NON-MEASURABLE SETS AND TRANSLATION INVARIANCE

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In this brief note, we prove a simple but quite general fact about translation invariant measures: *if μ is a finite non-trivial measure on a group G, then G has non-measurable subsets.* An immediate very well-known corollary is the existence of a set of reals which is not Lebesgue measurable. The most popular proofs of this latter result leave one with the impression that non-measurable sets of reals are connected with the density of the rationals, [R], the relatively small number of closed sets of reals, [M], or the identification of the reals with infinite binary sequences, [B].

We begin by fixing the familiar terminology.

Definition Suppose that S is a set and F is a σ-algebra of subsets of S. A *measure* over F is a function μ from F into $[0,\infty]$ such that
(1) $\mu(\emptyset) = 0$
and
(2) if $\{ X_n \in F : n \in \mathbf{N} \}$ is a family of pairwise disjoint sets, then

$$\mu\left(\bigcup_{n\in\mathbf{N}} X_n\right) = \sum_{n\in\mathbf{N}} \mu(X_n).$$

The subsets in F are said to be the *measurable* subsets of S. We say that μ is a *total* measure if
(3) $F = P(S)$, i.e. every subset of S is measurable.
A measure μ is *non-trivial* if $\mu(\{x\}) = 0$ for every $x \in S$, and *finite* if $\mu(S)$ is a positive real number. We say loosely that μ is a measure on S when we mean that the domain of μ is a σ-algebra of subsets of S.

Some well-known examples of non-trivial measures are Lebesgue measure on $\mathbf{R}^n$ and the Haar measure on a locally compact group. These measures are also translation invariant, in accordance with the following definition.

Definition Suppose that $G = (G,*)$ is a group. We say that a measure μ on G is (left-) translation invariant if $\mu(g * X) = \mu(X)$ for every g in G and every X in the domain of μ, where $g * X = \{ g * x : x \in X \}$.

Our first observation is a group-theoretic one.

Proposition 1. *Suppose that G is a group, A is a subgroup of G and X is a non-empty subset of G with $A * X \subseteq X$. Then there exists a subset E of X such that the following hold:*

(i) $X = \cup_{a\in A}(a * E)$;

(ii) if a and b are distinct elements of A, then $a * E \cap b * E = \emptyset$.
Proof: Define an equivalence relation R on X as follows: xRy if and only if $x * y^{-1} \in A$. Since A is a subgroup of G, it follows easily that R is an equivalence relation on X. So R partitions X into equivalence classes. Using the Axiom of Choice, choose a representative from each distinct class and let E be the set of these representatives. It is now straightforward to check that E satisfies (i) and (ii).
Corollary If A is a subgroup of G, then there exists a subset E of G such that the following hold:
(i) $G = \cup_{a\in A}(a * E)$;
(ii) if a and b are distinct elements of A, then $a * E \cap b * E = \emptyset$.
In fact, E is just a set of right coset representatives of A in G.

We now derive the main result from Proposition 1.

Theorem 2. *Suppose that μ is a finite non-trivial (left-) translation invariant measure on the group G. Then G has non-measurable subsets (so μ is not total).*

Proof: Since μ is finite, non-trivial and countably additive, it follows that G is an uncountable set. Let A be any countably infinite subgroup of G (just take the subgroup generated by some countably infinite subset of G; model theorists will apply the Downward

Loewenheim Skolem theorem). By the corollary, there is a subset E of G satisfying (i) and (ii). We claim that E is non-measurable. Well, suppose otherwise; then

$$\mu(G) = \mu(\bigcup_{a\in A} a * E) = \sum_{a\in A} \mu(a * E) = \sum_{a\in A} \mu(E),$$

where we have used (i), (ii), countable additivity and translation invariance. This is impossible since μ is finite and A is infinite. Hence E is non-measurable.

Corollary 3. *There exists a set of reals which is not Lebesgue measurable.*

Proof: Let G be the group $(0, 1]$ under addition modulo 1. Lebesgue measure restricted to G satisfies the hypotheses of Theorem 2.

Of course, everything goes through for (right-) translation invariant measures if one formulates an appropriate version of Proposition 1.

The use of the Axiom of Choice (AC) in Corollary 3 prompted mathematicians to study whether and how much choice was necessary. In 1970, Solovay, [S], published the following famous theorem:

Theorem. *Suppose that there exists an inaccessible cardinal. Then there is a model of ZF+DC+ "Every set of reals is Lebesgue measurable".*

The Axiom of Dependent Choice (DC) above is equivalent to the Baire Category Theorem, and is strictly weaker than AC. Matters rested here for a while, as logicians worried about the inaccessible cardinal. Then their cares were lifted when Shelah, [Sh], proved (among other things) that if all Σ^1_3 sets of reals are Lebesgue measurable, then the first uncountable cardinal is inaccessible in L, the universe of constructible sets. This, taken in conjunction with Solovay's theorem, established the equivalence of assertions about the consistency of the Lebesgue measurability of classes of reals and the consistency of large cardinal axioms, and

inspired a stream of equiconsistency results. It surprised the wider public to learn that holding unrestrained views about Lebesgue integrability of certain real functions was no different (in terms of consistency) from endorsing set-theoretic universes containing large cardinals.

The complexity of non-measurable sets of reals and their possible whereabouts in the analytic hierarchy of the subsets of **R** continue to form the focus of intensive research. The lecture notes, [B], of Bekkali present some of the developments in this area.

References

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