state to another is found. Instability is studied as a function of various parameters. For a certain perturbation behaviour similar to that predicted by Deininger (1981) was found.

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DERIVATIONS AND COMPLETELY BOUNDED MAPS ON C*-ALGEBRAS

A Survey

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The present paper summarises a series of lectures delivered at the Department of Mathematics of University College Cork in early spring 1990 which were supported by the ERASMUS programme. Aimed at the non-specialist, we intend to provide a general survey of the theory of completely bounded linear operators on C^*-algebras with a closer view of their relations to derivations. Most of the details we have omitted can be found in Paulsen's fine treatise [26], in fact the reader may use this paper as a guide to [26] under the particular aspect of applications to derivations on C^*-algebras. A more comprehensive state-of-the-art overview on completely bounded operators is given in the recent paper by Christensen and Sinclair [7], while Effros' address to the ICM 86 [11] emphasises the connections with cohomology theory of operator algebras.

Since the mid 1970's it emerged that the classes of completely bounded and completely positive operators are among the most important classes of (multi-)linear mappings on C^*-algebras, as they are intimately related to a number of structural properties, and several open questions can be phrased in terms of these operators. Here, we shall mainly concentrate on how the problem of innerness of derivations naturally leads to consider completely bounded maps. On the way we will also add some remarks on the role these operators play in the operator algebraic approach to quantum theory. Occasionally, proofs are outlined in order to illustrate the typical techniques.

1. Prerequisites on C^*-algebras

This section is of a preparatory nature; we will compile several facts from C^*-algebra theory that will be needed in the sequel.

Throughout H, K denote Hilbert spaces over the complex field $\mathbb{C}$ and $L(H,K)$ is the Banach space of all bounded linear operators from H into K endowed with the operator norm, $\|T\| = \sup\{\|T\xi\| \mid \xi \in H,\ \|\xi\| \leq 1\}$. The Banach algebra $L(H) = L(H,H)$ carries a natural involution, $T \mapsto T^*$ where T^* is the adjoint of T, and the algebraic and the metric structures are related by the important identity $\|T^*T\| = \|T\|^2$.

A *C^*-algebra* A is a closed *-subalgebra of $L(H)$. Due to the work of Gelfand and Naimark, which was completed by several other mathematicians, there is the following abstract characterisation of C^*-algebras: every Banach algebra with an involution * (i.e. an anti-multiplicative conjugate-linear bijection of order two) satisfying $\|x^*x\| = \|x\|^2$ for all x is (isometrically *-isomorphic to) a C^*-algebra. To have both pictures of a C^*-algebra is advantageous: often C^*-algebras arise without specification of an *a priori* Hilbert space, but to realise immediately a C^*-algebra given concretely as operators facilitates many arguments. If a C^*-algebra A is *non-unital*, i.e. does not contain a multiplicative identity 1, we can embed A as a closed ideal into the unital C^*-algebra $\dot{A} = \{a + \lambda 1_{L(H)} \mid a \in A, \lambda \in \mathbb{C}\}$ where $A \subseteq L(H)$.

Here are some of the basic examples of C^*-algebras:

$M_n = L(\mathbb{C}^n)$, the complex $n \times n$-matrices, provides the simplest non-commutative C^*-algebra (for $n > 1$) and will play an extraordinary role in the following;

$K(H)$, the compact operators on an infinite dimensional Hilbert space H, is a simple non-unital C^*-algebra;

$C(H) = L(H)/K(H)$, the Calkin algebra, closely related to Fredholm operators;

$C_0(X)$, the continuous complex-valued functions on a locally compact Hausdorff space X vanishing at infinity. This is the prototype of a commutative C^*-algebra, as, by the Gelfand-Naimark theorem from 1941, every commutative C^*-algebra A is isometrically *-isomorphic to $C_0(\hat{A})$, $\hat{A}$ being the set of all homomorphisms from A onto $\mathbb{C}$ with the weak* topology.

Of course, there are many other fundamental examples and a number of methods to obtain new C^*-algebras from given ones. The following is the most important for our purposes.

Definition 1.1. Let $A \subseteq L(H)$ be a C^*-algebra. For each $n \in \mathbb{N}$ the set $M_n(A)$ of all $n \times n$-matrices with entries from A is a *-subalgebra of $L(H^n)$ under the canonical operations and thus can be normed with the operator norm. From

$$\max_{1\leq i,j\leq n} \|a_{ij}\| \leq \|(a_{ij})\| \leq \sum_{i,j=1}^{n} \|a_{ij}\|$$

for all $(a_{ij}) \in M_n(A)$ we see that $M_n(A)$ is complete, hence a C^*-algebra on H^n. If we change the faithful representation of A we obtain an isometrically *-isomorphic matrix algebra over A, thus $M_n(A)$ can be considered as an abstract C^*-algebra, too.

Examples. $M_n(K(H)) = K(H^n)$, $M_n(C_0(X)) = C_0(X, M_n)$.

We can also view $M_n(A)$ as a tensor product. If $\{u_{ij} \mid 1 \leq i,j \leq n\}$ denotes the canonical matrix units in M_n, the mapping

$$(a_{ij}) \mapsto \sum_{i,j=1}^{n} u_{ij} \otimes a_{ij}, \qquad M_n(A) \to M_n \otimes A$$

is a *-isomorphism. Let α be any C^*-cross norm on $M_n \otimes A$ and $M_n \otimes_\alpha A$ be its completion. Since every *-isomorphism between C^*-algebras is an isometry it follows that $M_n(A)$ and $M_n \otimes A$ are isometrically *-isomorphic, in particular, $M_n \otimes A = M_n \otimes_\alpha A$. (This argument shows in addition that all C^*-cross norms on $M_n \otimes A$ coincide, i.e. M_n is *nuclear*, see Section 4.)

The process of iterating matrix algebras is simplified by the *canonical shuffle*. If $n, m \in \mathbb{N}$, then, as a consequence of associativity and commutativity of tensor products, we have that

$$\begin{aligned} M_m(M_n(A)) &= M_m \otimes (M_n \otimes A) \\ &= M_n \otimes (M_m \otimes A) \\ &= M_n(M_m(A)) \end{aligned}$$

which amounts to a permutation of the entries (cf. [26]).

Let A be a C^*-algebra. Decomposing $x \in A$ into its real and imaginary parts shows that A is the topological direct sum of A_{sa} and iA_{sa} where $A_{sa} = \{x \in A \mid x = x^*\}$ is the *real* Banach space of all self-adjoint elements in A. The latter becomes an ordered Banach space by putting

$$x \leq y \quad \text{if } y - x \in A_+ \qquad (x, y \in A_{sa})$$

where $A_+ = \{x \in A_{sa} |$ all spectral values of x are non-negative$\}$ is the proper closed generating cone of positive elements in A. By the Fukamiya-Kaplansky-Kelley-Vaught theorem, there is the following important intimate interrelation between the order and the algebraic structure: $A_+ = \{x^*x \mid x \in A\}$. If A is unital, the identity 1 also serves as an order unit and thus A_{sa} will be an *order unit space*. As a consequence, the unit ball A_1 of A can be described as $A_1 = \{x \in A \mid xx^* \leq 1\}$.

This last observation can be used to derive the following criterion for positivity of certain 2×2-matrices which will turn out to be crucial in the sequel.

Lemma 1.2. *Let a be an element in a unital C^*-algebra A. Then $\|a\| \leq 1$ if and only if $\left(\begin{smallmatrix} 1 & a \\ a^* & 1 \end{smallmatrix}\right) \geq 0$.*

Proof. If $\|a\| \leq 1$ then $1 - aa^* \geq 0$. Take $x \in A_{sa}$ such that

$x^2 = 1 - aa^*$. Then

$$\begin{aligned} 0 &\leq \begin{pmatrix} x & 0 \\ a^* & 1 \end{pmatrix}^* \begin{pmatrix} x & 0 \\ a^* & 1 \end{pmatrix} \\ &= \begin{pmatrix} x & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ a^* & 1 \end{pmatrix} \\ &= \begin{pmatrix} x^2 + aa^* & a \\ a^* & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix} \end{aligned}$$

proving the "only if"-part.

Conversely, if $\left(\begin{smallmatrix} 1 & a \\ a^* & 1 \end{smallmatrix}\right) = x^2$ is positive and $A \subseteq L(H)$, then, for all $\xi, \eta \in H$, we have

$$\begin{aligned} &|(a\xi \mid \eta)|^2 \\ &= \left|\left(\begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \xi \end{pmatrix} \,\middle|\, \begin{pmatrix} \eta \\ 0 \end{pmatrix}\right)\right|^2 = \left|\left(x\begin{pmatrix} 0 \\ \xi \end{pmatrix} \,\middle|\, x\begin{pmatrix} \eta \\ 0 \end{pmatrix}\right)\right|^2 \\ &\leq \left(x\begin{pmatrix} 0 \\ \xi \end{pmatrix} \,\middle|\, x\begin{pmatrix} 0 \\ \xi \end{pmatrix}\right) \left(x\begin{pmatrix} \eta \\ 0 \end{pmatrix} \,\middle|\, x\begin{pmatrix} \eta \\ 0 \end{pmatrix}\right) \\ &= \left(\begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \xi \end{pmatrix} \,\middle|\, \begin{pmatrix} 0 \\ \xi \end{pmatrix}\right) \left(\begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix} \begin{pmatrix} \eta \\ 0 \end{pmatrix} \,\middle|\, \begin{pmatrix} \eta \\ 0 \end{pmatrix}\right) \\ &= \|\xi\|^2 \|\eta\|^2 \end{aligned}$$

which implies that $\|a\| \leq 1$. □

Finally we introduce the order preserving mappings.

Definition 1.3. Let A and B be C^*-algebras. A linear map $\phi: A \to B$ is called *positive* if $\phi(A_+) \subseteq B_+$, *n-positive* if $\mathrm{id} \otimes \phi: M_n \otimes A \to M_n \otimes B$ is positive, and *completely positive* if ϕ is n-positive for all $n \in \mathbb{N}$. The convex cone of all completely positive maps from A into B will be denoted by $CP(A, B)$.

Under the identification $M_n \otimes A \to M_n(A)$, the mapping $\mathrm{id} \otimes \phi$ becomes

$$(\mathrm{id} \otimes \phi)(\sum_{i,j} u_{ij} \otimes a_{ij}) = \sum_{i,j} u_{ij} \otimes \phi(a_{ij}) \mapsto (\phi(a_{ij}))_{1 \le i,j \le n}.$$

The mapping $(a_{ij}) \mapsto (\phi(a_{ij}))$, $M_n(A) \to M_n(B)$ is denoted by ϕ_n.

Note that, by a simple application of the uniform boundedness principle, every positive linear map is automatically bounded. For some more grounding on completely positive maps and C^*-algebras we refer to [35].

2. Derivations and homomorphisms of C^*-algebras

In this section some of the basic properties of derivations of C^*-algebras as well as their relations to homomorphisms will be studied, and one of the fundamental problems on derivations is phrased.

A linear map $\delta: A \to A$ where A is a C^*-algebra is called a *derivation of* A if

$$\delta(ab) = a(\delta b) + (\delta a)\, b \qquad (a, b \in A).$$

A derivation is automatically continuous, by a result due to Sakai, and only non-commutative C^*-algebras allow non-zero derivations, which is an observation of I. Singer. For each $x \in A$ the derivation

$$\mathrm{ad}\, x: a \mapsto ax - xa$$

is described as an *inner* derivation, and all non-inner derivations are called *outer*. There is a temptation to concentrate on inner derivations since they are given concretely and therefore their properties, e.g. their spectra, can be described more easily. However, if for instance $A = K(H)$ and p is an infinite dimensional projection on H, then the restriction of $\mathrm{ad}\, p$ to $K(H)$ will be outer. It is therefore important to know under which conditions derivations become inner.

Here are two well known and important answers to this question (see [31]: every derivation of a W^*-algebra and every derivation of a simple unital C^*-algebra is inner. Recall that a W^*-algebra is a C^*-algebra which is isometrically isomorphic to the dual of another Banach space which yields an additional weak* topology and thus many nice structural properties. Some necessary and sufficient conditions for derivations to be inner can be found in [27].

Before we proceed to a more general question, let us see where derivations arise in the applications.

In the operator algebraic approach to quantum theory one uses the self-adjoint part A_{sa} of a (suitable) C^*-algebra A for the collection of all observables of a specific physical system, and the *state space* $S(A)$ (the set of all normalised positive linear functionals on A) for the set of all physical states of the system. In the more traditional theory, A was $L(H)$ and the states were identified with unit vectors in H (*vector states*). The dynamics of the system is then given by a continuous one-parameter group of unitary operators $u(t)$, $t \in \mathbb{R}$, on H: while time passes from $t = 0$ to $t = t_0$, the system evolves from the state φ into the state $u(t_0)\varphi$. Of special interest is the infinitesimal generator h of $\{u(t) \mid t \in \mathbb{R}\}$ which is a self-adjoint operator (it corresponds to the energy). The identity $u(t) = e^{ith}$ yields the *Schrödinger equation*

$$\frac{d}{dt}\, u(t)\, \varphi = i\, h\, u(t)\, \varphi \qquad (\varphi \in H).$$

In the *Heisenberg picture*, the dynamics is on the observables rather than on the states and thus given by the one-parameter group of *-automorphisms $x \mapsto u(t)^* x\, u(t)$, $x \in L(H)$, $t \in \mathbb{R}$, whose generator is the inner derivation $x \mapsto i(xh - hx)$.

As it emerged that the algebra $L(H)$ is not always appropriate for the physical model one had to choose more general C^*-algebras. In many cases there is no *a priori* specified Hilbert space, whence the use of the Heisenberg picture and a one-parameter group of *-automorphisms $U(t)$ on A is more convenient; the generator of $\{U(t) \mid t \in \mathbb{R}\}$ will again be a derivation δ. Of course, the *Schrödinger picture* still exists and is equivalent: the adjoint of $U(t)$ will map $S(A)$ onto $S(A)$. In fact, since $U(t)$ is *-preserving, δ is a *-derivation, i.e. $\delta A_{sa} \subseteq A_{sa}$. (We are somewhat sloppy about the domain of definition of δ which depends on the continuity of $t \mapsto U(t)$. To be precise, we have to assume uniform continuity throughout.) Conversely, given a *-derivation δ of A, one defines a one-parameter group of *-automorphisms on A by $U(t) = e^{t\delta}$, $t \in \mathbb{R}$.

Now, if $U(t)$ is of the form $U(t)\,x = u(t)^*\,x\,u(t)$ for a group of unitaries $u(t)$ in A, then one has observability of the energy. And if δ is inner, $\delta = \mathrm{ad}\,ih$, then $U(t)$ will be inner with $u(t) = e^{ith}$, another reason for the interest in inner derivations. For example, Olesen proved in 1974 that every norm continuous group of *-automorphisms of A is inner in A^{**} (the enveloping W^*-algebra of A).

While groups of *-automorphisms are sufficient to describe reversible evolutions of quantum systems, irreversible evolutions may be described by semigroups of completely positive contractive operators. There are a lot of good reasons for the choice of completely positive contractions, both mathematical and physical ones. First of all, the adjoint operators have to fix the state space; thus they must be positive, hence the original ones also have to be. Secondly, an invertible completely positive contraction whose inverse is a completely positive contraction is a *-automorphism; this fails for general positive maps. And even more important is the fact that two interacting systems are usually described by the tensor product of the corresponding C^*-algebras whence the tensor product of the dynamical operators should give the joint dynamics. Complete positivity ensures this, while mere positivity doesn't.

One of the strategies to understand irreversible evolutions (*open quantum systems*) has been to try to 'embed' them into larger reversible systems (*Hamiltonian systems*). This is known as dilation theory (see e.g. [13], [20]).

The generators of norm continuous semigroups of completely positive operators can be described precisely: let $L: A \to A$ be a self-adjoint bounded linear operator (i.e. $L\,A_{sa} \subseteq A_{sa}$). Then $T(t) = e^{tL}$, $t \geq 0$, defines a semigroup of completely positive operators if and only if L is *conditionally completely positive* (see [13]). For a large class of *von Neumann algebras* (weakly closed unital *-subalgebras of $L(H)$) a more detailed description of conditionally completely positive maps is possible; they can be viewed as perturbations of completely positive maps by generalised inner derivations of a certain type. It is expedient to extend the notion of a derivation as follows.

Definition 2.1. Let A be a C^*-algebra and E be a Banach A-bimodule (i.e. E is a Banach space and an A-bimodule with continuous module multiplications). A linear map $\delta: A \to E$ satisfying

$$\delta(ab) = a\,(\delta b) + (\delta a)\,b \qquad (a, b \in A)$$

is called an *E-valued derivation* of A. Every such derivation is a bounded operator as proved by Ringrose [29]. Again, δ is said to be *(E-)inner* if $\delta = \mathrm{ad}\,x$ for some $x \in E$. A linear map $d: A \to E$ is said to be a *generalised inner derivation* if $d(a) = ax + ya$ for some $x, y \in E$ and all $a \in A$. In this case, we write $d = d_{x,y}$. Note that, if A is unital, $d_{x,y}$ is nothing but an additive perturbation of $\mathrm{ad}\,x$ by left multiplication with $x + y$.

The following situation often arises. A derivation of a C^*-algebra A is not inner in A but will become inner when A is regarded as a C^*-subalgebra of another C^*-algebra B and B is viewed as an A-bimodule. For example, we observed above that a derivation of a simple C^*-algebra A need not be inner in A, but it will be inner in the multiplier algebra $M(A)$ (another result by Sakai [32]). As, by the Gelfand-Naimark theorem, each C^*-algebra A can be considered as a C^*-subalgebra of some $L(H)$, the following question naturally arises.

Problem 2.2. *Let $A \subseteq L(H)$. Is every derivation $\delta: A \to L(H)$ inner?*

This problem can be considered as the major open question in the theory of (bounded) derivations of C^*-algebras. So far a number of important contributions have been made, and it is widely conjectured that, at least for von Neumann algebras, the answer is always yes. An affirmative answer for type I and for hyperfinite von Neumann algebras was given by Johnson and Ringrose in 1972, and for the properly infinite case by Christensen in [4]. Problem 2.2 also serves as the motivation for our exposition of the interrelations between derivations and completely bounded maps. Our final purpose is to present Christensen's equivalent formulation of Problem 2.2 in terms of completely bounded maps, and to relate it to a number of other important structural properties and questions (see below and Chapter 8 of [26]). Note that it is tantamount to ask whether every derivation $\delta: A \to L(H)$ can be extended to a derivation on $L(H)$.

One of these applications is to a canonical decomposition of conditionally completely positive maps combining results by Lindblad from 1976 and Evans from 1977 [13].

Theorem 2.3. *The following conditions on a W^*-algebra A are equivalent.*

(a) *Whenever A is faithfully represented as a von Neumann algebra on a Hilbert space H, then every derivation $\delta: A \to L(H)$ is inner.*

(b) *Whenever A is faithfully represented as a von Neumann algebra on a Hilbert space H, then every conditionally completely positive ultraweakly continuous self-adjoint linear map $L: A \to L(H)$ can be decomposed as $L = \psi + d_{x,x^*}$ with $\psi: A \to L(H)$ completely positive and $x \in L(H)$.*

Completely positive maps not only are important in the applications to mathematical physics but also play a central role in the theory of tensor products of C^*-algebras (see Section 4), non-commutative harmonic analysis, and non-commutative probability theory where they serve as transition operators of non-commutative stochastic processes.

In addition to the relation between derivations and homomorphisms given by exponentiation, $\delta \mapsto e^\delta$, there is a more algebraic connection which has also been known for a long time.

Suppose that A is a C^*-subalgebra of a C^*-algebra B and let $\delta: A \to B$ be a derivation. Define a homomorphism $\rho: A \to M_2(B)$ by

$$\rho(a) = \begin{pmatrix} a & \delta(a) \\ 0 & a \end{pmatrix}, \qquad (a \in A).$$

If A is unital, then ρ will be unital, but ρ need not be a *-homomorphism if δ is a *-derivation. Actually, the following is easily obtained [26].

Proposition 2.4. *Let $A \subseteq L(H)$ be a C^*-algebra. The derivation $\delta: A \to L(H)$ is inner if and only if the canonically associated homomorphism $\rho: A \to L(H^2)$ constructed above is similar to a *-homomorphism, i.e. there is an invertible operator $S \in L(H^2)$ such that $a \mapsto S^{-1}\rho(a)S$ defines a *-homomorphism.*

This result turns out to play a key role in an attack to solve Problem 2.2 (see the following section). The question how different homomorphisms of C^*-algebras can be from *-homomorphisms has been investigated by many authors. For example, a result due to Gardner [31] stating that two C^*-algebras which are isomorphic as algebras are in fact *-isomorphic yields a factorisation of an isomorphism $\rho: A \to B$ between C^*-algebras A and B into a product of a *-isomorphism and an automorphism of the form e^δ, δ a derivation of A.

3. The similarity problem

In 1955 Kadison raised the question when a given homomorphism from a C^*-algebra A into $L(H)$ is similar to a *-homomorphism [19]. This was preceded by a related question whether a bounded representation of a topological group is similar to a unitary representation. The latter is certainly true for finite groups which is a classical result, and Dixmier [10] gave an affirmative answer for amenable groups. However, the result fails in general as was shown by Kunze and Stein in 1960. Kadison's question is still open, and in the present section we will develop the terminology to state a partial, but important answer due to Haagerup [16]. (In [16] the reader may find additional comments on the history of this problem.) From Haagerup's theorem a characterisation of inner derivations first given by Christensen in [5] is immediate (Theorem 3.3 below).

Let $\rho: A \to L(H)$ be a homomorphism. If there is a similarity $S \in L(H)$ such that $\pi(a) = S^{-1}\,\rho(a)\,S$ defines a *-homomorphism, then, since π is a contraction, ρ has to be bounded by $\|S^{-1}\|\,\|S\|$. Whether or not every homomorphism from a C^*-algebra is necessarily bounded was an open question since the beginning of the theory of C^*-algebras in the 1940's. Even in the commutative case the answer wasn't clear for many years, and had been one of the main stimuli in automatic continuity theory. One of the early answers is Gelfand's result stating that every homomorphism from a C^*-algebra into a semi-simple commutative Banach algebra is bounded, but it took some time until the assumption of semi-simplicity could be dropped (which was done by Laursen in 1987 for epimorphisms). The question for the case $A = C(X)$ was finally answered by Esterle in 1978. A good up-to-date account of this topic is given in Dales' paper [9].

If we extend ρ to $\rho_n: M_n(A) \to L(H^n)$, then $\rho((a_{ij})) = S_n\,\pi_n((a_{ij}))\,S_n^{-1}$ where S_n denotes the n-fold direct sum of S. Since π_n is a contraction and $\|S_n\| = \|S\|$, $\|S_n^{-1}\| = \|S^{-1}\|$ for all $n \in \mathbb{N}$, we still get that $\|\rho_n\| \leq \|S^{-1}\|\,\|S\|$. This stronger boundedness property motivates the following definition.

Definition 3.1. Let A and B be C^*-algebras, and for a linear map $\phi: A \to B$ let $\phi_n: M_n(A) \to M_n(B)$ be its extension as defined in Definition 1.3. Then ϕ is said to be *completely bounded* if $\sup_n \|\phi_n\| < \infty$, and in this case $\|\phi\|_{cb} = \sup_n \|\phi_n\|$ is called the *completely bounded norm* of ϕ. Moreover, ϕ is called *completely contractive*, respectively *completely isometric*, if $\|\phi\|_{cb} \leq 1$, respectively ϕ_n is an isometry for all $n \in \mathbb{N}$.

The set $CB(A, B)$ of all completely bounded linear maps from A into B is a Banach space under $\|\cdot\|_{cb}$, but is not complete under $\|\cdot\|$, in general; e.g. $(CB(A, L(H)), \|\cdot\|)$ is never complete and is topologically small, i.e. a rare subset, in $L(A, L(H))$ unless both A and H are finite dimensional [33].

The surprising result by Haagerup is that the complete boundedness of a homomorphism from a C^*-algebra not only is a necessary but also a sufficient condition for similarity to a *-homomorphism.

Theorem 3.2. (Haagerup 1983) *Let A be a unital C^*-algebra and $\rho: A \to L(H)$ be a unital homomorphism. Then ρ is similar to a *-homomorphism if and only if ρ is completely bounded. In this case, there exists a similarity S such that $a \mapsto S^{-1}\,\rho(a)\,S$ is a *-homomorphism and $\|\rho\|_{cb} = \|S^{-1}\|\,\|S\|$.*

This result was proved by Haagerup in [16]; a different proof given by Paulsen will be outlined in Section 5.

Suppose that $\delta: A \to L(H)$ is a derivation where $A \subseteq L(H)$. Since a derivation annihilates every central projection we may assume that A is unital whence the canonically associated homomorphism ρ: $A \to L(H^2)$ is unital. Using the canonical shuffle we easily obtain that $\|\delta_n\| \leq \|\rho_n\| \leq \|\delta_n\| + 2$ for all $n \in \mathbb{N}$, i.e. δ is completely bounded if and only if ρ is completely bounded. Combining Theorem 3.2 with Proposition 2.4 thus yields the following result (cf. [26]).

Theorem 3.3. (Christensen 1982) *Let $\delta: A \to L(H)$ be a derivation of a C^*-subalgebra A of $L(H)$. Then δ is inner if and only if δ is completely bounded.*

Christensen's original proof [5] rests on the ultrastrong continuity of a derivation defined on a properly infinite von Neumann algebra [4] as well as on an estimate relating the norm $\|\mathrm{ad}\, x_{|A}\|$ and the distance of $x \in L(H)$ to the commutant A'. It follows in particular that every derivation of an injective von Neumann algebra (for the terminology see Section 4) and of a C^*-algebra with cyclic vector is inner in $L(H)$. Both the arguments of Christensen and Haagerup use in some way Pisier's non-commutative Grothendieck inequality.

In the remainder of this section we will discuss some examples of completely bounded maps. The first result is a simple consequence of the fact that an element a in a C^*-algebra A is selfadjoint if $\varphi(a) \in \mathbb{R}$ for every state φ of A.

Proposition 3.4. *The following conditions on a unital homomorphism ρ between unital C^*-algebras are equivalent.*

(a) *ρ is contractive.*

(b) *ρ is completely contractive.*

(c) *ρ is a *-homomorphism.*

Together with the *-homomorphisms, the following are the prototypes of completely bounded maps. Let $a, b \in L(H,K)$. The mapping

$$M_{a^*,b}\colon L(K) \longrightarrow L(H), \quad x \mapsto a^*xb$$

is called a *two-sided multiplication*. Since $(M_{a^*,b})_n = M_{a_n{}^*,b_n}$, where $c_n \in L(H^n, K^n)$ is the n-fold direct sum of $c \in L(H,K)$, it is easily calculated that $M_{a^*,b}$ is completely bounded with

$$\|M_{a^*,b}\|_{cb} = \|a\|\,\|b\|.$$

In Section 4 we will discuss the representation theorems which state that every completely bounded (completely positive) linear map can be decomposed into a *-homomorphism and a (completely positive) two-sided multiplication. The completely positive multiplications can be described as follows.

Proposition 3.5. *The following conditions are equivalent.*

(a) *$M_{a^*,b}$ is positive.*

(b) *$M_{a^*,b}$ is completely positive.*

(c) *$M_{a^*,b} = M_{c^*,c}$ for some $c \in L(H,K)$.*

The proof given in [22] for the case $H = K$ is easily adopted to cover Proposition 3.5. Note in addition that the following polarisation identity holds which is useful in a deduction of Wittstock's decomposition theorem (Theorem 4.4 below)

(1) $$M_{a^*,b} = \tfrac{1}{4}\sum_{k=0}^{3} i^k\, M_{(b+i^k a)^*,\,b+i^k a}.$$

Some matrix calculations show that each bounded linear functional φ on a C^*-algebra is completely bounded with $\|\varphi\| = \|\varphi\|_{cb}$, and if φ is positive, then it is completely positive. As a result, bounded respectively positive linear mappings into commutative C^*-algebras are completely bounded respectively completely positive, and their norms coincide with the completely bounded norm (here, the identification $M_n(C(X)) = C(X, M_n)$ turns out to be useful). Likewise each positive linear map *from* a commutative C^*-algebra is completely positive which was already noted by Stinespring in 1955, however the corresponding result for bounded maps fails.

Finite-dimensionality has also its consequences on the behaviour of completely positive and completely bounded maps. For example, Choi proved that every n-positive linear map from M_n into a C^*-algebra is completely positive (cf. [26]), and Smith showed that $CB(A, M_n) = L(A, M_n)$ for every C^*-algebra A and that $\|\phi\|_{cb} = \|\phi_n\| \leq n\,\|\phi\|$ for each $\phi \in L(A, M_n)$ (cf. [26]). However, as Haagerup [17] observed, there is in general no $m \in \mathbb{N}$ such that $\|\phi_m\| = \|\phi\|_{cb}$ if $\phi \in L(M_n, B)$.

The next result is not unexpected.

Proposition 3.6. *For all C^*-algebras A and B we have*

$$CP(A,B) \subseteq CB(A,B),$$

and the norm and the completely bounded norm of a completely positive map coincide.

This can be deduced nicely from Lemma 1.2. Assuming without restriction that A is unital we take $a \in M_n(A)$ with $\|a\| \leq 1$ whence $\left(\begin{smallmatrix} 1 & a \\ a^* & 1 \end{smallmatrix}\right) \in M_{2n}(A)$ is positive. The complete positivity of ϕ yields that

$$\phi_{2n}\begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix} = \begin{pmatrix} \phi_n(1) & \phi_n(a) \\ \phi_n(a)^* & \phi_n(1) \end{pmatrix}$$

is positive which entails that

$$\|\phi_n(a)\| \leq \|\phi_n(1)\| = \|\phi(1)\|.$$

Therefore, ϕ is completely bounded with $\|\phi\|_{cb} \leq \|\phi\|$, and the other inequality is obvious.

In particular, the linear span of the completely positive operators is contained in the completely bounded operators, and it was an open question for some time whether equality always holds. This is in fact true for a certain class of C^*-algebras which will be discussed in the next section, but for instance not the case for $A = B = C[0,1]$, as proved by Smith [33].

So far we haven't provided any concrete examples of positive respectively bounded maps that are *not* completely positive respectively completely bounded. The easiest positive mapping which is not 2-positive is the transpose map on M_2, and an infinite dimensional analogue on $L(\ell^2)$ gives a bounded not completely bounded map (for details see [26]).

4. Representation and extension theorems

Two important features of bounded linear functionals on C^*-algebras are the Jordan decomposition and, of course, the Hahn-Banach theorem. The former was established by Grothendieck in 1957 and generalises the fact that every bounded regular Borel measure on a compact Hausdorff space is a linear combination of four positive measures, while the latter is clearly an indispensable tool of the theory. In the present section we will discuss possible extensions of these results to arbitrary completely bounded maps.

In order to be able to formulate the problems, we have to extend the notions of complete positivity and complete boundedness as follows.

Definition 4.1. Every subspace M of a C^*-algebra A is called an *operator space*, with the understanding that, for each $n \in \mathbb{N}$, $M_n(M)$ is regarded as a subspace of $M_n(A)$. Every self-adjoint subspace S of a unital C^*-algebra which contains the identity is called an *operator system*. Note that the self-adjoint part S_{sa} of S is a real ordered normed space with generating cone $S_+ = \{x \in S \mid x \geq 0\}$ since

$$x = \tfrac{1}{2}(\|x\| + x) - \tfrac{1}{2}(\|x\| - x) \qquad (x \in S_{sa}).$$

Again, $M_n(S)$ is endowed with the order inherited from $M_n(A)$. If B is another C^*-algebra and $\phi: M \to B$ is a linear map, then the notions of ***complete boundedness***, ***complete contractivity***, ***complete isometry***, and ***n-positivity*** respectively ***complete positivity***, if $M = S$ is an operator system, are defined analogously to the case $M = A$.

An abstract characterisation of operator spaces which goes parallel with Banach's abstract characterisation of the subspaces of $C(X)$ was given by Ruan [30] as follows. Let M be a normed complex vector space, and suppose that for each $n \in \mathbb{N}$ norms are provided on the matrix spaces $M_n(M)$ satisfying

$$\|\alpha x\| \leq \|\alpha\|\,\|x\|, \quad \|x\alpha\| \leq \|x\|\,\|\alpha\|,$$

and

$$\|x \oplus y\| = \max\{\|x\|, \|y\|\}$$

for all $x \in M_n(M)$, $y \in M_m(M)$ and $\alpha \in M_n$. Then M is (completely isometric to) an operator space.

The following generalisation of the Hahn-Banach theorem was proved for the completely positive case by Arveson [1] in 1969, and for the completely bounded case independently by Haagerup [14], Paulsen [24] and Wittstock [37] several years later. Wittstock's original proof used a Hahn-Banach theorem for set-valued mappings into $L(H)$ while Haagerup elaborated techniques previously available for completely positive maps only for completely bounded maps. Paulsen's proof reduces the problem to the completely positive case via the "off-diagonal technique" described below, and the proof of Arveson's theorem can be divided into two steps: first consider the finite-dimensional situation and then extend the result to the general case by exploiting the compactness of closed bounded subsets of $CP(A, L(H))$ in the BW-topology.

Theorem 4.2. *Every completely bounded (completely positive) linear map from an operator space (operator system) in a unital C^*-algebra A into $L(H)$ can be extended to a completely bounded (completely positive) map on A under preservation of the cb-norm.*

A C^*-algebra B is called *injective* if every completely positive linear map from an operator system S in some C^*-algebra A into B can be extended to a completely positive map from A into B. Thus, Arveson's extension theorem states that $L(H)$ is injective. From this and a result by Tomiyama (see e.g. [13] or [35]), it is easily deduced that $B \subseteq L(H)$ is injective if and only if there exists a projection of norm one from $L(H)$ onto B (a *conditional expectation*).

Injectivity is related to a number of other important structural properties of C^*-algebras which are compiled in the next theorem, thus revealing the significance of completely positive operators. It is here where the real sorcerers in the field used all their magic.

Theorem 4.3. *The following conditions on a C^*-algebra A are equivalent.*

(a) *A is nuclear.*

(b) *A has the* CPAP.

(c) *A^{**} is semi-discrete.*

(d) *A^{**} is injective.*

(e) *A is amenable.*

(f) $CB(A^{**}, A^{**}) = \operatorname{lin} CP(A^{**}, A^{**})$.

The various implications in this result are due to Connes [8], Choi and Effros [2], [3], Effros and Lance [12], and Haagerup [15], [17]. In order to explain the terminology we recall that a C^*-algebra A is said to be *nuclear* if for every C^*-algebra B all C^*-cross norms on $A \otimes B$ coincide, or equivalently, $A \otimes_{min} B = A \otimes_{max} B$ where

$$\|x\|_{min} = \sup\{\|\pi_1 \otimes \pi_2(x)\| \mid \pi_1,\ \pi_2 \text{ representations of } A,\ B\}$$

and

$$\|x\|_{max} = \sup\{\|\pi(x)\| \mid \pi \text{ a representation of } A \otimes B\}$$

are the minimal respectively the maximal C^*-cross norm. Among the class of nuclear C^*-algebras are all finite-dimensional and all commutative C^*-algebras, and inductive limits as well as tensor products of nuclear C^*-algebras are nuclear. The reduced group C^*-algebra $C_r^*(G)$ of a locally compact group G is nuclear if and only if G is amenable. For some more information, see e.g. [21] and [35]. A C^*-algebra A has the *completely positive approximation property* (CPAP) if the identity on the dual of A can be approximated by completely positive contractions of finite rank in the topology of simple convergence, while a W^*-algebra R is *semi-discrete* if the identity on R is approximated by normal completely positive contractions of finite rank in the topology of simple convergence on $(R, \sigma(R, R_*))$. Finally, A is *amenable* if every derivation $\delta: A \to E$, E a dual Banach A-bimodule, is inner. A recent discussion of Theorem 4.3 can be found in [28].

Injectivity also plays a role in the generalisation of the Jordan decomposition. The following result generally referred to as Wittstock's decomposition theorem was obtained independently in [14], [24], and [36].

Theorem 4.4. *Let A be a unital and B an injective C^*-algebra. Then $CB(A, B) = \operatorname{lin} CP(A, B)$. More precisely, if $\phi: A \to B$ is completely bounded, then there exists a completely positive map $\psi: A \to B$ with $\|\psi\|_{cb} \leq \|\phi\|_{cb}$ such that $\psi \pm \operatorname{Re}(\phi)$ and $\psi \pm \operatorname{Im}(\phi)$ are all completely positive.*

Here, the *real* and *imaginary parts* of a linear map ϕ are defined by $\operatorname{Re}(\phi)(x) = \frac{1}{2}(\phi(x) + \phi(x^*)^*)$ and $\operatorname{Im}(\phi)(x) = \frac{1}{2i}(\phi(x) - \phi(x^*)^*)$, respectively. The decomposition of a completely bounded linear map into a linear combination of completely positive maps is not always possible, e.g. if $A = B = C[0,1]$ [33]. If $A = B$ is a W^*-algebra, then the injectivity is also a necessary condition for the decomposition property as observed by Haagerup in [17].

It emerged that Theorem 4.4 is in fact an immediate consequence of the following representation theorem which was proved by Stinespring in 1955 for completely positive maps [34], and by Paulsen in 1984 for completely bounded maps [24].

Theorem 4.5. *Let A be a unital C^*-algebra and $\phi: A \to L(H)$ be a completely bounded (completely positive) linear map. Then there exist a representation (π, K) of A and $v_i \in L(H, K)$, $i = 1, 2$ such that*

$$\phi = M_{v_1^*, v_2} \circ \pi \tag{2}$$

and $\|v_i\| = \|\phi\|_{cb}^{\frac{1}{2}}$, $i = 1, 2$. If $\|\phi\|_{cb} = 1$, then v_1 and v_2 can be taken to be isometries, and if ϕ is completely positive, then v_1 and v_2 can be taken equal, equivalently, $M_{v_1^*, v_2}$ is completely positive.

Stinespring's paper from 1955 in which the notion of a completely positive map was introduced can be viewed as both the historical as well as the conceptual starting point of the whole theory. Originally intended as an extension of a dilation theorem due to Naimark, it also generalises the famous GNS-construction. In fact, if φ is a state of a C^*-algebra A, the GNS-construction yields a triple $(\pi_\varphi, H_\varphi, \xi_\varphi)$ consisting of a cyclic representation (π_φ, H_φ) with cyclic vector ξ_φ such that $\varphi(x) = (\pi_\varphi(x)\xi_\varphi \mid \xi_\varphi)$ for all $x \in A$, and by choosing $H = \mathbb{C}$, $K = H_\varphi$ and $v: H \to K$, $v1 = \xi_\varphi$ this translates into $\varphi = M_{v^*, v} \circ \pi_\varphi$. Generally, the triple (π, K, v) is called a *Stinespring representation* of the completely positive map ϕ, and it is easily seen that (π, K, v) is unique up to unitary equivalence if $\pi(A)vH$ is total in K. However, for the completely bounded case, no additional assumption is known making the above representation unique up to unitary equivalence. More information on this topic is contained in [26], [35], and also [13] where the Stinespring representation is derived from the Kolmogorov decomposition for positive-definite kernels.

From Theorem 4.5, Wittstock's decomposition theorem is quickly deduced (cf. [24] and [26]). To do this it suffices to take $B = L(H)$. If $\phi: A \to L(H)$ is completely bounded, then, by (1) and (2),

$$\phi = \tfrac{1}{4} \sum_{k=0}^{3} i^k \, M_{(v_2 + i^k v_1)^*, v_2 + i^k v_1} \circ \pi \tag{3}$$

whence ϕ can be linearly combined by four completely positive maps. A simple rearrangement of (3) shows that

$$\psi = \frac{1}{2}(M_{v_1^*, v_1} \circ \pi + M_{v_2^*, v_2} \circ \pi)$$

meets the conditions of Theorem 4.4.

The reader will have noticed the rather long time which passed after the representation and the extension theorems for completely positive maps until their counterparts for completely bounded maps were obtained. The reason for this was the lack of a method relating completely bounded maps to completely positive ones in a natural way. This was remedied by Paulsen's "off-diagonal technique" which concludes this section.

Lemma 4.6. (Paulsen 1982) *Let A and B be unital C^*-algebras, and let $M \subseteq A$ be an operator space. Define an operator system $S \subseteq M_2(A)$ by*

$$S = \left\{ \begin{pmatrix} \lambda & a \\ b^* & \mu \end{pmatrix} \;\middle|\; \lambda, \mu \in \mathbb{C}, a, b \in M \right\},$$

and for each linear map $\phi: M \to B$ a linear map $\Phi: S \to M_2(B)$ by

$$\Phi \begin{pmatrix} \lambda & a \\ b^* & \mu \end{pmatrix} = \begin{pmatrix} \lambda & \phi(a) \\ \phi(b)^* & \mu \end{pmatrix}.$$

Then ϕ is completely contractive if and only if Φ is completely positive.

In the surprisingly simple proof one uses first the canonical shuffle and a module property of ϕ_n to reduce to the case $n = 1$, i.e. to contractivity respectively positivity, and then an approximation as well as a factorisation argument to reduce further to consideration of $\left(\begin{smallmatrix} 1 & a \\ a^* & 1 \end{smallmatrix}\right)$ instead of arbitrary elements of S. Applying Lemma 1.2 twice accomplishes the proof.

This lemma is used for example in the proof of the extension theorem (Theorem 4.2) as follows. If $\phi: M \to L(H)$ is completely bounded with $||\phi||_{cb} = 1$, Lemma 4.6 yields a completely positive map $\Phi: S \to M_2(L(H)) = L(H^2)$ which can be extended to $\Psi \in CP(M_2(A), L(H^2))$ under preservation of the norm by Arveson's extension theorem. Letting w_1 respectively w_2 be the isometries from H onto $H \oplus 0$ respectively $0 \oplus H$ and $\iota: A \to M_2(A)$ the embedding into the upper left corner we obtain a complete contraction $\psi: A \to L(H)$ extending ϕ by

$$\psi = M_{w_1^*, w_2} \circ \Psi \circ M_{1, w_1 w_2^*} \circ \iota.$$

5. Completely bounded homomorphisms

This final section is devoted to a deduction of Haagerup's characterisation of those bounded unital homomorphisms which are similar to *-homomorphisms (Theorem 3.2) from the following result by Paulsen [25]. By an *operator algebra* we understand a unital subalgebra of some C^*-algebra.

Theorem 5.1. (Paulsen 1984) *For every completely bounded unital homomorphism $\rho: A \to L(H)$ on an operator algebra A there exists an invertible operator $S \in L(H)$ such that $||\rho||_{cb} = ||S^{-1}||\,||S||$ and $M_{S^{-1},S} \circ \rho$ is a completely contractive homomorphism. Moreover,*

$$||\rho||_{cb} = \inf\{||R^{-1}||\,||R|| \mid M_{R^{-1},R} \circ \rho \text{ is completely contractive}\}.$$

The main idea in the proof of this result is to use Theorem 4.2 to extend the homomorphism ρ to the C^*-algebra containing A and the representation theorem applied to the extended map in order to introduce a new norm on H which is equivalent to the original one such that ρ becomes completely contractive. Once this is done, Haagerup's theorem is immediate from Theorem 5.1 and Proposition 3.3.

At about the same time when Haagerup proved Theorem 3.2, Hadwin showed in [18] that a unital homomorphism from a C^*-algebra into $L(H)$ is similar to a *-homomorphism if and only if the homomorphism lies in the span of the completely positive maps. This together with Wittstock's decomposition theorem yields an alternative argument for Theorem 3.2, without giving the norm identity. Theorem 5.1 is also useful in other applications, for instance to Halmos' question whether every polynomially bounded operator is similar to a contraction, see [26].

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