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EXPLICIT RELATIONSHIPS BETWEEN ROUTH-HURWITZ AND SCHUR-COHN TYPES OF STABILITY

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Abstract: Given two linear systems of differential equations with real or complex coefficients, and of the same arbitrary dimension. Suppose both systems are stable, one in the Routh-Hurwitz sense and the other in the Schur-Cohn sense. We directly express the coefficients of each system in terms of those of the other. These relationships, being explicit, make it possible to convey any stability criterion of either of the two types to the other.

1. Introduction

The concept of stability in differential equations has been defined in many different ways. Among these various definitions are the well-known Routh-Hurwitz and Schur-Cohn types of stability. Given a linear system of differential equations, the classical Routh-Hurwitz problem is that of obtaining necessary and sufficient conditions for all eigenvalues of the system to lie in the left half of the complex plane. The Schur-Cohn problem is that of establishing necessary and sufficient conditions for all eigenvalues to lie within the unit circle. Solutions to these problems have been the subject of intensive research over the last few years [2], [3], [9], [12] and [14].

It is often noticed in the literature that some interesting results about stability, in the Hurwitz sense for example, triggers an

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interest in the corresponding problem in the Schur sense or vice versa. See for example the introduction in [11] and example 4.2 in [8].

Recently some notable attempts have been made to give a common interpretation to the algorithms for testing the stability of continuous-time (Routh-Hurwitz) and discrete-time (Schur-Cohn) systems of differential equations [6], [10] and [13]. An excellent survey is given in [1] for continuous-time and in [5] for discrete-time systems.

The search for a unified approach to the study of root distribution of complex polynomials with respect to the half plane for continuous systems, and with respect to the unit disc for discrete systems, has been advocated by many eminent researchers in the field, see for example [4]. An interesting way of looking at the two problems of stability is to relate them to each other through the bilinear transformation $z = (1+w)/(1-w)$, which is equivalent to $w = (z-1)/(z+1)$. This is a one-to-one mapping between the left half of the complex z-plane, i.e. the region $\Re(z) < 0$, and the unit disc $|w| < 1$ in the complex w-plane. For a general discussion of bilinear transformations in this context, see [7]. Such connections prove useful in gaining new insights into the nature of the different algorithms.

This paper is a further thrust towards a firm unified approach to the relevant testing procedures for both continuous-time and discrete-time systems. In section 2 we give some notations, and the main results of the paper are given in section 3.

2. Notations

If A is an $n \times n$ real or complex matrix, and $X(t)$ is an n-dimensional column vector function of t, let $X' = AX$ be a system of differential equations, with eigenvalues $z_1, z_2, \dots, z_n$. Then the characteristic polynomial of this system may be written in both factored and expanded forms as follows: $f(z) = \prod_{j=1}^{n}(z-z_j) = \sum_{j=0}^{n} a_j z^{n-j}$ where $a_0 = 1$ by definition. Similarly if $X' = BX$ is a system with eigenvalues $w_1, w_2, \dots, w_n$ (where w_j is related to z_j of the previous system by $w_j = (z_j-1)/(z_j+1)$), then its characteristic polynomial is $g(w) = \prod_{j=1}^{n}(w - w_j) = \sum_{j=0}^{n} b_j w^{n-j}$, with $b_0 = 1$.

The intimate relationship between Routh-Hurwitz and Schur-Cohn types of stability could best be expressed by the following:

Theorem 2.1. *The system $X' = AX$ is Schur-Cohn stable if and only if $X' = BX$ is Routh-Hurwitz stable.*

Proof: Suppose $z = \dfrac{1+w}{1-w}$, or equivalently $w = \dfrac{z-1}{z+1}$, where z and w are complex numbers. The following relationships can easily be established

$$w + \overline{w} = \frac{2(z\overline{z}-1)}{|z+1|^2} \quad \text{and} \quad z\overline{z} - 1 = \frac{2(w+\overline{w})}{|1-w|^2},$$

from either of which it follows that $|z| < 1$ if and only if $\Re w < 0$.

3. Main Results

If $X' = AX$ and $X' = BX$ are the two systems defined in section 2 with their corresponding characteristic polynomials, then

Theorem 3.1.

$$b_p = \frac{\displaystyle\sum_{t=0}^{n} \sum_{q=\max(t-p,0)}^{\min(n-p,1)} (-1)^q \binom{t}{q} \binom{n-t}{n-p-q} a_t}{\displaystyle\sum_{t=0}^{n} (-1)^t a_t}$$

for all $p = 1, \dots, n$.

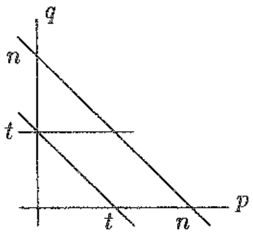
Proof: Consider $f(z) = \sum_{t=0}^{n} a_t z^{n-t}$ with zeros $z_1, \dots, z_n$ and $z_j = (1+w_j)/(1-w_j)$ for $j = 1, \dots, n$. Hence $w_1, \dots, w_n$ are the zeros of

$$f\left(\frac{1+w}{1-w}\right) = \sum_{t=0}^{n} a_t \left(\frac{1+w}{1-w}\right)^{n-t}$$
$$= \frac{1}{(1-w)^n} \sum_{t=0}^{n} a_t (1-w)^t (1+w)^{n-t}.$$

Therefore $w_1, \ldots, w_n$ are the zeros of the polynomial

$$\begin{aligned} h(w) &= \sum_{t=0}^{n} a_t (1-w)^t (1+w)^{n-t} \\ &= \sum_{t=0}^{n} a_t \sum_{r=0}^{t} (-1)^r \binom{t}{r} w^r \sum_{s=0}^{n-t} \binom{n-t}{s} w^s \\ &= \sum_{t=0}^{n} \sum_{r=0}^{t} \sum_{s=0}^{n-t} (-1)^r \binom{t}{r} \binom{n-t}{s} a_t w^{r+s}. \end{aligned}$$

We make the following transformation from the (r, s) plane to the (p, q) plane:



$$p = n - r - s, \quad q = r.$$

Then the rectangle in the (r, s) plane with sides $r = 0$, $r = t$, $s = 0$, $s = n-t$ is transformed into the parallelogram in the (p, q) plane with sides $q = 0$, $q = t$, $q = n - p$, $q = t - p$. Hence

$$h(w) = \sum_{t=0}^{n} \sum_{p=0}^{n} \sum_{q=\max(t-p,0)}^{\min(n-p,t)} (-1)^q \binom{t}{q} \binom{n-t}{n-p-q} a_t w^{n-p}.$$

Write $h(w) = \sum_{p=0}^{n} N_p w^{n-p}$, where

$$N_p = \sum_{t=0}^{n} \sum_{q=\max(t-p,0)}^{\min(n-p,t)} (-1)^q \binom{t}{q} \binom{n-t}{n-p-q} a_t.$$

In the polynomial $h(w)$, the leading coefficient is

$$N_0 = \sum_{t=0}^{n} (-1)^t \binom{t}{t} \binom{n-t}{n-t} a_t = \sum_{t=0}^{n} (-1)^t a_t.$$

Now the two polynomials

$$\frac{1}{N_0} h(w) = \sum_{p=0}^{n} \frac{N_p}{N_0} w^{n-p} \quad \text{and} \quad g(w) = \sum_{p=0}^{n} b_p w^{n-p}$$

being both monic and having the same set of zeros are identical, leading automatically to the desired conclusion.

The converse of Theorem 3.1 states the following

Theorem 3.2.

$$a_p = \frac{\displaystyle\sum_{t=0}^{n} \sum_{q=\max(t-p,0)}^{\min(n-p,t)} (-1)^{p+q-1} \binom{t}{q} \binom{n-t}{n-p-q} b_t}{\displaystyle\sum_{t=0}^{n} b_t}$$

for all $p = 1, \ldots, n$.

The proof of this theorem is omitted as it is similar to that of Theorem 3.1.

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UNDERGRADUATE PROJECTS

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Abstract: The rationale for, operation of and assessment of undergraduate projects at the University of Ulster are discussed. Specimen project titles are provided.

Introduction

A debate on undergraduate mathematics teaching in Ireland has recently been started through this Bulletin [1], [2]. It has been continued at a conference organized by the Sub-Commission for Mathematical Instruction of the Royal Irish Academy and held in Dublin in September 1991 (RIA-91), [3].

O'Reilly [1] questioned how we teach mathematics at tertiary level, leaving readers with many "focusing questions" and "questions for exploration". Dickenson et al. [2] described innovative methods of teaching, learning and assessment used at the University of Ulster, and went some way to answering O'Reilly's questions. Ted Hurley (UCG) continued the debate in his plenary lecture "Mathematics at Third Level" at RIA-91. In his lecture he pointed out that

(i) the number of honours graduates in mathematics from Irish Universities per capita is 3.5 times smaller than the number per capita from British Universities;

(ii) 45% of these Irish graduates entered further study compared to 14.5% of British graduates.

He concluded that this was an unsatisfactory state of affairs and made some suggestions for remedying the situation such as putting greater emphasis on the links between mathematics and computing.