C*-DYNAMICAL SYSTEMS AND COVARIANCE ALGEBRAS

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The last two decades have seen extraordinary progress in the theory of operator algebras and an enormous increase in the range and power of its applications. In this paper we shall look at that part of the theory which deals with the interaction of C*-algebras and groups of their automorphisms. From the purely theoretical point of view, the motivation for studying this area is that it enables the construction of new interesting C*-algebras from old. Further motivation is provided by the sheer depth and elegance of the ideas of the theory, which involve a beautiful interplay of C*-algebras and harmonic analysis, and concern some of the deepest (and hardest) results of the theory of operator algebras. Historically, however, the main impetus to the development of the subject came from its applications in mathematical physics. For this reason we shall occasionally motivate a point by a brief reference to quantum physics.

§1. Simple and primitive C*-algebras.

We begin by reviewing some basic terminology. Let A be an algebra (all vector spaces and algebras are complex). An involution on A is a conjugate-linear map, $a \mapsto a^*$, such that $(ab)^* = b^*a^*$ and $a^{**} = a$ $(a, b \in A)$. A C*-algebra is an algebra endowed with an involution and a complete norm such that $\|ab\| \leq \|a\| \|b\|$ and $\|a^*a\| = \|a\|^2$ $(a, b \in A)$. Obviously the complex field $\mathbb{C}$ is a C*-algebra. Less trivially, if Ω is a locally compact Hausdorff space, then the set $C_0(\Omega)$ of all complex-valued continuous functions on Ω vanishing at infinity is a C*-algebra (the operations are defined pointwise and the norm is given by $\|f\|_\infty = \sup_{\omega \in \Omega} |f(\omega)|$). By

the Gelfand representation, every commutative C*-algebra is of this form, up to isomorphism.

If H is a Hilbert space, let $B(H)$ denote the set of (bounded linear) operators on H. This is a C*-algebra with the operator norm and the involution defined by the usual adjoint. If A is a norm-closed subalgebra of $B(H)$ such that $T^* \in A$ whenever $T \in A$, then it is a C*-algebra. The Gelfand-Naimark theorem asserts that every C*-algebra is of this form (up to isomorphism).

A fundamental technique used in analysing a C*-algebra A is to represent it on various Hilbert spaces. A representation of A is defined to be a pair (H, φ), where H is a Hilbert space and $\varphi : A \to B(H)$ is a *-homomorphism, that is, a linear map preserving multiplication and involution. We say that (H, φ) is nondegenerate if H is the closed linear span of all elements $\varphi(a)\eta$ $(a \in A, \eta \in H)$; and we say that (H, φ) is irreducible if the only closed vector spaces K of H such that $\varphi(a)K \subseteq K$ $(a \in A)$ are $K = 0$ and $K = H$.

There are two classes of C*-algebras that play the role of "building blocks" in the theory—the simple and the primitive C*-algebras (their description as building blocks has to be taken *cum grano salis*). A primitive C*-algebra is one which admits an irreducible representation (H, φ) with φ injective. For example, $B(H)$ is primitive, but $C_0(\Omega)$ is not, unless Ω is a single point (in which case $C_0(\Omega) = \mathbb{C}$). A C*-algebra A is simple if its only closed ideals are the trivial ones, 0 and A. Simple C*-algebras are primitive, but not conversely. For instance, $B(H)$ is simple only when H is finite-dimensional. The ideal of compact operators is always simple.

In general it is a non-trivial task to exhibit examples of simple and primitive C*-algebras. The covariance algebras that we introduce in the next section play a vital role in the construction of many such examples.

§2. C*-dynamical systems and covariance algebras.

An automorphism of a C*-algebra A is a bijective *-homomorphism from A onto itself. We denote by Aut A the group of automorphisms of A.

A C^*-dynamical system is a triple (A, G, α), where A is a C^*-algebra, G is a locally compact group, and the map $\alpha : G \to$ Aut A, $x \mapsto \alpha_x$, is a homomorphism that is continuous in the sense that the map $G \to A$, $x \mapsto \alpha_x(a)$, is continuous for each $a \in A$.

The terminology derives from the applications. In quantum physics the observables are non-commuting operators on a Hilbert space. In some models they "form" a C^*-algebra A (more precisely, they form its self-adjoint part $A_{sa} = \{a \in A \mid a^* = a\}$). Time evolution and spatial translation of the observables are then described by a C^*-dynamical system.

If A is abelian, we can write $A = C_0(\Omega)$. In this case, the analysis of (A, G, α) relates to ergodic theory, since we get a corresponding action of G on Ω by a group of homeomorphisms α_x^t, where the map $\alpha_x^t : \Omega \to \Omega$ is determined by the equation

$$(\alpha_x f)(\omega) = f(\alpha_{x^{-1}}^t(\omega)) \qquad (x \in G, \ \omega \in \Omega, f \in A).$$

When $G = \mathbf{R}$, $\mathbf{Z}$ or $\mathbf{T}$ (the circle group), the study of $(C_0(\Omega), G, \alpha)$ is in essence classical topological dynamics. The motivation to work with A non-abelian came from the quantum physicists, who have to deal with non-commutating observables.

A unitary representation of G is a pair (H, U), where H is a Hilbert space, the map

$$U : G \to B(H), \quad x \mapsto U_x,$$

is a homomorphism into the group of unitary operators on H, and U is continuous in the sense that for arbitrary $\eta, \eta' \in H$ the function

$$G \to \mathbf{C}, \quad x \mapsto \langle U_x \eta, \eta' \rangle,$$

is continuous.

The analogous object to a representation of a C^*-algebra A is a covariant representation of a C^*-dynamical system (A, G, α). This is a triple (H, φ, U), where (H, φ) is a representation of A, the pair (H, U) is a unitary representation of G, and φ, U interact via the condition

$$\varphi(\alpha_x(a)) = U_x \varphi(a) U_x^* \quad (a \in A, \ x \in G).$$

We can now introduce the covariance algebra of (A, G, α). The connection of this C^*-algebra with (A, G, α) is that there is a natural one-one correspondence between its non-degenerate representations and the covariant representations of (A, G, α), so that, at least to some extent, the theory of covariant representations is reduced to that of ordinary representations.

Let m and Δ denote the left Haar measure and the modular function of G respectively. Denote by $K(G, A)$ the vector space of continuous maps from G to A having compact support. We endow $K(G, A)$ with a (convolution-type) multiplication and an involution defined by

$$(f * g)(y) = \int f(x) \alpha_x(g(x^{-1}y)) \, dm(x)$$
$$f^*(x) = \Delta(x)^{-1} \alpha_x(f(x^{-1}))^*$$

for f, $g \in K(G, A)$ and $x, y \in G$.

By rather indirect means, one also equips $K(G, A)$ with a suitable norm making it almost a C^*-algebra—the only requirement that is not satisfied is completeness. This defect is remedied simply by completing $K(G, A)$ and extending its operations by continuity to get a C^*-algebra, denoted by $C^*(A, G, \alpha)$ or $A \times_\alpha G$ and called the covariance algebra of (A, G, α), or the crossed product of A with G (under the action α).

A primary motivation for this construction is that $C^*(A, G, \alpha)$ can be made simple or primitive by imposing suitable conditions on (A, G, α). Examples of simple and primitive C^*-algebras are important not only for theoretical reasons, but also for applications. The algebras occurring in physics are often of this type—as D. Kastler remarks, nature does not have ideals. In physics the algebra of quantum observables is frequently obtained from the commutative algebra of the classical observables by taking something like the crossed product with the group generated by a set of "conjugate" variables of the classical variables.

A particular case of the crossed product construction is of great importance in the theory of unitary representations of locally compact groups. If G is one of these groups, we get a C^*-

dynamical system $(\mathbf{C}, G, \alpha)$ by letting G act trivially on $\mathbf{C}$. The covariance C*-algebra $\mathbf{C} \times_\alpha G$ is denoted by $C^*(G)$ and called the group C*-algebra of G. The theory of the unitary representations of G then becomes a part of the representation theory of C*-algebras, since they correspond to the representations of $C^*(G)$ (for details, see [2]). If G is abelian, then $C^*(G) = C_0(\hat{G})$, where $\hat{G}$ is the Pontryagin dual group of G, but in the non-abelian case the analysis of $C^*(G)$ can be very difficult.

Another class of C*-algebras that arise from the crossed product construction is the class of the irrational rotation algebras. These have been extensively studied. One reason for their importance is that they are motivating examples for the non-commutative differential geometry being developed by the Fields medalist Alain Connes.

Let $A = C(\mathbf{T})$ and let $u : \mathbf{T} \to \mathbf{C}$ be the inclusion function (u generates A). If we fix an irrational number θ in [0,1], then there is a unique automorphism α_1 of A such that $\alpha_1(u) = e^{i2\pi\theta}u$. Setting $\alpha_n = \alpha_1^n$, we get a C*-dynamical system $(A, \mathbf{Z}, \alpha)$ whose covariance C*-algebra is denoted by A_θ and called an irrational rotation algebra. The action of $\mathbf{Z}$ on $\mathbf{T}$ corresponding to α on $C(\mathbf{T})$ is given by rotation through the irrational angle θ, hence the name. We shall return to these algebras in the next section.

§3. Ergodicty and simplicity.

Although the crossed product is the most powerful device for getting new C*-algebras, the process is very elusive and a great deal of effort has been required to give general conditions which imply it is simple or primitive. In this and the next section we discuss some of these conditions (there are others which are not suitable for inclusion here due to their complexity).

We shall make the following assumption:

In this and the next section, (A, G, α) is a C-dynamical system for which A is separable and G is countable, discrete and abelian.*

Moreover, in this section only, we further assume that A is abelian.

Thus, we may write $A = C_0(\Omega)$. If $\omega \in \Omega$, its orbit is the set of all points $\alpha_x^t(\omega)$ ($x \in G$). We say that α is ergodic if every orbit is dense in Ω. We write $f < g$ in A to mean $f(\omega) \leq g(\omega)$ for all $\omega \in \Omega$ and $f \neq g$. We define α to be free if for all non-zero elements $x \in G$ and all elements $f > 0$ in A, there exists an element $g > 0$ in A such that $g < f$ and $\alpha_x(g) \neq g$.

The following important result is due to E. Effros and F. Hahn.

Theorem [4]. *If (A, G, α) is as assumed above, and the action α is ergodic and free, then the crossed product $A \times_\alpha G$ is simple.*

Despite the considerable restrictions imposed, this is still a very useful result. We illustrate it by applying it to the C*-dynamical system $(C(\mathbf{T}), \mathbf{Z}, \alpha)$ associated to the irrational rotation algebra A_θ: As is well known, the only closed subgroups of $\mathbf{T}$ are the finite ones and $\mathbf{T}$ itself. The irrationality of θ implies that the set $\{e^{i2\pi n\theta} \mid n \in \mathbf{Z}\}$ is infinite, and therefore the closed subgroup generated by $e^{i2\pi\theta}$ is equal to $\mathbf{T}$. It follows that every orbit is dense in $\mathbf{T}$, that is, α is ergodic. If f is an element of $C(\mathbf{T})$ such that $\alpha_n(f) = f$ for some non-zero integer n, then $\alpha_{mn}(f) = f$ for all $m \in \mathbf{Z}$. Hence, $f(e^{i2\pi mn\theta}) = f(1)$, and therefore, by density of the set $\{e^{i2\pi mn\theta} \mid m \in \mathbf{Z}\}$ in $\mathbf{T}$, the function f is constant. This easily implies that α is free. Since all the conditions of the Effros-Hahn theorem hold, we conclude that A_θ is simple.

§4. The Olesen-Pedersen spectral theory.

A subset S of A is said to be G-invariant if $\alpha_x(S) = S$ ($x \in G$). If A is abelian, the ergodicity condition defined in the preceding section means that the only G-invariant closed ideals of A are the trivial ideals 0 and A. When A is not (necessarily) abelian, we use the term G-simple for this reformulated condition. We say that A is G-prime if every pair of non-zero G-invariant closed ideals of A have a non-zero intersection.

The Arveson spectrum $Sp(\alpha)$ of α is the set of all $\gamma \in \hat{G}$ such that there exists a sequence of unit vectors a_n in A for which

$$\lim_{n \to \infty} \|\alpha_x(a_n) - \gamma(x)a_n\| = 0 \quad (x \in G).$$

(The $\gamma(x)$ are "joint approximate eigenvalues" of α_x.) The appropriate spectral object for C^*-dynamical systems is not this spectrum, however, but rather another spectrum derived from it which we now describe. If B is a G-invariant C^*-subalgebra of A, we get a new C^*-dynamical system $(B, G, a|B)$ by restriction of α to B. The Connes spectrum of α is

$$\Gamma(\alpha) = \cap_B Sp(\alpha|B),$$

where B runs over all non-zero G-invariant hereditary C^*-subalgebras of A (B is hereditary if $BAB \subseteq B$). The computation of $\Gamma(\alpha)$ is helped by the fact that it is a closed subgroup of $\hat{G}$, but nevertheless its calculation is in general a non-trivial task.

The following result is due to Olesen and Pedersen.

Theorem [7] ,[8]. *If (A, G, α) satisfies the assumption in section 3 the following conditions are equivalent:*
(a) $A \times_\alpha G$ is primitive (respectively, simple);
(b) A is G-prime (respectively, G-simple) and $\Gamma(\alpha) = \hat{G}$.

This is a difficult result, involving a beautiful duality theory for C^*-dynamical systems due to Takesaki and Takai that is a sort of C^*-analogue of the Pontryagin duality theory for locally compact groups. We do not attempt a statement of what this duality involves, as it would require a disproportionate amount of detail.

§5. Crossed products by semigroups.
A question that is begged by the theory we have outlined above is what kind of results hold if we replace groups by semigroups. This situation has been analysed by a number of mathematicians in recent years. We shall briefly outline here some results of a theory developed by the author [5], [6]. Surprisingly (or perhaps not), the situation turns out to be radically different, but nevertheless we get new examples of primitive C^*-algebras and, indirectly, of simple C^*-algebras.

We redefine a C^*-dynamical system to be a triple (A, G, α), where A is a C*-algebra, G is a cancellative abelian semigroup

with zero, and the map $\alpha : G \to \text{Aut}\, A$ is a homomorphism. To avoid trivialities we assume that A and G are non-zero. The previous construction of the crossed product using $K(G, A)$ does not work in this setting, but this difficulty is surmounted by constructing $A \times_\alpha G$ as something like the solution to a universal mapping problem (when G is a group, our crossed product is the same as before). The details are omitted as they are technical.

For G arbitrary, we can get a C^*-dynamical system $(\mathbf{C}, G, \alpha)$ by letting G act trivially on $\mathbf{C}$; we then denote $\mathbf{C} \times_\alpha G$ by $C^*(G)$. Observe that $C^*(\mathbf{Z}) = C(\mathbf{T})$, which is not something new. However, $C^*(\mathbf{N})$ is a much more complicated and interesting C^*-algebra. It is called the Toeplitz C^*-algebra, as it is (isomorphic to) the C^*-algebra generated by all Toeplitz operators with continuous symbol on the unit circle $\mathbf{T}$. It plays an important role in K-theory, as indeed does the algebra $A \times_\alpha \mathbf{N}$, for any C^*-dynamical system $(A, \mathbf{Z}, \alpha)$ (this algebra is isomorphic to the generalised Toeplitz algebra of α as defined by Pimsner and Voiculescu in [10]). If G is an ordered group, that is, an abelian group endowed with a total order $\leq$ such that $x \leq y \Rightarrow x + z \leq y + z$, and if $G^+ = \{x \in G \mid 0 \leq x\}$, then $C^*(G^+)$ was shown to be primitive in [5]. A special case of these algebras was first studied by Douglas in [3], where he showed that for G a subgroup of $\mathbf{R}$ with the induced order, not only is $C^*(G^+)$ primitive, but in this case the commutator ideal (the closed ideal generated by all $ab - ba$) is simple.

Let (A, G, α) be a C^*-dynamical system and suppose that G is an ordered group. We get a new (non-classical) C^*-dynamical system (A, G^+, α) by restricting α to G^+. There is a canonical *-homomorphism from $A \times_\alpha G^+$ to $A \times_\alpha G$. We let $K(A, G, \alpha)$ denote its kernel.

The algebra $A \times_\alpha G^+$ is never simple, but we can still get new simple C*-algebras by indirect means from this construction, and it seems in some ways to be easier to get primitive C*-algebras using $A \times_\alpha G^+$.

The following theorem is the main result of [6].

Theorem. *If (A, G, α) is as above, then*

(a) *If A is primitive, so is $A \times_\alpha G^+$;*

(b) *If A is simple and G is a subgroup of $\mathbf{R}$ with the induced order, then $K(A, G, \alpha)$ is simple.*

A useful feature of this result is that one does not have to compute a Connes spectrum—this makes the hypothesis easy to verify.

Concluding remarks.

We have said nothing about the related theory of W*-dynamical systems. This involves the revolutionary Tomita-Takesaki theory and the deep results of Connes on factors. The reader wishing to learn about this vast subject can consult [9], or, for a quick survey of Tomita-Takesaki theory, Lance's preface to [1].

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