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SOME QUESTIONS CONCERNING THE VALENCE OF ANALYTIC FUNCTIONS

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In this short note we discuss, and illustrate by means of some examples, certain questions concerning the *valence* of analytic functions of one complex variable, that is, the number of times such functions take their values. We present a theorem which asserts the existence of certain constants relating to the valence of analytic functions in the unit disc, and conclude the note by raising some questions regarding these constants for the reader.

We begin with a definition. Suppose a function f is analytic in a domain D in the complex plane. We say that f is *p-valent* in D, p a positive integer, if (*i*) f takes no value more than p times in D, and (*ii*) f takes at least one value exactly p times in D. If $p = 1$ we have, of course, a *univalent* (or one-to-one) function. The following result for univalent functions is elementary and known:
(1) *If f is analytic in the unit disc $U = \{z : |z| < 1\}$ and univalent in the annulus*

$$A(\delta) \equiv \{z : \delta < |z| < 1\},$$

where $0 < \delta < 1$, then f is univalent in the full disc U.
This result is an easy consequence of Darboux's theorem [1, p. 115]: If f is analytic on and inside a simple closed curve γ, and f takes no value more than once on γ, then f is univalent inside γ.

It is natural to attempt to generalize (1) and to ask whether there is an analogous result for p-valent functions when $p > 1$. (This question was first posed by A. W. Goodman in a seminar in Tampa many years ago and this author's interest in these problems dates — albeit discontinuously — from that occasion.) We note immediately that the direct analogue of (1), namely

(2) *f analytic in U and p-valent in* $A(\delta)$, $0<\delta<1$ *and* $p>1$
$\Rightarrow f$ *is p-valent in U*,

is false for every $p > 1$. We illustrate this here for the case $p = 2$ with an example which shows that, given any δ in $(0, 1)$, there exists a polynomial which is 2-valent in $A(\delta)$, but which is not 2-valent in U.

Example 1. *Let* $P_n(z) = z(z^2 - \alpha_n)$, *where* $\alpha_n = 1 - 1/4n^2$. *Then, for* $n \geq 2$, P_n *is 2-valent in the annulus* $A(\frac{1}{n})$ *and 3-valent in U.*

To understand this example — simple as it is — it is helpful to examine the image of the unit circle $C = \{z : |z| = 1\}$ under the mapping $w = P_n(z)$.

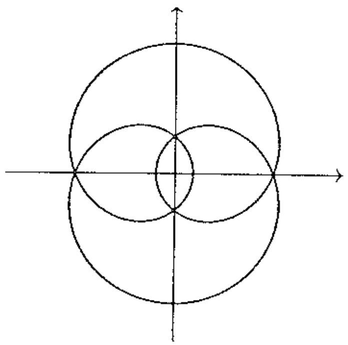

Fig. 1 $P_n(C)$

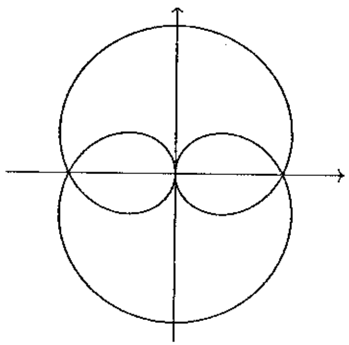

Fig. 2 $P(C)$

See Fig. 1. Now if β is any point inside the bounded component O_n of $E_n = \mathbf{C} \setminus P_n(C)$ that contains the origin, then

$$\frac{1}{2\pi}\Delta arg\{P_n(z) - \beta\} = 3,$$

where Δarg denotes the net change in the argument as z traverses C in the positive sense. Hence, by the *argument principle* [1, p. 104], every such value β is taken exactly three times in U by P_n. For similar reasons, every value in each of the other four bounded components of E_n is taken exactly once or twice only. The component O_n shrinks to the empty set as $n \to \infty$ (see Fig.2 for $P(C)$, where $P(z) = z^3 - z = \lim_{n\to\infty} P_n(z)$), and, as $P_n(0) = 0$, it is clear that, for each $n \geq 2$, there is a disc D_n centred at the origin with radius ε_n (decreasing to zero as $n \to \infty$) such that $P_n(D_n) \supset O_n$. But then P_n can take no value more than twice in $U \setminus D_n$ and (assuming that $U \setminus D_n$ contains the two non-zero zeros of P_n) is thus 2-valent in the annulus $A(\varepsilon_n)$. We leave it to the reader to *prove* that this is so with $\varepsilon_n = \frac{1}{n}$ for $n \geq 2$.

A function f satisfying the conditions in (2) is not necessarily p-valent in U, therefore, but it is the case (and easy to prove) that such a function is q-valent in U for *some* positive integer q. The value of q can be arbitrarily large, however. Indeed, as our next example shows, given $q \geq p \geq 2$, there exists an analytic function which is p-valent in $A(\delta)$, for some δ in $(0, 1)$, and q-valent in U.

Example 2. *Let p and q be integers with* $q \geq p \geq 2$ *and set* $F(z) = exp(q\pi z)$. *Then F is p-valent in the annulus*

$$\{z : \sqrt{1 - (\frac{4p-3}{4q})^2} < |z| < 1\}$$

(for instance), and q-valent in U.

This example, as the reader will readily verify, is an easy consequence of the standard periodicity property of the exponential function.

Example 2 leaves open the possibility that if f is any function satisfying the conditions in (2), and q is an integer greater than p, then f is at most q-valent in U, *provided* δ *is small enough.* This, finally, is indeed — with a qualification — essentially what our theorem asserts.

Theorem 3. *Suppose that p,q are integers with* $p \geq 2$ *and* $q \geq 2p$, *and that f is analytic in U. There exists a (largest) number*

$r^*(p,q)$ in $(0,1)$ such that if f is p-valent in an annulus $A(\delta)$, and $0 < \delta < r^*(p,q)$, then f is at most q-valent in U.

The author's proof of this result — which is based on a *normal family* [1, p. 213] argument, as a complex analyst reader might anticipate — is somewhat technical in detail and sheds no light on how the questions raised by the theorem might be answered, so we do not include it here. One question which arises is whether the theorem is true if we replace the condition '$q \geq 2p$' with '$q > p$', but a more fundamental question is :

What is the value of $r^*(p,q)$ for each permissible pair (p,q)?

We conclude by leaving these open questions, unclouded by any conjectures, for the reader.

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ON A QUESTION POSED BY GRAHAM HIGMAN

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Consider a function f of the non-negative integers given by the following rules:

$$f(3n) = 4n$$
$$f(3n+1) = 4n+1$$
$$f(3n+2) \text{ is undefined} \qquad (n = 0, 1, 2, 3, \ldots).$$

Since $f(0) = 0$ and $f(1) = 1$, the function may be repeatedly and indefinitely applied to 0 and 1; that is, for $z = 0$ and 1, $f^k(z)$ is defined for all $k > 0$.

Question: Is there any integer $z > 2$ such that $f^k(z)$ is defined for all $k > 0$?

This function was introduced by Professor Graham Higman [1] during a lecture on explicit embeddings of finitely presented groups. He posed the question and he conjectured that the answer was "No". To be precise, he declared "No" to be his "first best guess".

In this paper, we will not prove Higman's conjecture but we will produce a good deal of evidence in its favour. Neither will we discuss the group theoretic context in which the question was raised. Instead we present an exploration of the problem as an example of computer-aided mathematics suitable for secondary school and college level students.

We use elementary programs in BASIC to obtain data on the function and we use this data in further development of the problem, leading to more efficient programming. Our suggestion is that students' knowledge and understanding of mathematics is