HIGHER ORDER SYMMETRY OF GRAPHS*

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Symmetry in analogues of set theory.

This article gives background to and results of work of my student John Shrimpton [18, 19, 20]. It advertises the joining of two themes: groups and symmetry; and categorical methods and analogues of set theory.

Groups are expected to be associated with symmetry. Klein's famous *Erlanger Programm* asserted that the study of a geometry was the study of the group of automorphisms of that geometry.

The structure of group alone may not give all the expression one needs of the intuitive idea of symmetry. One often needs structured groups (for example topological, Lie, algebraic, order,...). Here we consider groups with the additional structure of a *directed graph*, which we abbreviate to *graph*. This type of structure appears in [13] and [17].

We shall associate with a graph A a group $\mathrm{AUT}(A)$ which is also a graph. The vertices of $\mathrm{AUT}(A)$ are the automorphisms of the graph A and the edges between automorphisms give an expression of "adjacency" of automorphisms. The vertices of this graph form a group, and so also do the edges. The automorphisms of A adjacent to the identity will be called the *inner automorphisms* of the graph A. One aspect of the problem is to describe these inner automorphisms in terms of the internal structure of the graph A.

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The second theme is that of regarding the usual category of sets and mappings as but one environment for doing mathematics, and one which may be replaced by others. We use the word "environment" here rather than "foundation", because the former word implies a more relativistic approach.

The other environment we choose here is the category of directed graphs and their morphisms. We define this category, and then use methods analogous to those of set theory within this category. This allows set-theoretic intuition to be used to generate new results and methods, and is possible because of the "good" properties of this category of graphs. The background here is that of topos theory, which has given methods for considering many other environments for mathematics, and for comparing these environments.

Topos theory takes a relative rather than absolute viewpoint towards sets. The topos of sets is obviously an important, standard and basic kind of topos, but suffers from the defect of being somewhat boring, reflecting the fact that the objects of the topos, namely the abstract sets, are devoid of structure. The topos theory approach allows not only other versions of the category, or topos, of sets, but also allows comparison of different versions, through the notion of functor and natural transformation.

Thus different notions of set, or graph, can be evaluated by comparing the properties of the associated category. This global viewpoint has proved fruitful. One point of appearance was in topology, where the standard category of topological spaces was found not to have a function space with convenient properties. So different categories of topological spaces were proposed with "better" or more convenient function spaces.

The idea of emphasizing the categorical aspects of sets is not so familiar outside of category theory. For example, the article [1] does not mention any categorical approach. The traditional viewpoint is that sets are defined by the membership relation. There is, however, a strong argument that this approach is counter intuitive, since for many sets we wish to use, such as that of real numbers, it is very difficult to get one's hands on any but a small fraction of their members.

The categorical approach is that sets are defined by the relations between them, namely the functions, and this view has been strengthened by the success of topos theory. The book [14] is a good introduction to topos theory for those with a foundation in category theory. For an article relating the history of topos theory to notions of the foundations of sets, see [15]. The author emphasizes that the notion of topos was defined by Grothendieck as a replacement for the notion of topological space. Thus it was intrinsic to the definition that many different topoi were to be considered.

In the work of Lawvere, categories of structures other than sets are regarded as having intuitive value equal to that of the category of sets. That is, the category of sets is not regarded as a foundation for mathematics. Some words have to be said on the advantages of categorical methods, whose objectives and methodology have failed to be realized by some. The book by Reid [16] even writes: "The study of category theory for its own sake (surely one of the most sterile of intellectual pursuits) also dates from this time; Grothendieck himself can't necessarily be blamed for this, since his own use of categories was very successful in solving problems".

This quotation has aspects which should be noted. One is that it derides some vaguely specified group of colleagues as essentially unprofessional. A second is its lack of adventure. Let me propose a game: "I can think of a more sterile intellectual pursuit than you". A third is that it is hardly sensible to think of "blaming" Grothendieck for developments in mathematics. A fourth is its avoidance of historical analysis and of supporting evidence. This should be contrasted with McLarty's article [15].

A fifth is the view that the aim of mathematics is the solution of problems, which, by implication, are already formulated. By contrast, a historical view shows that the value of mathematics for other subjects, and for its own ends, is that it has developed language for:

- the study of patterns and structures;
- the formulation of problems;
- the development of methods of calculation and deduction.

The solution of problems is often a byproduct of this wider process and these wider aims. In this process, the study of an area for its own sake is often a necessary developmental stage. Judgements on the sterility or otherwise of such a study can be a matter of timing, or of gossip and snobbery, and are not always based on comparison and scholarship.

Does our education of mathematicians train them in the development of faculties of value, judgement, and scholarship? I believe we need more in this respect, so as to give people a sound base and mode of criticism for discussion and debate on the development of ideas.

The origins of category theory help to explain its utility. It arose from attempting to explain the meaning of the word "natural" in mathematics, and with a strong impetus from the axiomatic approach to homology theories, developed by Eilenberg and Steenrod, [6]. The original paper on the subject by Eilenberg and Mac Lane, [5], has an interesting discussion of the word "natural" in terms of the map $V \to V^{**}$ of a vector space into its double dual. To define natural required a definition of functors, and to define functors required a definition of category. This itself reflected also the growing realization that whenever a structure has been defined, it is usually necessary to consider also the morphisms of that structure.

By now, the general notions of limit and colimit, whose formulation was possible with the use of categories, and the later notion of adjoint functor, must be regarded as basic tools in mathematics. For example, the fact that a functor which is a left adjoint preserves colimits, while a right adjoint preserves limits, is a useful computational tool in many aspects of algebra and even combinatorics. Graduate books will probably have to give initial sections on basic concepts of category theory, in the same way as they have given basic sections on set theory, algebra and topology.

Category theory has been found useful for

- a global approach: i.e. constructions are defined by universal properties, which give the relation of the constructed object to all other objects;
- formulating definitions and theorems;

- carrying out proofs;
- discovering and exploiting analogies between various fields of mathematics.

Grothendieck's work on the foundations of algebraic geometry led him to develop a vast range of new categorical concepts. It is significant that his first important work was in analysis, and he brought to algebraic geometry a local-to-global approach. In algebraic geometry, it seems that "local" means "at a given prime p", and "global" means "over the integers". His approach was also to take concepts seriously, recognizing the effort required to "bring new concepts out of the dark" ([7]), and to spend a lot of effort in turning difficult results into a series of tautologies.

As one other recent example, and an indication of a wide literature, the paper by Joyal and Street, [8], illustrates how an algebraic development initially formulated for metamathematical reasons, and almost for its own sake, namely the notion of monoidal, or tensored, category, has found striking applications in concrete problems in knot theory, and string theory in physics.

One of the attractions of category theory is that the same algebraic tools are found applicable at several levels, and in a variety of areas. This feature is also found in groupoid theory, of which a survey was given by me in [3]. This notion has allowed the formulation of important extensions of group theory and of notions of symmetry.

Thus category theory is *par excellence* the method which enables the recognition and exploitation of many forms of analogy and comparison of structures. The point is that the algebraic study of the structure of a theory involves studying the categories and functors associated to the theory, and such a study leads to new algebraic notions of interest in their own right.

Applications to graph theory.

There are several unfamiliar aspects of this approach as applied to graph theory.

1) In this approach, it is essential to use a category of graphs and their morphisms. By contrast, it is not so easy to find a book on graph theory which defines a morphism of graphs.

2) An important categorical method used is that of *universal property*. In our setting, this defines a construction on graphs by the relation of the construction to all graphs. This may seem curious and far from logical. In fact, a construction by universal properties is analogous to a program, which when given an input of particular graphs, or graphs and morphisms between them, gives an output, namely new graphs and new morphisms. This analogy to programming is one reason why computer scientists have found the methods of category theory useful.

3) We lift to the category of graphs standard methods available in the category of sets and functions.

There are many possible definitions of graph and morphism of graph. We take one which gives for our purposes the "best" properties of the corresponding category. This again is an example of a "global" approach, and is simply a step or so up from a common approach in mathematics of considering for example all numbers, or all the symmetries of a square.

We deal here only with directed graphs. So for us a *graph* will mean a set A_E of edges, a set A_V of vertices and three functions $s, t : A_E \to A_V$, $\epsilon : A_V \to A_E$ such that $s\epsilon = 1$, $t\epsilon = 1$. Here s and t are the *source* and *target* maps. If $x, y \in A_V$, then $A(x, y)$ denotes the set of edges with source x and target y. Such an edge a is also written $a : x \to y$. A *loop* is an edge with the same source and target.

This defines in essence a directed graph in which each vertex v has an *associated loop* ϵv at that vertex. This extra structure makes no difference to the combinatorics of an individual graph, but makes a considerable difference to the allowable graph morphisms. The associated loop at a vertex v is often written • and given the vertex label v. Thus one of the simplest graphs, denoted I, is pictured as

$$0\bullet \longrightarrow \bullet 1.$$

A *morphism* of graphs $f : A \to B$ is a pair of functions $f_E : A_E \to B_E$, and $f_V : A_V \to B_V$ preserving the source and target maps, and ϵ. The implication is that f maps edges to edges, vertices to vertices, and f can map a general edge to the loop associated to a vertex. In effect, this means edges may be mapped to vertices.

The category $\mathcal{DG}$ of directed graphs has objects the graphs and arrows the morphisms, and $\mathcal{DG}(A, B)$ denotes the set of graph morphisms $A \to B$. Lawvere in [12] calls this the category of *reflexive* graphs.

This category has a *terminal object*, written $\bullet$, with the property that, for any graph A, the vertices of A are naturally bijective with $\mathcal{DG}(\bullet, A)$. The edges of any graph A are naturally bijective with $\mathcal{DG}(I, A)$.

Continuing with the categorical approach, we define the product of graphs.

A *product* of graphs A and B consists of a graph $A \times B$ with morphisms $p : A \times B \to A$, $q : A \times B \to B$ such that for any graph C the function

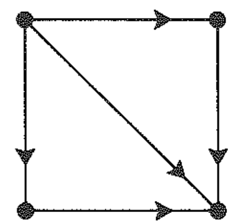
$$\mathcal{DG}(C, A \times B) \to \mathcal{DG}(C, A) \times \mathcal{DG}(C, B)$$

$$f \mapsto (pf, qf)$$

is a bijection. This says that a morphism to $A \times B$ is completely described by its component morphisms to A and B. The definition is also analogous to the law for numbers $(ab)^c = a^c b^c$.

It may be proved from the definition that the vertices of $A \times B$ are pairs of vertices from A and B, and the edges of $A \times B$ are pairs of edges from A and B. One way of proving this is to show that if $\mathcal{SETS}$ denotes the category of sets and functions, then the two functors $\mathcal{DG} \to \mathcal{SETS}$ given by the edges and the vertices have left adjoints, and so preserve limits, and in particular products. This deduction is one example of the "comparison" of environments referred to earlier. An important aspect of this procedure is that the product is defined by the universal property, which is the property that is most often used, and then a specific construction is deduced from the universal property. This verifies existence of the product.

As a typical example of the product of graphs, associated with the simplest graph I we have the product $I \times I$, illustrated by the following diagram:



Given sets B and C there is a set C^B of functions $B \to C$. In our category of graphs, the analogous construction is of course a *graph of morphisms* DIGRPH(B, C).

In the category of sets we have the standard *exponential law*

$$C^{A \times B} \cong (C^B)^A.$$

This corresponds to the law for numbers $c^{ab} = (c^a)^b$. In graph theory, we have the analogous law:

For graphs A, B and C, there is a natural bijection

$$\mathcal{DG}(A \times B, C) \cong \mathcal{DG}(A, \mathrm{DIGRPH}(B, C)).$$

Here the morphism graph DIGRPH(B, C) is in effect defined by this formula. From this formula, we can deduce the specific construction as follows.

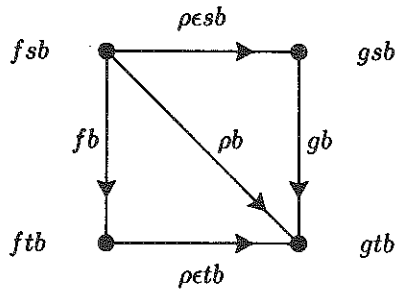
Let B and C be graphs. The graph DIGRPH(B, C) is to have vertices the morphisms of graphs $B \to C$ and to have edges the triples (ρ, f, g) such that f and g are morphisms of graphs $B \to C$ and $\rho : B \to C$ is a function from edges to edges such that if b is an edge of B then

$$s\rho b = fsb, \quad t\rho b = gtb.$$

Define

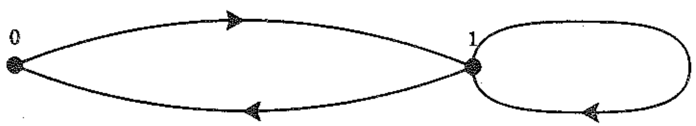
$$s(\rho, f, g) = f, \quad t(\rho, f, g) = g, \quad \epsilon(f) = (f, f, f).$$

Then each edge b of B yields the diagram



Comments.

1. If you define a directed graph by omitting ϵ, then product and morphism graph are defined, but the vertices of the morphism graph are not the morphisms of graphs. Instead, the morphisms correspond to the loops at vertices. From the categorical viewpoint, this is not surprising. The morphisms $B \to C$ should correspond to the morphisms $\bullet \to \mathrm{DIGRPH}(B, C)$, where $\bullet$ is the *terminal object* in the category, i.e. the graph such that there is exactly one morphism $A \to \bullet$ for any graph A. If the associated loop is omitted from the definition of graph, then the terminal object again has one vertex and one loop, and the morphisms of graphs are then not the vertices of the morphism graph, but are instead the loops of this morphism graph. The relations between these two categories of directed graphs are considered by Lawvere in [12].
2. There is another analogy between the category $\mathcal{DG}$ and the category of sets and functions. We can define in $\mathcal{DG}$ a graph Ω



and a morphism of graphs

$$\bullet \xrightarrow{\text{true}} \Omega,$$

called the *sub-object classifier* because it classifies subgraphs in a manner analogous to the way the inclusion

$$\{1\} \to \{0, 1\}$$

in sets classifies subsets via the characteristic function of a subset.

With this sub-object classifier, with the constructions defined earlier, and with the construction of limits (a more general notion than product), $\mathcal{DG}$ becomes what is called a *topos.* The name is due to Grothendieck, and was envisaged by him as a replacement of the notion of topological space by the category of sheaves on that space.

For our purposes, the idea is to carry out arguments in the topos $\mathcal{DG}$ as if it were the category of sets and functions, but never to use the law of the excluded middle. The reason for this is that the lattice of subgraphs of a given graph is not Boolean, since for example the complement $A \setminus (A \setminus B)$ of the complement $A \setminus B$ of a subgraph B is usually not the original subgraph B. Thus this theory is intuitionistic, an approach which is seen in this context as a practical mathematical tool for dealing with situations where the notion of membership is not the primary aspect. In the case of graphs, the "elements" have to be the vertices, but these capture only a small part of the structure. For more information on this approach in graph theory, see [12], while for the general body of theory, see the book by Mac Lane and Moerdijk, [14].

The exponential law in $\mathcal{DG}$ has a number of consequences. One is that there is a composition morphism

$$\mathrm{DIGRPH}(B, C) \times \mathrm{DIGRPH}(A, B) \to \mathrm{DIGRPH}(A, C)$$

which is associative and with identity. Hence

$$\mathrm{END}(A) = \mathrm{DIGRPH}(A, A)$$

has the structure of both a monoid and a graph. In the category of sets, monoids have maximal subgroups. This is also true in a topos. In the case of graphs, the maximal subgroup of the monoid $\mathrm{END}(A)$ is called

$$\mathrm{AUT}(A).$$

It is a group which is also a graph, or a graph which is also a group. Its set of vertices is the group $\mathrm{Aut}(A)$ of automorphisms of A.

Example: Let Δ_n be the complete graph on n vertices, and let D_n be the discrete graph on n vertices. These graphs have the same automorphism group, S_n, the symmetric group on n letters. But $\mathrm{AUT}(\Delta_n)$ is the complete graph, while $\mathrm{AUT}(D_n)$ is discrete.

In the graph $\mathrm{AUT}(A)$, the automorphisms adjacent to the identity form a normal subgroup of $\mathrm{Aut}(A)$: these automorphisms are called the *inner automorphisms* of A.

This raises the problem of describing the inner automorphisms of a graph in terms of internal properties of the graph. The solution is given by Shrimpton, [19, 20], in terms of the notion of inner subgraph.

A subgraph B of a graph A is *inner* if it is maximal with respect to the following properties:

1. complete (i.e. the sets $B(x, y)$ have the same cardinality for all $x, y \in B_V$);
2. full (i.e. $B(x, y) = A(x, y)$ for all $x, y \in B_V$);
3. any automorphism of B extends to an automorphism of A which is the identity on the complement $A \setminus B$ of B in A.

Claim [19, 20]. *Any vertex belongs to a unique inner subgraph.*

Theorem [19, 20]. *An automorphism of a graph is inner if and only if it restricts to an automorphism of each inner subgraph.*

This suggests that the inner subgraphs are a kind of *atom* of symmetry of the graph.

The consideration of group-graphs leads to another new notion, the *centre* of a graph.

A *group-graph* is defined by Ribenboim in [17] to consist of groups G_E and G_V and morphisms $s, t : G_E \to G_V$, $\epsilon : G_V \to G_E$ such that $s\epsilon = t\epsilon = 1$. This concept has occurred elsewhere, for example as a 1-truncated simplicial group [13], and as part of the structure of a group-groupoid, as in Brown and Spencer, [4], where this is called a $\mathcal{G}$-groupoid. Loday in [13] found it natural to consider the subgroup $[\ker s, \ker t]$ and to say that the group-graph is a *cat*^{1}*-group* if this subgroup is trivial. If it is not trivial, we can form the quotient $\gamma G_E = G_E/[\ker s, \ker t]$ with the induced morphisms to G_V giving γG the structure of cat^1-group. We call

$$(\ker s) \cap (\ker t) \subseteq \gamma G_E$$

the *second homotopy group* of G and write it $\pi_2 G$.

In particular, if $G = \mathrm{AUT}(A)$, then $\pi_2(G)$ is called the *centre* $Z(A)$ of the graph A. The centre is always an abelian group, and in fact is a module over $\mathrm{Out}(A) = \mathrm{Aut}(A)/\mathrm{Inn}(A)$. The aim is to describe this centre, in the case that A is finite, in terms of the structure of A.

To this end, we introduce in the following proposition an equivalence relation on the edges of a graph.

Proposition [19, 20]. *If A is a graph, then there is an equivalence relation on the edges of A given by x is equivalent to y if and only if there are inner subgraphs I and J of A such that sx, sy lie in I and tx, ty lie in J.*

Theorem [19, 20]. *If A is finite, then the centre $Z(A)$ of A is a direct sum of copies of the cyclic group of order 2, the number of copies being the number of equivalence classes of edges of A which contain multiple edge sets.*

Conclusion.

We have now shown that the study of categorical aspects of graph theory can lead to new problems, questions, and insights, and that it gives an interesting example of the relative viewpoint on set theory as exemplified by topos theory. Further work that might be done is in the area of "actions" of group-graphs, as well as the investigation of higher dimensional versions of $\mathrm{AUT}(A)$, such as the notion of automorphisms of ordered simplicial complexes.

The category $\mathcal{DG}$ is an example of what is called a *presheaf category*, namely a functor category $\hat{\mathcal{C}} = (\mathcal{SETS})^{\mathcal{C}^{op}}$ for a small category $\mathcal{C}$. The specific constructions outlined above for directed graphs are special cases of the fact that any such presheaf category $\hat{\mathcal{C}}$ is a topos (see Mac Lane and Moerdijk, [14]). These and other topoi yield a range of other "environments" for mathematics, or for a particular study, while types of categories other than topoi may be more suitable for other aims.

The notion of an internal group object in a category or in a topos is quite old. Thus the surprise is that the detailed study of this particular example, and the elucidation of the properties of the automorphism group-graph, had not been considered earlier. This suggests that there may be considerable mileage to be had from applying in new ways and in new places these and other concepts and methods of category theory.

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