REMARKS ON A PROBLEM OF FINBARR HOLLAND CONCERNING TRIGONOMETRIC POLYNOMIALS

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Let P_n denote the set of all non-negative trigonometric polynomials of degree at most n, normalized to have constant term equal to 1. Thus a typical element of P_n has the form

$$p(t) = 1 + \sum_{j=1}^{n}(a_j \cos jt + b_j \sin jt) \geq 0 \quad \text{for all real } t.$$

A problem posed by Holland [1, Problem 4.26] essentially asks for the value of

$$\Lambda_n = \sup_{p \in P_n} \frac{1}{2\pi} \int_0^{2\pi} (p(t))^2 \, dt.$$

A much simpler problem is the determination of

$$M_n = \sup_{p \in P_n} \max p(t).$$

This was solved by Fejér [4] (or see [7; pp. 78-79]). It will be helpful to discuss this first, for it leads easily to rough bounds for Λ_n. Fejér showed that $M_n = n+1$; for a short proof see [2; §3.2]. He also showed that M_n is an attained supremum: in fact if

$$q_n(t) = 1 + \frac{2}{n+1}(n \cos t + (n-1)\cos 2t + \cdots + \cos nt), \tag{1}$$

then $q_n \in P_n$ (for an easy calculation shows that

$$q_n(t) = \frac{1}{n+1}\left(\sin\{(n+1)\frac{t}{2}\}/\sin\frac{t}{2}\right)^2 \geq 0 \quad (0 < t < 2\pi))$$

and

$$\max q_n(t) = q_n(0) = 1 + \frac{2}{n+1}(n + (n-1) + \cdots + 1) = n+1.$$

Goldstein and McDonald [6] observed that Fejér's result leads to bounds on Λ_n as follows. If $p \in P_n$, then

$$\frac{1}{2\pi}\int_0^{2\pi}(p(t))^2 dt \leq \frac{1}{2\pi} \max p(t) \int_0^{2\pi} p(t) dt = \max p(t) \leq n+1.$$

On the other hand,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}(q_n(t))^2 dt &= 1 + \frac{2}{(n+1)^2}(n^2 + (n-1)^2 + \cdots + 1^2)\\ &= 1 + \frac{n(2n+1)}{3(n+1)}\\ &> \frac{2}{3}(n+1).\end{aligned}$$

Hence $2/3 < \Lambda_n/(n+1) \leq 1$. In [6] there is further evidence favouring the conjecture that $(\Lambda_n/(n+1))$ converges to a limit $C \in [2/3, 1]$. In fact, a proof of this conjecture, yielding the value $C = 0.68698\ldots$, is implicit in earlier work of Garsia, Rodemich and Rumsey [5]. In work based partly on [5], Brown, Goldstein and McDonald [2, Theorem 2] showed further that $(n+1)C \leq \Lambda_n < 1 + (n+1)C$ for all $n \geq 1$. Quite intricate arguments are used in both [5] and [2], and it seems worthwhile to give an elementary, self-contained, and comparatively short proof of the existence of the limit C.

Theorem. *The sequence $(\Lambda_n/(n+1))$ converges to a limit C in $[2/3, 1]$ and*

$$C = \inf_{n \geq 1} \Lambda_n/(n+1). \tag{2}$$

The main step in our proof is to establish the inequality

$$\frac{\Lambda_{nk+k-1}}{nk+k} \leq \frac{\Lambda_n}{n+1} \qquad (k \geq 2, n \geq 1). \tag{3}$$

Suppose for the moment that (3) is true and let C be defined by (2). Fix $\epsilon > 0$ and let N be such that $\Lambda_N/(N+1) < C+\epsilon$. If $n > N+1$ and $k(n)$ is the least integer such that $(N+1)k(n) > n$, then $(N+1)k(n) \leq n+N+1$, and hence using (3) and the obvious fact that (Λ_n) is non-decreasing, we obtain

$$\begin{aligned}\frac{\Lambda_n}{n+1} &\leq \frac{\Lambda_{Nk(n)+k(n)-1}}{(N+1)k(n)} \cdot \frac{(N+1)k(n)}{n+1} \\ &\leq \frac{\Lambda_N}{N+1}(1+\frac{N}{n+1}),\end{aligned}$$

so that $\limsup \Lambda_n/(n+1) < C+\epsilon$ and hence $\Lambda_n/(n+1) \to C$.

We write

$$J(p) = \frac{1}{2\pi}\int_0^{2\pi}(p(t))^2\,dt.$$

To prove (3), it suffices to show that if $p \in P_{nk+k-1}$, then

$$J(p) \leq k\Lambda_n. \tag{4}$$

Let such a function p be given by

$$p(t) = 1 + \sum_{j=1}^{nk+k-1}(a_j\cos jt + b_j \sin jt).$$

Since $p \geq 0$,

$$\begin{aligned}0 &\leq \frac{1}{2\pi}\sum_{m=1}^{k-1}\int_0^{2\pi} p(t)p(t+2m\pi/k)\,dt \\ &= k-1+\frac{1}{2}\sum_{j=1}^{nk+k-1}\left\{(a_j^2+b_j^2)\sum_{m=1}^{k-1}\cos(2mj\pi/k)\right\} \\ &= k-1-\frac{1}{2}\sum_{j=1}^{nk+k-1}(a_j^2+b_j^2)+\frac{1}{2}k\sum_{\ell=1}^{n}(a_{\ell k}^2+b_{\ell k}^2);\end{aligned}$$

the last-written equation follows from the fact that

$$\sum_{m=1}^{k-1}\cos(2mj\pi/k) = \begin{cases} k-1 & \text{if } k|j \\ -1 & \text{if } k\not| j.\end{cases}$$

Hence

$$J(p) = 1+\frac{1}{2}\sum_{j=1}^{nk+k-1}(a_j^2+b_j^2) \leq k\left(1+\frac{1}{2}\sum_{\ell=1}^{n}(a_{\ell k}^2+b_{\ell k}^2)\right). \tag{5}$$

Note that

$$1+\frac{1}{2}\sum_{\ell=1}^{n}(a_{\ell k}^2+b_{\ell k}^2) = J(q), \tag{6}$$

where

$$q(t) = 1+\sum_{\ell=1}^{n}(a_{\ell k}\cos \ell t - b_{\ell k}\sin \ell t). \tag{7}$$

If we can show that q is non-negative, then we shall have $q \in P_n$ and hence $J(q) \leq \Lambda_n$. From (5) and (6) it will then follow that $J(p) \leq kJ(q) \leq k\Lambda_n$, and (4) and hence (3) will be established.

To show that q is non-negative, we first associate to p the harmonic polynomial h defined by

$$h(re^{it}) = 1+\sum_{j=1}^{nk+k-1} r^j(a_j\cos jt + b_j\sin jt).$$

Let Δ denote the unit disc. Since $h(e^{it}) = p(t) \geq 0$ for all $t \in [0,2\pi]$, we have $h \geq 0$ on $\partial\Delta$ and hence, by the minimum principle, $h \geq 0$ on Δ. Also define K on Δ by

$$K(re^{it}) = 1+2\sum_{\ell=1}^{\infty} r^\ell \cos \ell t. \tag{8}$$

It is easy to verify that

$$K(re^{it}) = \frac{1-r^2}{1-2r\cos t + r^2} > 0 \qquad (re^{it} \in \Delta).$$

(In fact, K is the Poisson kernel of Δ with pole 1.) Since the series in (8) is locally uniformly convergent on Δ, we have for all $r \in (0,1)$ and all real θ,

$$0 \le \frac{1}{2\pi}\int_0^{2\pi} h(re^{it})K(re^{i(kt+\theta)})dt$$
$$= 1 + \frac{1}{\pi}\sum_{\ell=1}^{\infty} r^\ell \left(\sum_{j=1}^{nk+k-1} r^j \int_0^{2\pi} (a_j \cos jt + b_j \sin jt)\cos(\ell kt + \ell\theta)dt\right)$$
$$= 1 + \sum_{\ell=1}^{n} r^{\ell+\ell k}(a_{\ell k}\cos \ell\theta - b_{\ell k}\sin \ell\theta).$$

Letting $r \to 1-$, we find that the function q defined by (7) is indeed non-negative and, as explained earlier, (3) now follows and therefore $(\Lambda_n/(n+1))$ converges to the limit C given by (2). The bounds on Λ_n obtained from Fejér's work show that $2/3 \le C \le 1$.

Calculations using Mathematica and based on a representation of Λ_n obtained by Goldstein and McDonald [6, Corollary 2] suggest the values given in the table below. I am grateful to Tony Wickstead for his help with these calculations. Our values for $\Lambda_2, \ldots, \Lambda_5$ confirm those obtained in [6, p.87], except for a small discrepancy in the value of Λ_3.

n	Λ_n	$\Lambda_n/(n+1)$
1	1.5	.75
2	2.142857142...	.714285714...
3	2.808840165...	.702210041...
4	3.4834502.....	.6966900.....
5	4.1622565.....	.6937094.....
6	4.8434275.....	.6919182.....
7	5.5260645.....	.6907580.....
8	6.2096738.....	.6899637.....
9	6.8939613.....	.6893961.....

To the best of my knowledge, the conjecture that $(\Lambda_n/(n+1))$ is decreasing remains open.

One obvious generalization of Holland's question concerns

$$\Lambda_{n,\alpha} = \sup_{p\in P_n} \frac{1}{2\pi}\int_0^{2\pi} (p(t))^\alpha dt \qquad (\alpha > 0).$$

If $0 < \alpha < 1$, then Hölder's inequality shows that $\Lambda_{n,\alpha} \le \Lambda_{n,1} = 1$, and since we can always take $p(t) \equiv 1$, it follows that $\Lambda_{n,\alpha} = 1$ for all n. If $\alpha > 1$, then there exists a positive constant c_α such that

$$c_\alpha(n+1)^{\alpha-1} \le \Lambda_{n,\alpha} \le (n+1)^{\alpha-1}.$$

Here the upper bound is obtained from Fejér's result $M_n = n+1$ and the lower bound is obtained by estimating

$$\int_0^{2\pi} (q_n(t))^\alpha dt,$$

where q_n is given by (1). It seems plausible that $\lim_{n\to\infty}(n+1)^{1-\alpha}\Lambda_{n,\alpha}$ exists when $\alpha > 1$, but this appears to be an open question, except for $\alpha = 2$.

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