \tag{5.11}$$

where $R(t)$ is a polynomial with only non-negative coefficients.

We note in passing two facts that can be read off immediately from this pair of statements. If we set $t = 1$ in the first one, we see that any (Morse) function f has at least $\sum_{i=0}^{n} b_i$ critical points. If we set $t = -1$ in the second one then we see that

$$M_{-1}(f) = P_{-1}(M) = \sum_{i=0}^{n}(-1)^i b_i = \chi(M), \tag{5.12}$$

where $\chi(M)$ is the Euler characteristic of M. Note that the first of these facts describes a property of f, while the second is completely independent of f and is only a property of M.

A proof of the Morse inequalities usually uses the level sets of the function f: these are the sets $f^{-1}(c) = \{x \in M : f(x) = c\}$. We shall briefly sketch the part that they play in determining the topology of M. In Morse theory one constructs a half space M_c out of level sets where

$$M_c = \{x \in M : f(x) \leq c\}. \tag{5.13}$$

The topology of M begins to emerge when we consider M_c as a function of c. What happens is that, as c varies, the topology of

M_c is unchanged until c passes through a critical point, when it either acquires or sheds a cell of dimension λ, where λ is the index of the critical point. More precisely we have

Theorem (Bott–Morse–Smale) *M_a is diffeomorphic to M_b if there is no critical point in the interval $[a, b]$. Alternatively, if (a, b) contains just one critical point of index λ then $M_b \simeq M_a \cup e_\lambda$.*

The notation $M_a \cup e_\lambda$ means that a cell of dimension λ has been attached to M_a; also $M_b \simeq M_a \cup e_\lambda$ means that the two spaces have the same homotopy type. Thus, as far as the homotopy type of M is concerned (and this will be sufficient, for example, for computing the cohomology of M) one can think of M as being 'decomposed' into a set of cells

$$M = \bigcup_{\lambda} e_\lambda, \tag{5.14}$$

the number of these cells being equal to the number of critical points and the dimension of the cells being given by the index of the critical points. This decomposition is known as a *stratification* of M.

§6. Singular functions: the complex case

Now suppose that f is *complex valued* instead of real valued i.e. we have

$$f : M \longrightarrow \mathbf{C} \tag{6.1}$$

A corresponding complex analogue of Morse theory exists, known as *Picard-Lefschetz* theory. The content of the theory is quite different: Clearly the complex values of f render it impossible to define the index of a critical point any more; not surprisingly, in view of this, the critical points cease to provide a stratification M using the level sets $f^{-1}(c)$. In fact the level sets no longer undergo a topological change as c passes through a critical point—they are actually all homeomorphic to one another.

In the complex case what one does instead of passing *through* a critical point is to deform one's path to go *round* it; the obvious topology relevant in this setting resides in the winding number of a *closed path*, or cycle, round the singularity or critical point.

This results in integrals round closed cycles which in turn are continuous functions on the parameter space; the analysis of such an object is known as the *monodromy* of the singularity More technically, the level sets over a small circle surrounding a singular point form a fibre bundle (since they are all topologically identical) over S^1, and the monodromy is then the holonomy of the fibre corresponding to going round this circle once.

§7. Singular maps

Next, suppose that we replace $\mathbf{C}$ by a *manifold* N (both M and N are, for the moment, assumed to be real manifolds) giving the map

$$f : M \longrightarrow N, \quad \dim M = n, \dim N = m \tag{7.1}$$

Let us use local coordinates $(f^1, f^2, \ldots, f^m)$ to represent $f(x)$ on N, and $(x^1, x^2, \ldots, x^n)$ to represent x on M. A singularity of f is now defined using its Jacobian matrix

$$J = \left[\frac{\partial f^i}{\partial x^j}\right]_{m\times n} \tag{7.2}$$

rather than the operator d: a singularity of f is a point on M where J has less than its *maximal rank*. In this setting, the topology of the theory involves the Stiefel-Whitney characteristic classes $w_i(M) \in H^i(M; \mathbb{Z}_2)$ of M and the pullback, via f^*, of those of N. Universal polynomials known as *Thom polynomials* provide calculational formulae for these pullbacks. If we generalize to the case where M and N are complex manifolds then the Stiefel-Whitney classes are replaced by the Chern classes $c_i(M) \in H^{2i}(M; \mathbf{Z})$ of M and those of N.

§8. Singular points in infinite dimensions

A brief mention now, as promised, of some examples where M is infinite dimensional. The original problem of Morse, [6], was to study the critical points of the energy functional E defined by

$$E(\gamma) = \int_0^1 \left|\frac{d\gamma(t)}{dt}\right|^2 dt \equiv \int_0^1 g_{ij}\frac{d\gamma^i(t)}{dt}\frac{d\gamma^j(t)}{dt}dt \tag{8.1}$$

where $\gamma(t)$ is a parametrized path on M with end points p and q labelled by 0 and 1, and g_{ij} is the Riemannian metric on M. E is a function or functional of $\gamma(t)$. Hence E is a positive real valued function on the space $PM(p,q)$ of paths on M from p to q. More formally we can represent E as

$$\begin{aligned} E : PM(p,q) &\longrightarrow \mathbf{R} \\ \gamma &\longmapsto E(\gamma) \end{aligned} \tag{8.2}$$

The space $PM(p,q)$ is of course infinite dimensional. The critical points of E are easily seen to be the geodesics joining p to q with the usual equation

$$\frac{d^2\gamma^i}{dt^2} + \Gamma^i_{jk}\frac{d\gamma^j}{dt}\frac{d\gamma^k}{dt} = 0, \tag{8.3}$$

where Γ^i_{jk} are the components of the Christoffel symbol for the metric g_{ij}. To consider *closed* geodesics we simply require γ to be a loop, on M; this means that we take elements of $Map(S^1, M)$ instead* of $PM(p,q)$. Now we regard E as a functional of the form

$$E : Map(S^1, M) \longrightarrow \mathbf{R} \tag{8.4}$$

Now, for the case where M is the sphere S^k, Morse tackled the infinite dimensionality of $Map(S^1, M)$ by approximating the loops by geodesic polygons with n vertices $p_1, \ldots, p_n$. This makes $E(\gamma)$ a function of the n variables $p_1, \ldots, p_n$ instead of γ, i.e. $E = E(p_1, \ldots, p_n)$. If we denote the space of these $\{p_i\}$ by $Map_n(S^1, S^k)$, then $Map_n(S^1, S^k)$ is to be viewed as a finite dimensional subset of the infinite dimensional $Map(S^1, S^k)$. The idea then is to compute the topology of $Map_n(S^1, S^k)$ and to understand its dependence on n. This allows the passage to the limit $n \to \infty$ where one eventually deduces results such as the existence of an infinite number of closed geodesics on S^k and that E is a *perfect* Morse function; this latter property means that the

* $Map(S^1, M)$ is the space of loops on M, i.e. it is the space of continuous maps from S^1 to M.

Morse inequalities have become *equalities*. For more details cf. Klingenberg, [7].

Much later, the construction of a general Morse theory in infinite dimensions was achieved by Palais, Smale and many others cf. Palais, [8, 9, 10]; still more recently Floer, [11, 12], and Taubes, [13] have successfully tackled infinite dimensional problems which are outside the scope of the Palais-Smale framework. These are problems in Yang-Mills gauge theories but have consequences far outside theoretical physics: for example in Floer's case his work constructs a new class of highly interesting homology for 3-manifolds; for more information cf. Nash, [14].

§9. Classification of singular points of functions and Lie algebras

We come now to a most interesting topic: namely the *classification* of singular points of functions. There is a remarkable correspondence between the classification of singularities of functions and that of simple Lie algebras. There is no space to do justice to it here but some salient features can be mentioned.

Let f be a function with possibly degenerate critical points, with the multiplicity of a critical point being labelled by μ. Now f belongs to the infinite dimensional space $\mathcal{F}$ of functions $\mathcal{F} = Map(M, \mathbf{R})$, say, and, from the abstract viewpoint, the classification of the singular points corresponds to the finding of the orbits of the action of the group $Diff\,(M)$ of diffeomorphisms of M on $\mathcal{F}$.

In practice what happens is that one learns that functions may be transformed, by elements of $Diff\,(M)$, into certain polynomials known as *normal forms*.

The basic idea is to build up a picture of the functions as a subset of $\mathcal{F}$. So first one considers 1 parameter families of functions in $\mathcal{F}$ and analyses their possible singular points, then one considers 2 parameter families and so on.

For example, if $\dim M = n$, then *near* a singular point all 1 parameter families of functions are equivalent under $Diff\,(M)$ (i.e. after a suitable change of variables) to the *normal form*

$$f(x) = x_n^3 + \lambda x_n + q(x) \qquad (9.1)$$

where λ is the parameter and $q(x)$ is a non-degenerate quadratic form in the remaining variables given by

$$q(x) = -x_1^2 - x_2^2 - \cdots - x_j^2 + x_{j+1}^2 + \cdots + x_{n-1}^2.$$

In 1 dimension this becomes simply

$$f(x) = x^3 + \lambda x, \qquad (9.2)$$

where such a result is not so hard to prove. If we have 2 parameters λ_1 and λ_2 then the normal form is

$$f(x) = x_n^4 + \lambda_1 x_n^2 + \lambda_2 x_n + q(x) \qquad (9.3)$$

and more generally for k parameters the normal form is

$$f(x) = x_n^{k+1} + \lambda_1 x_n^{k-1} + \lambda_2 x_n^{k-2} + \cdots + \lambda_{k-1} x_n + \lambda_k + q(x) \quad (9.4)$$

The polynomial

$$x_n^{k+1} + \lambda_1 x_n^{k-1} + \lambda_2 x_n^{k-2} + \cdots + \lambda_{k-1} x_n + \lambda_k \qquad (9.5)$$

that emerges here is recognizable to Lie group experts as being isomorphic to the orbit space of the reflection group known as the Weyl group A_k for the simple Lie algebra $su(k+1)$.

There are also normal forms corresponding to the Weyl groups $D_k \simeq so(2k)$ and the exceptional set E_6, E_7 and E_8 for the exceptional algebras. Thus we have the whole of the so called A, D, E series, [15].

Example *Manifolds with boundary*

If M is a manifold with a non-empty boundary ∂M then $\mathcal{F} = Map(M, \mathbf{R})$ now can contain functions whose singular or critical points are on ∂M itself. These functions turn out, [16], to have normal forms which correspond to the Weyl groups B_k, C_k and F_4 i.e. to the remaining simple Lie algebras $so(2k+1)$, $sp(k)$ and F_4 respectively. Notice, however, that there is precisely one simple Lie algebra missing from the classification above—it is the last exceptional algebra G_2—it too can play a rôle cf. [17].

The indices on the various series A, D, E etc. label the multiplicity of the degenerate singularities of the family, for example all the most degenerate singularities with normal form A_k clearly have the same multiplicity k. Hence the index labels the $\mu =$ constant strata inside $\mathcal{F}$, and, since the value of μ gives the number of parameters of the family, this value of μ also is equal to the codimension of this stratum inside $\mathcal{F}$.

All the classifications above describe singularities which are called *simple*: small perturbations bifurcate them into only finite numbers of new singularities. There are also those which associate a continuum to the singularity: one then says that the singularity has moduli. These are the complement to the discrete series just discussed, i.e. all simple singularities occur in the lists given above.

§11. Applications of dynamical systems

We now give an idea of how diverse the subject is by mentioning some of the problems where ideas from dynamical systems can be applied.

Morse theory provides us with many examples and they are impressive and widespread; a few notable examples are the proof by Morse, [6], that there exist infinitely many geodesics joining a pair of points on a sphere S^n endowed with any Riemannian metric, Bott's proof of his celebrated periodicity theorems on the homotopy of Lie groups, [18], Milnor's construction, [19], of the first exotic spheres, and the proof by Smale of the Poincaré conjecture for $\dim M \geq 5$, [20].

Morse theory has also found a variety of applications in physics; this is not too surprising in view of the central position occupied by the variational principle in both classical and quantum physics. Some of these are described in Nash and Sen, [21].

Gradient dynamical systems were used by Thom, [21, 22, 23], in his work on what is now called *Catastrophe Theory*. Thom took the system

$$\frac{d\gamma(t)}{dt} = \operatorname{grad} V(\gamma(t)) \tag{10.1}$$

where V is a potential function. Next, for families of such V containing up to four parameters, Thom classified the possible critical points into seven types known as the seven elementary catastrophes; he then proposed to use these dynamical systems as models for the behaviour of a large class of physical, chemical and biological systems. In many of these cases the models are not at all adequate; nevertheless, there are some successes. On the mathematical side the classification into seven categories misses some singularities when one has three and four parameter families, cf. part II of the books under review; the seminal nature of Thom's work is clear though, as it is the beginning of the classification theory for singularities.

A vast body of the theory of dynamical systems concerns *Hamiltonian systems*. These of course have their origin in ordinary dynamics but exist now in a much wider context. To have a Hamiltonian system one needs to satisfy some requirements: M must must be even dimensional and must possess a closed non-degenerate 2-form ω known as a symplectic form; a Hamiltonian function

$$H : M \longrightarrow \mathbf{R} \tag{10.2}$$

then provides a vector field V on M via the equation

$$i(V)\,\omega = dH \tag{10.3}$$

where $i(V)$ denotes contraction, or interior product, with the vector V. It is easy to check that H is conserved along the orbits of V and this corresponds to the conservation of energy in the physical cases. The perturbation theory of these systems underwent an enormous development with the work particularly of Kolmogorov, Arnold and Moser resulting in what is now called KAM theory.

The blossoming of ergodic theory also owes some debts here. Ergodic theory originates largely in nineteenth century studies in the kinetic theory of gases. However it has now been axiomatized, expanded, refined and reformulated so that it has links with many parts of mathematics as well as retaining some with physics. Some dynamical systems exhibit ergodic behaviour, a notable class of examples being provided by *geodesic flow* on surfaces of constant negative curvature. This involves too the study of the flows by a discrete encoding known as symbolic dynamics, use of one dimensional interval maps, the zeta functions of Ruelle, the Patterson measure and so on, cf. [25]. Classical and quantum chaos, and the distinction between the two, are also studied in this context.

The last application that we shall mention is that of Vasil'ev to knot theory, [26, 27]. Vasil'ev's work constitutes a big step forward in knot theory but should also be regarded as a big step forward in the tackling of global problems in singularity theory as his methods are not limited just to knot theory.

Vasil'ev constructs a huge new class of knot invariants and we shall now give a sketch of what is involved.

A knot is a smooth embedding of a circle into $\mathbf{R}^3$. Thus a knot gives a map

$$f : S^1 \longrightarrow \mathbf{R}^3, \tag{10.4}$$

so that f belongs to the space $\mathcal{F}$ where $\mathcal{F} = Map(S^1, \mathbf{R}^3)$. Not all elements of $\mathcal{F}$ give knots, since a knot map f is not allowed to self-intersect or be singular. Let Σ be the subspace of $\mathcal{F}$ which contains either self-intersecting or singular maps. Then the subspace of knots is the *complement*

$$\mathcal{F} - \Sigma \tag{10.5}$$

Now any element of Σ can be made smooth by a simple one parameter deformation, hence Σ is a *hypersurface* in $\mathcal{F}$ and is known as the *discriminant.* As the discriminant Σ wanders through $\mathcal{F}$ it skirts along the edge of the complement $\mathcal{F} - \Sigma$ and divides it into many different connected components. Clearly knots in the same connected component can be deformed into each other and so are equivalent (or isotopic).

Now any knot *invariant* is, by the previous sentence, a function which is *constant* on each connected component of $\mathcal{F} - \Sigma$. Hence the task of constructing all (numerical) knot invariants is the same as finding all functions on $\mathcal{F} - \Sigma$ which are constant on each connected component. But topology tells us at once that this is just the 0-cohomology of $\mathcal{F} - \Sigma$. In other words,

$$H^0(\mathcal{F} - \Sigma) = \text{the space of knot invariants.} \tag{10.6}$$

Vasil'ev, [27], provides a method for computing most, and possibly all, of $H^0(\mathcal{F} - \Sigma)$.

Because of the immense importance of this breakthrough we give a brief summary of the steps involved in the construction of [27]. Vasil'ev deals with the infinite dimensionality of $\mathcal{F}$ by approximating its elements by trigonometric polynomials of degree n giving a finite dimensional space $\mathcal{F}^n$ of dimension $3n$. But $\mathcal{F}^n$ is clearly contractible, so Alexander duality gives us

$$H^i(\mathcal{F}^n - \Sigma) \simeq H_{3n-i-1}(\mathcal{F}^n \cap \Sigma). \tag{10.7}$$

Hence the cohomology of the knot space $\mathcal{F}^n - \Sigma$ is computable from the homology of the subsets Σ_n of the discriminant Σ given by

$$\Sigma_n = \mathcal{F}^n \cap \Sigma \tag{10.8}$$

The singularities present in Σ_n give a stratification* of Σ allowing the computation of its homology. This stratification of Σ provides a filtration from which a standard spectral sequence then flows. The spectral sequence is roughly an algebro-topological analogue of a Taylor series and, as for a Taylor series, one must demonstrate convergence and absence of remainder in the limit $n \to \infty$.

The construction then provides us with a hierarchy of knot invariants V_n—the Vasil'ev invariants—which looks like

$$V_0 \subset V_1 \subset \cdots V_n \subset \cdots \subset H^0(\mathcal{F} - \Sigma) \tag{10.9}$$

where each V_n is finite dimensional and already completely constructed for $0 \le n \le 8$.

The convergence of the spectral sequence has been conjectured by Vasil'ev and, if proved, would mean that the Vasil'ev invariants distinguish *any* two inequivalent knots. It is already known, Birman [28], that they distinguish more knots than the other well known knot polynomials, namely the Alexander, Jones,

* This is the great advantage of working with Σ_n instead of with $\mathcal{F}^n - \Sigma$; this latter space contains only smooth maps and provides us with no natural way of constructing a stratification.

Homfly and Kaufmann polynomials. Kontsevich, [29], has given a 'universal integral' which associates to each knot an element of an algebra of 'Feynman diagrams' (cf. also Bar-Natan, [30]), from which one calculates the Vasil'ev invariants for the knot; this work uses results of Knizhnik and Zamolodchikov, [31], from the physics literature.

§11. Conclusion

We now wind things up with a return to the books under review. These two volumes certainly cover a wide range of material on singularity theory and, although, they belong to a section of the encyclopaedia on dynamical systems there is much material here for anyone with an interest in singularity theory, not just those who work on dynamical systems.

Part I begins with basic notions concerning singular smooth maps and introduces normal forms. It then moves on to complex functions and Picard-Lefschetz theory to which it devotes a considerable amount of space—about a hundred pages. Next comes a chapter on singularities of smooth maps in general; and the final chapter is on the global singularity theory relevant for maps and deals with the subject of Thom polynomials and related matters.

Part II is a mixture of applications and material on classification of singularities. However part II is largely intended to be independent of part I. The first chapter deals with the singularities and normal forms for functions on a manifold with boundary. This is followed by a chapter on applications including a section on catastrophe theory. Then one moves on to singularities on the boundaries of function spaces. Chapter four is about monodromy and Picard-Lefschetz theory and contains a remarkable early monodromy result of Newton from his *Principia*: For an ellipse, with origin at a focus, this is that the area swept out in time t by the radius vector r is a transcendental function of the tangent of the angle between r and the x-axis. The book then finishes with a chapter on deformation of real singularities and their lacunae, including a discussion of the use of computer algorithms to obtain some of the results.

The style of both volumes is definitely mathematical rather

than physical and so some physicists will find the text heavy going. Cross referencing within the text is done fairly well; and this encyclopaedia does not indulge in the annoying practice of referring one to equations present in other volumes as if one had the desk space, or the money, to have them all at hand; readers of Dieudonné's admirable six volume *Treatise on Analysis* may remember that it continually suffers from that drawback. The bibliography is very good and extremely large in both cases. It is interesting to note, however, that Vasil'ev's paper [27] is in the bibliography but is, unfortunately, not discussed; a comparison of the dates of the Russian original and the English translation is consistent with the fact that the reference entered only at the translation stage.

The price of both books is DM 141 which is about 58 punts and is a little on the expensive side for books of 250 odd pages, though they are produced up to the usual high standards of Springer. Price notwithstanding, I do recommend them both particularly as library purchases, and because they can be read independently of the other volumes of the encyclopaedia.

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