Abstract of Doctoral Thesis

MINIMALITY VIA ORDER AND TOPOLOGY

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The family of all topologies definable for an infinite set X is a complete, atomic and complemented lattice (under inclusion) and is often denoted by $LT(X)$. Given a topological invariant P, a member $\mathcal{T}$ of $LT(X)$ is said to be minimal (maximal) P if and only if $\mathcal{T}$ possesses property P but no weaker (stronger) member of $LT(X)$ possesses property P.

The underlying theme of this work, then, is an investigation of minimality with respect to certain topological properties, including that of sobriety. (A space has the latter property precisely when every non-empty closed irreducible subset is a unique point-closure.) Motivation for such an investigation is provided by realizing that it is in seeking to identify those members of $LT(X)$ which minimally satisfy an invariant property that we are, in a very real sense, examining the topological essence of the invariant. In the recent past, questions of this nature have been considered by Andima and Thron [1], Larson [4], McCartan [5] and Johnston and McCartan [2,3].

The thesis comprises three main sections, the first two of which are devoted to a discussion of minimality using a purely topological approach. In particular, we consider sobriety, an invariant which has seen striking application in such seemingly diverse areas as the theory of continuous lattices and theoretical computer science (notably domain theory). Indeed, recent developments in these fields would justify the claim that life without

Hausdorff is not only possible but that it is imperative. Research efforts in domain theory have been significantly advanced by the recognition of the limitations imposed by a traditional insistence on Hausdorff.

The final section represents a shift in focus where we give an order-theoretic interpretation of the previously established minimality structures. The order referred to here is the natural partial order induced on X by any T_0-member of $LT(X)$, thus:

$$x \leq y \text{ if and only if } x \in \overline{\{y\}} \qquad x,\, y \in X.$$

Given a topological space then, we may invoke the inherent partial order and rewrite point-closures and point kernels order-theoretically as:

$$\overline{\{y\}} = \{z \in X : z \leq y\}$$
$$\overset{\frown}{\{y\}} = \{z \in X : z \geq y\}, \qquad y \in X.$$

Thus, we have at our disposal a partial order which may be readily exploited and indeed lends a welcome visual aspect to the discussion. In particular, the structure of invariants expressed solely in terms of point-closures is reflected in the nature of the induced partial order. Thus they may be interpreted as properties of partial orders, which offers new insight into the problem of minimality. This is the tack taken by Andima and Thron in [1] where they consider certain 'order-induced' topological properties; that is, a topological property P which has an associated order property K such that a topology has property P if and only if its induced order has property K. Thus in searching for minimality with respect to such a property, we are now concerned with establishing the appropriate K-order structure.

Of course, many invariants are not of this type but nevertheless, an awareness of the effect of the invariant on the behaviour of the underlying partial order provides valuable insight into the methods employed to establish minimality.

An important aspect of this work is the development of certain techniques to solve many of the minimality problems under consideration. We illustrate their potential in characterizing and, where possible, identifying certain minimal structures. Further, while these methods are introduced in a purely topological setting, we show that they have a strong order-theoretic appeal. Their topological significance has a direct order-theoretic translation when we regard the space as a partially-ordered set.

References

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Abstract of Doctoral Thesis

DIMENSIONS OF COMMUTATIVE MATRIX ALGEBRAS

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Let F be a field and let $M_n(F)$ be the algebra of $n \times n$ matrices over F. Let $A, B \in M_n(F)$ with $AB = BA$ and let $\mathcal{A}$ be the algebra generated by A and B over F. A theorem of Gerstenhaber [Ann. Math. 73: 324–348 (1961)] states that the dimension of $\mathcal{A}$ is at most n. Gerstenhaber's proof uses the methods of algebraic geometry. In Chapter I of this thesis, we obtain a purely matrix–theoretic proof of the result, constructing in the process a basis for the algebra, $\mathcal{A}$. We also examine when equality occurs. The case where F is algebraically closed and $\mathcal{A}$ is indecomposable (under similarity) holds the key to the general situation. In this case, we obtain a Cayley–Hamilton–like theorem expressing B^k as a polynomial in $I, B, \ldots, B^{k-1}$ with coefficients in $F[A]$, where k denotes the number of blocks in the Jordan form of A. If all Jordan blocks of A have the same size, we say $\mathcal{A}$ is homogeneous. In this case we obtain a nonderogatory–like condition on B which is equivalent to $\dim_F \mathcal{A} = n$. We also show that in this case, $\dim_F \mathcal{A} = n$ is equivalent to the maximality of $\mathcal{A}$ as a commutative subalgebra of $M_n(F)$.

In Chapter II we examine the dimensions of three–generated commutative subalgebras of $M_n(F)$. Let A, B and $C \in M_n(F)$ be pairwise commutative, and let $\mathcal{A}$ be the algebra generated by A, B and C over F. It is an open question whether or not the dimension of $\mathcal{A}$ is bounded above by n. Again, the case where