8. **Elections**

The following were elected, unopposed, to the Committee (Re-election to the Committee is denoted by ∗):

Committee member	Proposed	Seconded
P. Mellon* (Secretary)	R. Timoney	M. Ó Searcóid
M. Vandyck (Treasurer)	D. Hurley	J. Pulé
E. Gath*	J. Pulé	G. Lessells
R. Gow	M. Ó Searcóid	D. Tipple
C. Nash	M. Ó Searcóid	P. Mellon
R. Timoney*	B. Goldsmith	S. Dineen

The following have one more year of office: B. Goldsmith (President), D. Hurley (Vice-President), G. Lessells, B. McCann, M. Ó Searcóid, J. Pulé.
The Committee is to co-opt one member from UCG.
The following have left the Committee: G. Ellis, D. Tipple
The President thanked the out-going Secretary for his services over the last four years.

9. **September meeting**

S. Dineen and S. Gardiner are organizing the 1994 annual meeting at UCD for 5th and 6th September. It will be followed by a three day conference on polynomials and holomorphic functions. Accommodation will be available on the UCD campus at £13.50 per night.

The 1995 annual meeting will be held at the University of Limerick.

10. There was no **other business**.

The meeting closed at 1.00 pm.

Graham Ellis
University College
Galway

A VIEWPOINT ON MINIMALITY IN TOPOLOGY

P. T. Matthews and T. B. M. McMaster

Introduction.

Given a family $\mathcal{F}$ of topological spaces whose point-sets all have the same cardinality, and a particular space X in $\mathcal{F}$, what should we mean by saying that X is *minimal* in $\mathcal{F}$, and what use can be made of such a concept?

Well, it depends both on the nature of the family and, critically, on the ordering relation between the spaces which belong to it. If for example we take $\mathcal{F}$ as a collection of spaces all having the same underlying point-set S, and order them by refinement of topology (writing $(S,\tau_1) \leq (S,\tau_2)$ if and only if every τ_1-open set is τ_2-open) then we are looking at part of the lattice of topologies on S. Here the *interpretation* of minimality is entirely unambiguous and straightforward: (S,τ) is minimal in $\mathcal{F}$ if, whenever $(S,\tau') \leq (S,\tau)$ and $(S,\tau') \in \mathcal{F}$, then $\tau = \tau'$. The *techniques* required, however, to access minimal objects in this context can on occasions be extremely complex and subtle (see, for instance, Larson [3], Johnston and McCartan [5,6], McCluskey [10], McCluskey and McCartan [11,12,13]) and the resulting insights correspondingly deep: indeed, as has been persuasively argued, "In seeking to identify those [topologies on S] which minimally satisfy an invariant property, we are, in a very real sense, examining the topological essence of the invariant" [11].

Nevertheless there are aspects of general topology for which this approach to minimality is not appropriate. It is often the correct practice (especially when the discussion is in any sense categorical) to co-identify spaces which are homeomorphic to one another, in effect working with the homeomorphism classes in preference to the multitude of examples within each class. This device is not readily compatible with 'refinement of topology' since it is easy to exhibit on any infinite set two homeomorphic topologies one of which is strictly finer than the other, and in any case the 'same underlying point-set' phrase is rendered meaningless by focussing on classes. A different ordering relation is therefore needed here, and the one most readily available is that of 'embeddability as a subspace', the binary relation *sub* defined by X sub Y iff X is homeomorphic to a subspace of Y. This has several desirable features, such as respecting homeomorphic equivalence, relating nicely to hereditary classes of spaces, and being reflexive and transitive. But it is not antisymmetric (the open and closed intervals (0,1) and [0,1] are by no means the same space, yet (0,1) sub [0,1] and [0,1] sub (0,1) are both true) and herein lies the difficulty: how can we assign a meaning to minimality of elements in a set which is not *partially* ordered but only *quasi*-ordered? And, of course, why should we bother to do so?

This note considers two suggestions for answering the first question. One is effectively that adopted in Ginsburg and Sands' paper [2] and in the unresolved 'Toronto problem' which is associated with it. We use the other to establish a proposition, previously unnoticed so far as we have been able to determine, about that best-known of all topological spaces, the real line; it will then be seen to play a key role in characterizing the circumstances in which Bankston's 'Anti-' operation [1] exhibits a certain behaviour. Hopefully these findings will be perceived as a partial answer to the second question above!

We thank the referee for helpful and perceptive criticisms of this article.

Strong and weak quasi-minimality.

Take an infinite cardinal α, $\mathcal{T}(\alpha)$ to denote the family of all topological spaces on α points, $\mathcal{F}$ a subfamily of $\mathcal{T}(\alpha)$, X a member of $\mathcal{F}$ and sub as described in the Introduction. Let us agree to call X *strongly quasi-minimal in* $\mathcal{F}$ if

$$Y \text{ sub } X, Y \in \mathcal{F} \text{ imply } Y \text{ homeomorphic to } X,$$

and *weakly quasi-minimal in* $\mathcal{F}$ if

$$Y \text{ sub } X, Y \in \mathcal{F} \text{ imply } X \text{ sub } Y.$$

The abbreviations sqm and wqm will be employed, and the following remarks are immediate:

Proposition.

(i) *sqm in $\mathcal{F}$ implies wqm in $\mathcal{F}$ (for any $\mathcal{F}$).*

(ii) *The converse is not always valid (consider the two-space counterexample* $\mathcal{F} = \{(0,1), [0,1]\}$).

(iii) *In any $\mathcal{F}$ which is partially ordered by sub (after identification of homeomorphic spaces) the sqm, wqm and minimal elements coincide.*

Of particular interest for our applications is the case $\mathcal{F} = \mathcal{T}(\alpha)$, so we shall compactify our notation further and write 'X is sqm' (or wqm) rather than 'X is sqm in $\mathcal{T}(\alpha)$' (or wqm in $\mathcal{T}(\alpha)$). So an sqm space is homeomorphic to each of its equicardinal subspaces, a wqm space is embeddable into each of its equicardinal subspaces. What do these spaces look like?

Well, in the case $\alpha = \aleph_0$ Ginsburg and Sands give a complete and remarkably tidy answer [2]. They observe that on the set of positive integers the discrete, trivial, cofinite, initial-segment and final-segment topologies give sqm spaces, and they demonstrate that every infinite space contains a copy of one or more of these five (let us call them GS spaces).

Theorem (Ginsburg and Sands).

(i) In $\mathcal{T}(\aleph_0)$ the sqm and the wqm spaces are precisely the five GS spaces.

(ii) $\mathcal{T}(\aleph_0)$ is "supported" by its wqm members, in the sense that for each X in $\mathcal{T}(\aleph_0)$ there is some wqm Y in $\mathcal{T}(\aleph_0)$ such that Y sub X.

Corollary. *In the class $T_2 \cap \mathcal{T}(\aleph_0)$ of denumerable Hausdorff spaces, only the discrete space is sqm (or wqm) and this space on its own supports $T_2 \cap \mathcal{T}(\aleph_0)$.*

At present it is far from clear what happens to these results when $\aleph_0$ is replaced by a larger cardinal. Certainly not every uncountable Hausdorff space contains a discrete equicardinal subspace, so for part (ii) of the theorem to be generalizable to $\aleph_1$ or beyond we should require uncountable non-discrete Hausdorff sqm spaces; the Toronto problem [14, p.15] asks whether such objects exist, and it has yet to be answered. Our contribution to the debate is to observe that if they do exist, then there are "not enough of them to support their colleagues" in the above sense, even if we relax sqm to wqm. More precisely, we show (subject to set-theoretic assumptions) that for some uncountable cardinals α,

(a) the wqm spaces in $\mathcal{T}(\alpha)$ do not support $\mathcal{T}(\alpha)$,
(b) the wqm spaces in $T_2 \cap \mathcal{T}(\alpha)$ do not support $T_2 \cap \mathcal{T}(\alpha)$,
(c) any subfamily of $\mathcal{T}(\alpha)$ which does support the entire family must have fairly large cardinality.

The proof, embodied largely in the following three lemmas, consists of a transfinite-induction construction combined with a variant of a standard argument relating the weight of a space to the number of its autohomeomorphisms.

Lemma A. *Let α be a regular cardinal, X a set of cardinality α, and $\{S_\beta : \beta < \alpha\}$ a family of α subsets of X each having α elements. Then there is an α-element subset T of X which contains none of the sets S_β.*

Proof: Suppose for convenience that X *is* α. We define a strictly increasing transfinite sequence $(x_\beta, \beta < \alpha)$ in such a way that $x_\beta \in S_\beta$ and $x_{\beta'} > (x_\beta)'$ for each β, the "dash" indicating successor in α. Initialize by choosing

$$x_0 = \text{the least element of } S_0.$$

Now assuming (for typical non-zero $\beta < \alpha$) that the x_γ for $\gamma < \beta$ have been chosen in accordance with the desired criteria, we note that the set $\{x_\gamma : \gamma < \beta\}$ has smaller cardinality than α and must therefore be bounded above in (regular) α. Select an upper bound u for it, notice that S_β cannot be bounded in α, and choose

$$x_\beta = \text{the least element of } S_\beta \text{ strictly greater than } u'.$$

Now that induction guarantees the existence of the required $(x_\beta, \beta < \alpha)$, we observe that the set of successors of its terms

$$T = \{x'_\beta : \beta < \alpha\}$$

includes none of the x_β, and thus contains none of the S_β.

Lemma B. *Suppose that for cardinals α and β we have*

$$\alpha \le 2^\beta \text{ and}$$

$$\gamma < \beta \text{ implies } 2^\gamma \le \beta.$$

Then there is a Hausdorff topology on a set of cardinality α in which every subspace has a dense subset of cardinality β or less.

Proof: We first consider the power set $P(\beta)$ of β. For each $\gamma < \beta$ and each subset G of γ put

$$[\gamma, G] = \{H \in P(\beta) \ : \ H \cap \gamma = G\}.$$

Whenever $\gamma_1 \le \gamma_2$ in β we see that

$$[\gamma_1, G_1] \cap [\gamma_2, G_2] = \begin{cases} [\gamma_2, G_2] & \text{if } G_2 \cap \gamma_1 = G_1, \\ \phi & \text{otherwise} \end{cases}$$

and it follows that $\mathcal{B} = \{[\gamma, G] \ : \ \gamma < \beta, G \subseteq \gamma\}$ is a base for a topology τ on $P(\beta)$. Given distinct elements A, B of $P(\beta)$ it will always be possible to find $\gamma < \beta$ such that γ belongs to exactly one of A and B; then $[\gamma', A \cap \gamma']$ and $[\gamma', B \cap \gamma']$ are disjoint τ-open neighbourhoods of A and B, so τ is Hausdorff. The cardinality of $\mathcal{B}$ is $\sum_{\gamma<\beta} 2^\gamma$ which, under the stated supposition, is β. Every subspace of $P(\beta)$ therefore has a base (and consequently a dense subset) with at most β members. Now any set of cardinality α can be injected into $P(\beta)$ to inherit a topology with the same property.

Lemma C. *Suppose that X is a Hausdorff space of regular cardinality α whose subspaces all have dense subsets of β or fewer points, and that $\alpha^\beta = \alpha$. Let there be given a family $\{S_\gamma : \gamma < \alpha\}$ of α subsets of X each possessing α elements. Then every α-element subspace Y of X has an α-element subspace into which none of the S_γ can be homeomorphically embedded.*

Proof: Fix Y. For each $\gamma < \alpha$, any homeomorphism h from S_γ onto a subset of Y is completely determined by its values on a dense subset of S_γ; since this subset may be taken to comprise no more than β elements, there cannot be more than $\alpha^\beta = \alpha$ of these homeomorphisms, for which reason the number of subsets of Y that are homeomorphic to any of the various S_γ is at most $\alpha \cdot \alpha = \alpha$. Lemma A now assures us that Y has an α-element subset containing no homeomorph of any S_γ.

Notice that if we put $S_\gamma = Y$ for every $\gamma < \alpha$ in Lemma C, it tells us that Y is not wqm. Consider now the following composite assertion $Q\min(\alpha)$ concerning an uncountable cardinal α:

(a) neither $\mathcal{T}(\alpha)$ nor $T_2 \cap \mathcal{T}(\alpha)$ is supported by its wqm members, and

(b) any subfamily of $\mathcal{T}(\alpha)$ or of $T_2 \cap \mathcal{T}(\alpha)$ which does support the whole family must have more than α members .... [$Q\min(\alpha)$]

Our three lemmas now permit us to probe the relationship between set-theoretic axioms and the values of α for which this is a valid conclusion, thus:

Theorem.
(i) The assumption $\aleph_1^{\aleph_0} = \aleph_1$ gives us $Q\min(\aleph_1)$.
(ii) If $\mathfrak{c} = 2^{\aleph_0}$ is regular [note: this is a consequence of Martin's Axiom (MA), see [4, p.284]] then we get $Q\min(\mathfrak{c})$.
(iii) The continuum hypothesis $CH(2^{\aleph_0} = \aleph_1)$ implies $Q\min(\aleph_1)$.
(iv) If the generalized continuum hypothesis $GCH(2^\alpha$ is the successor of α for each $\alpha \geq \aleph_0)$ is assumed, we get $Q\min(\alpha)$ for every successor cardinal α.

Proof: Both (i) and (ii) follow directly from Lemma C without recourse to Lemma B, since the real line (or an $\aleph_1$-element subset of it) will suffice for the space X, choosing $\beta = \aleph_0$. (iii) follows immediately from (ii). If we assume GCH, then every successor cardinal α is of the form 2^β (where β is its immediate predecessor) and Lemma B supplies the Hausdorff space needed by Lemma C; also $\alpha^\beta = (2^\beta)^\beta = \alpha$ to complete the evidence for (iv).

Corollary (to Lemma C). *CH (or MA) implies that the real line contains no Toronto space (sqm uncountable non-discrete Hausdorff space).*

Application to Bankston's "Anti-".

Paul Bankston [1] developed a procedure, based on the connected/totally disconnected relationship, for converting any given topological invariant P into another, "anti-P": a space X is *anti-P* when the only subspaces Y of X that are P are those for which *every* topology on a set of Y's cardinality is P. A comprehensive survey of this topic up to 1989 will be found in an earlier issue of this Bulletin [7], to which we refer the reader for details.

Suppose now that P is a given hereditary (non-universal) property and that λ_P denotes the smallest cardinality of the non-P spaces. Matier and McMaster have identified the circumstances in which there is a hereditary invariant Q satisfying anti-$Q = P$ (a *hereditary pre-anti* for P) [8,9] and in the case $\lambda_P = \aleph_0$ they use the Ginsburg and Sands theorem to identify amongst these properties Q one which is logically strongest. The question they leave unresolved, of whether it is possible to do this also when $\lambda_P > \aleph_0$, will now be shown to depend on the existence of 'enough' wqm spaces of cardinality λ_P.

Let us first examine the special case in which no space of cardinality λ_P or more is P. Topologically this is of extreme triviality, since P then is the property of having fewer than λ_P points (such an invariant has been referred to as *cardinally decisive!*) but it turns out to provide an adequate illustration of techniques and results so far obtained.

Lemma D. *When P is cardinally decisive, then Q is a hereditary pre-anti for P if and only if*
(i) Q is hereditary,
(ii) $\lambda_Q = \lambda_P$,
(iii) $\mathcal{T}(\lambda_P)$ is supported by $Q \cap \mathcal{T}(\lambda_P)$.

Proof: Almost immediate from the definitions.

Proposition. *A cardinally decisive property P possesses a strongest hereditary pre-anti if and only if $\mathcal{T}(\lambda_P)$ is supported by its wqm members.*

Proof: Supposing that $\mathcal{T}(\lambda_P)$ is so supported, we define S to comprise all spaces on fewer than λ_P points together with all wqm spaces on exactly λ_P points. Using Lemma D, this is easily checked to be one of the hereditary pre-antis for P. If Q is any of the latter properties and X is wqm in $\mathcal{T}(\lambda_P)$ then Y sub X for some Q space Y in $\mathcal{T}(\lambda_P)$, and consequently X sub Y also, which shows X to be Q; hence every S space is Q, so S is indeed the strongest such invariant.

Conversely, suppose that there is a space X in $\mathcal{T}(\lambda_P)$ none of whose equicardinal subspaces is wqm. Given any hereditary pre-anti Q for P, there must be a space Y in $Q \cap \mathcal{T}(\lambda_P)$ such that Y sub X, and since Y is not wqm we can find Z in $\mathcal{T}(\lambda_P)$ with Z sub Y but not (Y sub Z). We define:

$$Q^* = \{T \in Q : \text{ not } (Y \text{ sub } T)\}.$$

Another appeal to Lemma D readily shows Q^* to be a hereditary pre-anti for P which, since it excludes the Q space Y, is strictly stronger than Q. We conclude that no strongest hereditary pre-anti for P can exist.

Corollary.

(i) [9] The strongest hereditary pre-anti for the class of finite spaces comprises the five GS spaces together with all the finite spaces.

(ii) Assuming CH, the class of countable spaces has no strongest hereditary pre-anti.

(iii) Assuming GCH, for each successor cardinal α the class of spaces having cardinalities less than α has no strongest hereditary pre-anti.

No very radical transformation of the argument above is needed to generalize from the cardinally decisive case to that in which some P spaces have λ_P or more elements. We obtain the following conclusions:

Lemma E.

(i) A wqm space of cardinality λ_P is anti-P if and only if it is non-P.

(ii) If X sub Y where Y is wqm and X has the same cardinality as Y, then X is wqm.

Theorem. *Let P be a topological invariant which possesses hereditary pre-antis. There exists a strongest hereditary pre-anti for P if and only if the class of non-P spaces in $\mathcal{T}(\lambda_P)$ is supported by its wqm members. When it exists, it consists of the wqm non-P spaces of cardinality λ_P together with all spaces of smaller cardinality.*

Amongst the questions so far unresolved in our investigations of this topic, the following appear to be most pressing:

Problem 1. Find a wqm space which is not sqm. More generally, for which values of α are wqm and sqm in $\mathcal{T}(\alpha)$ equivalent?

Problem 2. Will any reasonable set-theoretic assumptions enable us to prove or disprove $Q\min(\alpha)$ where α is an uncountable limit cardinal?

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P. T. Matthews and T. B. M. McMaster
Department of Pure Mathematics
The Queen's University of Belfast
Belfast BT7 1NN

EPIMORPHISMS ACTING ON BURNSIDE

Des MacHale and Robert Sheehy

The Burnside group $B(r,n)$ is the group of exponent n, generated by r elements $x_1, x_2, \ldots, x_r$. It is well known that $B(r,n)$ is finite for $n = 2, 3, 4$ and 6 for all r but that for $n \geq 665$ and n odd, $B(r,n)$ is infinite when $r > 1$. In addition, it has recently been shown that for $n \geq 2^{48}$, $B(r,n)$ is infinite for $r > 1$, [1].

Let $\mathcal{B}$ be the set of all positive integers n for which $B(r,n)$ is finite for all r. Since the relation $g^n = 1$ can be written as $g^{n+1} = g = (g)I$ where I is the identity automorphism, we ask the following question.

Suppose G is a finitely generated group and the map α given by $g\alpha = g^k$ for all $g \in G$ and a fixed positive integer k, is an automorphism of G. What values of k force G to be finite?

In fact, in what follows, we can replace 'automorphism' by 'epimorphism', that is, an endomorphism of G onto G, and prove the following result.

Theorem. *Suppose that n belongs to $\mathcal{B}$ and that G is a finitely generated group such that the map α given by $g\alpha = g^{n+1}$ for all $g \in G$ is an epimorphism of G. Then G is finite.*

Proof: For all a and b in G, $(ab)\alpha = (ab)^{n+1} = a^{n+1}b^{n+1}$, so by cancellation $(ba)^n = a^n b^n$. Then $(ba)^{n+1} = (ba)^n ba = a^n b^n ba$, whence $b^{n+1}a^n = a^n b^{n+1}$. Since α is onto, $ga^n = a^n g$ for all a and g in G, and so $a^n \in Z(G)$ for all $a \in G$, where $Z(G)$ denotes the centre of G.

Now $G/Z(G)$, being a factor group of a finitely generated group, is finitely generated of exponent n and since $n \in \mathcal{B}$, $G/Z(G)$ is finite. Thus $Z(G)$, being a subgroup of finite index in a finitely generated group, is a finitely generated abelian group.