A NOTE ON MINIMAL INFINITE SUBSPACES OF A PRODUCT SPACE

D. J. Marron* and T. B. M. McMaster

Abstract Ginsburg and Sands, [1], have identified the five topological spaces which are 'minimal infinite' in the sense that each is homeomorphic to all of its own infinite subspaces. Given a finite family of spaces, and knowledge of which of the five may be embedded in each of them, we show how to obtain the same information concerning their product.

On the set $\mathbf{N}$ of positive integers, let τ_1, τ_0, τ_{cf}, $\tau(\uparrow)$ and $\tau(\downarrow)$ denote respectively the discrete, trivial and cofinite topologies, the topology of final segments and that of initial segments. If τ is any one of these five and Y is an infinite subset of $\mathbf{N}$, then clearly the subspace (Y, τ_Y) is homeomorphic to the whole space $(\mathbf{N}, \tau)$. More significantly, Ginsburg and Sands [1] have demonstrated using Ramsey's theorem that every infinite space contains a homeomorphically embedded copy of at least one of these five, so that they are necessarily the only spaces (up to homeomorphism) enjoying this kind of minimality, and they may be perceived from a certain viewpoint as the 'atoms' of infinite topology. There have been a number of recent applications of this idea: we refer, for example, to Matier and McMaster's uses of it in exploring total negations of properties enjoyed by all finite spaces but not by all countable ones, [3], [4].

For brevity, let us take the phrases 'atom of a space X' to mean a minimal infinite space capable of being embedded into X, and 'atomic structure of X' to mean the list of the atoms of

X. Once we know the atomic structure of finitely many spaces X_1, X_2, ..., X_n, then that of their disjoint union is immediately obtainable just by conflation. It might be expected that the same observation would apply to their product, but the truth is rather more interesting. Certainly, each atom of one of the X_i must be an atom of the product space, but the following example shows the converse to fail:

Example The diagonal line $\{(x, x) : x \in \mathbf{N}\}$ in the product space $(\mathbf{N}, \tau(\downarrow)) \times (\mathbf{N}, \tau(\uparrow))$ is discrete: as it also is in $(\mathbf{N}, \tau(\downarrow)) \times (\mathbf{N}, \tau_{cf})$. So $(\mathbf{N}, \tau_1)$ is a 'surprise atom', occurring in the atomic structures of these two products but absent from the factor spaces. We shall see that there is a uniqueness about these two examples: surprise atoms in finite products are always attributable to one or other of them.

Lemma 1. *Let A be any infinite subset of $\mathbf{N} \times \mathbf{N}$. There is an infinite subset B of A which satifies one of the following conditions*
(a) all elements of B have the same first coordinate
(b) all elements of B have the same second coordinate
(c) no two elements of B have the same first coordinate and the second coordinate is a strictly increasing function of the first.

Proof: This is simple enough to demonstrate directly, but easier still as an application of Ramsey's theorem, [2, p.19]. Classify each two-element subset $\{(x_1, y_1), (x_2, y_2)\}$ of A as red, green, blue or yellow according as $x_1 = x_2$, $y_1 = y_2$, $(x_1 - x_2)(y_1 - y_2) > 0$ or $(x_1 - x_2)(y_1 - y_2) < 0$. Then there is an infinite subset B of A which is monochromatic (that is, every two-element subset of B has the same colour). The red, green and blue cases give (a), (b) and (c) respectively, while yellow leads to a contradiction since each positive integer has only finitely many predecessors.

Lemma 2. *Let τ, τ' be among the five 'minimal' topologies on $\mathbf{N}$ and satisfy $\tau \geq \tau'$, and let C be an infinite subset of $\mathbf{N}$. Then every increasing injective map f from the subspace (C, τ_C) to $(\mathbf{N}, \tau')$ is continuous.*

Proof: It suffices to check the case $\tau = \tau'$. When $\tau = \tau' = \tau_1$ or τ_0 the conclusion is trivial, in the τ_{cf} case continuity follows from

injectivity, in the remaining two cases the increasing nature of f is what is needed.

Proposition 1. *Let* $(\mathbf{N}, \tau)$ *and* $(\mathbf{N}, \tau')$ *be two (not necessarily distinct) of the minimal infinite spaces and suppose that* $\tau \geq \tau'$. *Then there are no 'surprise atoms' in* $(\mathbf{N}, \tau) \times (\mathbf{N}, \tau')$.

Proof: Suppose that the subset A of $\mathbf{N} \times \mathbf{N}$ is homeomorphic to an atom. Choose B as in Lemma 1; in cases (a) and (b), B will be a homeomorph of an infinite subspace of $(\mathbf{N}, \tau')$ or $(\mathbf{N}, \tau)$, and therefore via minimality, A itself must be a copy of one of these two. In case (c), B is the graph of an increasing injective function

$$f : (C, \tau_C) \to (\mathbf{N}, \tau')$$

whose domain is an infinite subspace of $(\mathbf{N}, \tau)$ and whose continuity Lemma 2 establishes. It follows that C and B are homeomorphic, and a further appeal to minimality shows A to be a copy of $(\mathbf{N}, \tau)$.

Next we show how the atomic structure of the product of two arbitrary spaces X and Y is determined by those of X and Y separately.

Proposition 2. *Let* M *be an atom of* $X \times Y$ *but neither of* X *nor of* Y. *Then*
(i) M *is discrete, and*
(ii) one of X *and* Y *has* $(\mathbf{N}, \tau(\downarrow))$ *as an atom, the other one has either* $(\mathbf{N}, \tau(\uparrow))$ *or* $(\mathbf{N}, \tau_{cf})$ *as an atom.*

Proof: Take a subspace M' of $X \times Y$ which is a homeomorph of M. Arguing as in the proofs of Lemma 1 and Proposition 1, we find an infinite subset B of M' no two of whose points have either the same first or the same second coordinate. The first projection $\pi_1(B)$ contains a copy E of an atom of X. The second projection $\pi_2(\pi_1^{-1}(E) \cap B)$ of the points of B lying 'vertically above' E remains infinite, and contains a copy F of an atom of Y. So now $E \times F$ is a product of copies of atoms and encloses an infinite subspace of minimal M'. If the topologies of E and F are comparable, Proposition 1 yields a contradiction. If not, we are dealing with the product of $(\mathbf{N}, \tau(\downarrow))$ by either $(\mathbf{N}, \tau(\uparrow))$

or $(\mathbf{N}, \tau_{cf})$; that $(\mathbf{N}, \tau_1)$ is a surprise atom here has already been noted in the example, and another argument like that in the proof of Proposition 1 readily confirms that it is the only one.

Using induction, it is now routine to extend this proposition to apply to any finite numbe of factor spaces. We conclude:

Theorem. *The atomic structure of the product of a finite number of spaces is merely the conflation of their individual atomic structures unless:*
$(\mathbf{N}, \tau(\downarrow))$ *is an atom of one factor space,*
$(\mathbf{N}, \tau(\uparrow))$ *or* $(\mathbf{N}, \tau_{cf})$ *is an atom of another, and*
no factor space has $(\mathbf{N}, \tau_1)$ *as an atom.*
In the exceptional case, $(\mathbf{N}, \tau_1)$ *is to be appended to the conflation of the lists.*

References

[1]   J. Ginsburg and B. Sands, *Minimal infinite topological spaces*, Amer. Math. Monthly **86** (1979), 574-576.

[2]   R. L. Graham, B. L. Rothschild and J. H. Spencer, *Ramsey Theory*. John Wiley and Sons: New York, 1980.

[3]   J. Matier and T. B. M. McMaster, *Total negation of separability and related properties*, Proc. Royal Irish Acad. **90a** (1990), 131-137.

[4]   J. Matier and T. B. M. McMaster, *Minimal countable spaces and anti-properties*, Rend. Circ. Mat. Palermo (2) **29** (1992), 553-561.

D. J. Marron and T. B. M. McMaster,
Department of Pure Mathematics,
The Queen's University of Belfast,
Belfast BT7 1NN,
Northern Ireland.