## References


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## Book Review

### Around Burnside

Translated from the Russian by James Wiegold

A. I. Kostrikin


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**Reviewed by Seán Tobin**

This is in many ways an extraordinary book, and it is likely to become a collector's item even for those who are not primarily concerned with work on the Burnside problems. Commencing with a curious title, and ending with an eccentric index, it is full of stylistic quirks while also packed with information on the work by Kostrikin and his school which has led recently to the total solution by Efim Zelmanov, [5],[6], of the Restricted Burnside Problem. In his translator's preface Professor Wiegold remarks '...this book has been an interesting challenge to the translator. It is most unusual, in a text of this type, in that the style is racy with many literary allusions and witticisms: not the easiest to translate, but a source of inspiration to continue through material that could daunt by its computational complexity'.

In his preface to this English edition of his book, Professor Kostrikin says, 'Problems of Burnside type have become singularly popular in Moscow and Novosibirsk ... and it is of course advisable for [Russian algebraists] to share their knowledge with Western colleagues'. Our thanks are certainly due to both author and translator for their efforts to make this knowledge available to us, in particular through the medium of this book. Two other recent books worth mentioning in this context are Vaughan-Lee's [4] on the Restricted Problem and Zelmanov's [7] on problems of Burnside type.

Kostrikin's book commences with a brief but careful account of the history of developments in the study of the group problems posed by Burnside in his famous paper of 1902, [1]. These are (in our terminology):

(a) Is a finitely generated periodic group necessarily finite?

(b) Must such a group be finite if the periods are bounded?

(c) If finite, what is the order (in terms of obvious parameters)?

Question (b)–with its corollary (c)–has come to be known as THE Burnside Problem. Having stated (a), Burnside ignored it in his paper, possibly wisely since in 1964 E.S.Golod showed that the answer is "No". Burnside and, later, others obtained positive answers to (b) in a few special cases. If we let $B(n,k)$ represent the quotient of a free group of rank $n$ over the subgroup generated by the $k$–th powers of all its elements, then by 1958 it was known that for all $n$, the group $B(n,k)$ is finite when $k = 2$, 3, 4, or 6; and the precise order was known for $k = 2$, 3, or 6 (for $k = 4$, it is still not known). These are the only positive results; but in 1968 Novikov and Adian showed that $B(2,k)$ is infinite for large ($> 4380$) odd values of $k$. Since then this bound has been lowered, and other negative results have been obtained, but they do not concern us here.

Already in the nineteen-forties and fifties, a related problem was gaining attention. Even if $B(n,k)$ is infinite for a certain pair $n,k$, it could well happen that it has a largest finite quotient group $R(n,k)$ such that every finite homomorphic image of $B(n,k)$ is a homomorphic image of $R(n,k)$–equivalently, $B(n,k)$ might have just a finite number of non-isomorphic finite factor-groups. The question of the existence of $R(n,k)$ was called by Magnus, [3], the RESTRICTED Burnside Problem, and it is a problem to which Kostrikin has addressed himself for many years, showing what has been described as 'a heroic capacity for computation'. Incidentally the reviewer attended a course of lectures on Kostrikin's work given by James Wiegold in Canberra during the (Australian) winter of 1965, in which he displayed a similar heroic capacity–which must have stood him in good stead when coping with Kostrikin's book.



Kostrikin published his first result on this problem, a proof that $R(2,5)$ exists, in 1955 and then in 1958 he succeeded in proving that $R(n,p)$ exists for all $n$ and all primes $p$. The final step, replacing $p$ by any power of $p$, eluded him despite intense efforts. His book was written to chart the course of this passionate pilgrimage, to point out pitfalls and possible improvements, and to act as a kind of *vade mecum* for others engaged in the same task. And indeed it has done so splendidly, for his co-worker E. Zelmanov, building on the ideas developed by and with Kostrikin, has proved the existence of $R(n,k)$ whenever $k$ is a prime power. Taken in conjunction with the Reduction Theorem proved by P.Hall and G.Higman, [2], in 1956 (modulo certain conjectures about finite simple groups, which have since been confirmed by the Classification Theorem) this has the consequence that $R(n,k)$ exists for all $n$ and all $k$. Ironically, this result was obtained just as the book under review was being published, and could only be referred to in the translator's preface.

The book is concerned largely with Lie algebras which satisfy certain Engel identities. We will write $[x,y]$ for the product of $x$ and $y$ in a Lie algebra $L$, and $[x,n.y]$ for the left-normed product $[x,y,...,y]$, where $y$ appears $n$ times. An Engel algebra, or more specifically an $E(n)$–algebra, is a Lie algebra in which the law $[x,n.y] = 0$ is satisfied for some fixed integer $n$. This is linear in $x$ and $y$ when $n = 1$, but more generally an effort is made to extract multilinear laws (involving terms $[x,u,v,...,w]$) as a consequence, in order to exploit linearity properties of the algebra. The plan of attack on the Restricted problem for prime exponent $p$ is as follows: consider a finitely generated group $G$ with exponent $p$, and construct an associated Lie ring $L(G)$ using for example the descending central series of $G$. Then $L(G)$ has characteristic $p$ and may be regarded as an algebra over the integers modulo $p$. By a theorem of Magnus, this algebra satisfies the Engel identity $[x,(p-1).y] = 0$. Since $G$ is nilpotent, and therefore a finite $p$-group, if and only if $L(G)$ is nilpotent, the problem now becomes this: show that a finitely generated $E(p-1)$–algebra over the field of $p$ elements is nilpotent, (and calculate its class and order). An explanation of this technique is given in Chapter 7, which surveys

'...the shape of the linear methods in finite group theory that have a bearing on our principal theme...'.

That this is the last chapter rather than the first illustrates the curiously non-linear layout of the book, which skips forward and backwards, looking at the same result reached by different methods, occasionally pointing out cul-de-sacs which had been explored to no avail. Before commenting on the style, however, let us look at the contents.

The first four chapters are devoted to the work on $R(n,p)$ and are intended '...to make available...a text that is easily checked and does not pretend to the deceptive brevity of the original paper'. This is just one of the disarming comments scattered through the text. The author certainly takes pains to get his ideas across, but the checking is not a task for the faint-hearted.

Some shorter alternative proofs are discussed in Chapter 5, while Chapter 6 gives a number of results on nilpotency of other Engel algebras, in particular Razmyslov's theorem on the existence of non-solvable $E(p-2)$–algebras of characteristic $p$ and Zelmanov's theorem on the nilpotency of Engel algebras of characteristic zero. Appendix I gives an argument due to Zelmanov furnishing a recursive bound $f(s,t)$ for the nilpotency class of an $s$-generator $E(t)$–algebra of characteristic $p$, where $p > t$. Finally, Appendix II gives a brief biography of Burnside.

The work of Kostrikin and Zelmanov on the Restricted problem depends on studies of 'sandwiches'. An element $s$ in a Lie algebra $L$ is called a 'thin sandwich' if $[x,s,s] = [x,s,y,s] = 0$ for all $x$ and $y$ in $L$; it is 'thick' if also $[x,s,y,s,z] = 0$ for all $x$, $y$ and $z$ in $L$. The lengthy argument for exponent $p$ runs as follows: (i) if $L$ is an $E(n)$–algebra over a field $F$ of characteristic $p$, where $n < p$, then $L$ contains a (non-zero) thin sandwich; (ii) if $L$ has a thin sandwich, it must also contain a thick sandwich; (iii) if $L$ has a thick sandwich it must also contain a non-trivial abelian ideal; (iv) now let $R$ be the locally nilpotent radical of $L$, so that $L/R$ is also an $E(n)$–algebra over $F$. This must be zero, otherwise (iii) is contradicted; thus $L = R$ and we have done.

Now for a few words on the style of this remarkable book. Frankly, this reviewer found the heavy-handed humour and the forced jocular remarks quite tiresome (although, to be fair, the translation of humour is rarely a happy event). On the other hand, the book abounds with interesting and illuminating comments on each chapter; and one must admire an author who can laugh at himself, as for example on page 158: 'Theorem 1.1 refutes a strange conjecture to be found in the survey article [142]'–the reference here is to an article by Kostrikin himself published in 1974. He also pokes a little gentle fun at himself in the Epilogue, where he says: 'The method of sandwiches lies at the heart of this book. Unfortunately, the book itself is written in the form of a sandwich, with layers of similar material in different sections and even in different chapters...Perhaps it would be worthwhile to produce a slight rearrangement...'



Incidentally, the reference here to 'Theorem 1.1' is in fact a misprint, and the text has a small number of obvious misprints. As for the index, it was presumably compiled by a computer and unseen by human eye–how else to account for entries such as 'Bugaboo threat to Lemma 3.1'? There is a fine list of 288 references, and a separate Erratum pamphlet which reproduces this list but with the very useful addition of references to reviews in the Zentralblatt and to English versions (where available) of the many papers in Russian.

To sum up: this book sets out to explain clearly the great contributions of Kostrikin and his Russian collaborators, and to make them available to other workers in the field. It is unusual among books at this level in that the author has stamped his personality upon it, and indeed he comes across as a warm and likeable man. The book succeeds in its purpose, and anyone who wishes to understand the work which has led ultimately to the solution of the Restricted Burnside Problem should acquire a copy of 'Around Burnside'.

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